DIFFERENTIAL EQUATIONS

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Chapter 1

INTRODUCTION TO DIFFERENTIAL EQUATIONS

1.1 What differential equations are, and what it means to solve them

1.1.1 The description of functions from $\mathbb{R}$ to $\mathbb{R}^n$ in terms of differential equations.

If $\mathbf{x}(t)$ denotes the position of a particle of mass m at time t in $\mathbb{R}^3$, and if there is a force $\mathbf{F}(\mathbf{x})$ acting on the particle when its position is $\mathbf{x}$, then Newton's Second Law tells us that

$$\mathbf{x}''(t) = \frac{1}{m}\mathbf{F}(\mathbf{x}(t)) \ . \tag{1.1}$$

For example, if the force on the particle is due to the gravitational interaction with a point mass M located at the origin, then according to Newton's Universal Theory of Gravitation,

$$\mathbf{F}(\mathbf{x}) = -GMm\frac{\mathbf{x}}{\|\mathbf{x}\|^3} \ ,$$

where G is the gravitational constant, and then (1.1) becomes

$$\mathbf{x}''(t) = -GMm\frac{\mathbf{x}(t)}{\|\mathbf{x}(t)\|^3} \tag{1.2}$$

Newton showed that the only continuously twice differentiable curves in $\mathbb{R}^3$ that satisfy (1.2) are ellipses and parabolas, and from his further analysis of the solution curves was able to deduce Kepler's Laws of Planetary Motion – one of the most significant advances in modern science.

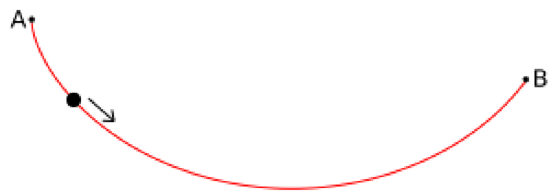
As in this example, many functions of interest – the functions parameterizing planetary trajectories in this case – are specified by natural laws in terms of some relation, expressed in terms of an equation, between the function and its various derivatives. The equation (1.2) specifies just such a relation between function and its the second derivative.

The theory of *differential equations* is concerned with deducing as much information as possible about the functions that satisfy the required relation. In many cases, as in this example, the functions that solve the equation can be viewed as representing curves, and this gives a geometric perspective on the subject that turns out to be very useful.

In some cases, one can find an explicit formula for the class of solution curves. In others cases one cannot do this, but one can still deduce answers to important questions about the set of solutions. For example, one can ask whether for a given force law $\mathbf{F}$, there exist *periodic* solutions of (1.1). For the specific force law in (1.2), we know the answer is "yes": The elliptical solutions are periodic, but the parabolic solutions are not. One way to see this is to explicitly find *all* solutions of (1.2), which can be done in this case. But we shall see that there are other ways to answer such questions without finding all of the solutions explicitly, which, in general, is not possible.

Before making general definitions, we turn to another example. Consider two points $\mathbf{A} = (x_0, y_0)$ and $B = (x_1, y_1)$ in the x, y plane. We suppose that $x_0 < x_1$ and $y_1 < y_0$ so that the straight line segment from $\mathbf{A}$ to $\mathbf{B}$ "runs downhill to the right".

Connect the points A and B with a "wire" that traces out the graph of a function $y(x)$ with $y(x_0) = y_0$ and $y(x_1) = y_1$, giving a "track" running from $\mathbf{A}$ to $\mathbf{B}$:



Depending on the shape of the wire, specified by the function $y(x)$, a bead sliding along the wire, started from rest at $\mathbf{A}$ and acted upon only by gravity (and the forces keeping it on the wire) will take a certain time T to reach $\mathbf{B}$. The *brachisochrone problem* is to find the function $y(x)$ that yields the minimum time T of travel. We shall see that if such a curve exists, it must satisfy the differential equation

$$y(x)\sqrt{1 + |y'(x)|^2} = C \tag{1.3}$$

for some constant C.

The equation (1.3) is another example of a differential equation; it specifies a relation between the function $y(x)$ and its first derivative $y'(x)$. As we shall see, when $y(x)$ satisfies (1.3), its graph is a *cycloid*.

Now let us write (1.3) in another way. Introduce the function

$$F(y, z) = y\sqrt{1 + z^2} - C\ .$$

Then (1.3) is equivalent to $F(y(x), y'(x)) = 0$, or, more briefly,

$$F(y, y') = 0 \tag{1.4}$$

Though it was not the case in this example, in other examples, we will have a relation among x, $y(x)$ and $y'(x)$.

We shall solve this problem later on and see that there is indeed such a minimizing curve, but it is not obvious at this point that such a curve exists.

Definition 1 (First order differential equation for a real valued function)**.** *Let G be a real valued function on $(a,b)\times\mathbb{R}\times\mathbb{R}$. A differentiable real valued function $y(x)$ defined on (a,b)* solves the differential equation

$$G(x,y,y') = 0 \tag{1.5}$$

if it is that case that $G(x,y(x),y'(x)) = 0$ for all $x\in(a,b)$. (We allow the cases $a=-\infty$ and $b=\infty$.) The set of all functions solving the equation is its solution set.

If the function $G(x,y,z)$ is continuously differentiable, we can often write $G(x,y,y') = 0$ in a better form. The equation $G(x,y,z) = 0$ specifies a surface in $\mathbb{R}^3$. Even more simply, if x is not present, as in our example from the brachistochrone problem, $F(y,z) = 0$ specifies a curve in $\mathbb{R}^2$.

In fact, in the case of the brachistochrone problem, we can easily solve $F(y,z) = 0$ for z as a function of y: We find

$$z = \pm\sqrt{(C/y)^2-1}\ .$$

This gives us *two* differential equations of a more explicit form

$$y' = -\sqrt{(C/y)^2-1} \quad \text{and} \quad y' = \sqrt{(C/y)^2-1}\ .$$

The equation on the left, on which y' is negative, describes the "downhill" part of the track, and the one of the right, on which y' is positive, describes the "uphill" part of the track, as we shall see.

While in this case we could solve $F(y,z) = 0$ for z as a function of y, we can more generally invoke the Implicit Function Theorem to write $z = f(y)$ for some continuously differentiable function f. The Implicit Function Theorem says that if $F(y,z)$ is continuously differentiable, and if

$$F(y_0,z_0) = 0 \qquad \text{and} \qquad \frac{\partial}{\partial z}F(y_0,z_0) \neq 0\ , \tag{1.6}$$

then there exists a continuously differentiable function $f(y)$ with $f(y_0) = z_0$ and an open neighborhood of (y_0,z_0) in which

$$F(y,z) = 0 \qquad \Longleftrightarrow \qquad z = f(y)\ .$$

Thus, under the conditions (1.6), there is an open neighborhood of (y_0,z_0) in which the differential equation $F(y,y') = 0$ is equivalent to an equation of the form

$$y' = f(y) \tag{1.7}$$

in the sense that they both have the same set of solution curves in this neighborhood.

For the more general equation $G(x,y,y') = 0$, the same sort of reasoning leads to a local equivalence with a differential equation of the form

$$y' = g(x,y)\ . \tag{1.8}$$

Definition 2 (First order autonomous and non-autonomous equations for real valued functions)**.** *The differential equation (1.7) is the general* first order autonomous differential equation *in standard form. The more general differential equation (1.8) is the general* first order non-autonomous differential equation *in standard form.*

By "standard form" we mean that the equation *explicitly* gives the derivative y' as a function of y (in the autonomous case, in which the derivative only depends on the value of y itself) or as an explicit function of x and y (in the non-autonomous case, in which the derivative depends on x as well as the value of the function y itself). When, a first order equation for a real valued function y is given in the general form (1.5), then it may require some work to express y' as a function of y, and, if necessary, x. It may only be possible to do this locally, and implicitly, using the implicit function theorem. However, in almost all of the cases we consider here, the equation shall arise from applications already in standard form, or else shall be easily put into standard form, as in our example concerning the brachistochrone problem.

Example 1 (A very simple equation)**.** *Let $f(x)$ be any given continuous function on $\mathbb{R}$. The differential equation*

$$y'(x) = f(x)$$

is very simple since the unknown function y does not appear on the right hand side. The Fundamental Theorem of Calculus says that the equation is satisfied if and only if

$$y(x) = y(x_0) + \int_{x_0}^{x} f(z)\mathrm{d}z \ .$$

or, in other words, if F is an anti-derivative of f, then the solutions set of this equation consists of the functions

$$y = F(x) + c$$

where c is an arbitrary constant. For example, with $f(x) = x^2$, we have

$$y(x) = \frac{1}{3}x^3 + c \ ,$$

and this is the general solution to this equation.

Example 2 (Solving a first order non-autonomous equation)**.** *Consider the equation*

$$x + yy' = 0 \ . \tag{1.9}$$

We could put this in standard form

$$y' = -\frac{x}{y} \ ,$$

taking into account that trouble may arise at any x for which $y(x) = 0$. We cannot directly apply the Fundamental Theorem of Calculus since now the right hand side involves the unknown function y.

However, we can reduce to this case by a change of variables*. Introduce the new variable*

$$z(x) = \frac{1}{2}y^2(x) \ .$$

Then $z' = yy'$ and our equation becomes

$$z' = -x \ .$$

This can be solved by the Fundamental Theorem of Calculus: Integrating both sides, we find

$$z = -\frac{1}{2}x^2 + c \ ,$$

or, in terms of y*,*

$$x^2 + y^2 = 2c \ . \tag{1.10}$$

We could solve this for y *as a function of* x*, but the geoemtric natrue of the graph of* $y(x)$ *is already evident in this form: Equation (1.10) describes a curve if and only if* $c > 0$*, in which case it describes a centered circle of radius* $\sqrt{2c}$*. The family of such circles is the family of solution curves of this equation, where by "solution curves", we mean the curves obtaned by graphing the functions in the solution set.*

The way we solved (1.9) in Example 2 may look *ad hoc*, but the equation in this example falls into several categories for which there are methods of solution, among which is the class of *separable equations*. We treat the general case, and a specific example, next.

A *separable equation* is a first order equation that can be written in the form

$$f(y)y' = g(x)$$

for some continuous functions f and g. Then, if F is any antiderivative of f, the equation can be written as

$$(F(y(x)))' = g(x) \ .$$

Then if G is any antiderivative of g, we can integrate both sides to obtain

$$F(y(x)) = G(x) + c$$

which may then be solved to find $y(x)$. This is a very simple class of differential equations in that it may be solved quite directly by the Fundamental Theorem of Calculus.

Example 3 (A separable equation)**.** *Consider the equation* $y' = e^{x+y}$*. Since* $e^{x+y} = e^x e^y$*, the equation can be written as*

$$e^{-y}y' = e^x \ .$$

Integrating both sides,

$$-e^{-y(x)} = e^x + c \ .$$

Evidently the constant c *must be negative. Solving for* $y(x)$ *we find*

$$y(x) = -\ln(-c - e^x) \ .$$

The solution is only defined for $x < \ln(-c)$*, and*

$$\lim_{t\to\ln(-c)} y(x) = \infty \ .$$

For each $y_0 \in \mathbb{R}$*, if* $c = -e^{-y_0} - 1$*,* $y(0) = y_0$*, and the value of* x *at which this solution "blows up" is* $x = \ln(e^{-y_0} + 1)$*, which is very small for large* y_0*.*

Much of the rest of this chapter will be concerned with the first order case, which is a cornerstone of the whole theory. Before turning to that, we discuss two generalizations which give some hint about the fundamental nature of the first order case.

Let $\mathbf{v}$ be a continuous function on some open subset $U \subset \mathbb{R}^n$ with values in $\mathbb{R}^n$. (We allow the case $U = \mathbb{R}^n$.) In many ways, the most fundamental equation of the whole subject is the equation

$$\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t)) \tag{1.11}$$

To *solve* this equations means to find the set of all continuously differentiable functions $\mathbf{x}(t)$ defined on some time interval (a, b) with values in U for which (1.11) is satisfied for all $t \in (a, b)$. The set of curves satisfying this equation is its *solution set.*

Given a continuously differentiable curve $\mathbf{x}(t)$, it is easy to check whether or not it is a solution of (1.11) : Simply differentiate $\mathbf{x}(t)$, and then substitute $\mathbf{x}(t)$ into $\mathbf{v}(x)$. If the result is the same for all t, $\mathbf{x}(t)$ is a solution, and otherwise not. Finding solutions is usually not so easy as checking whether a given function is or is not a solution.

The function $\mathbf{v}(\mathbf{x})$ is a *vector field*, and can be thought of as specifying the velocity $\mathbf{v}$ of the curve as it passes through the point $\mathbf{x}$. This the problem of solving (1.11) is the problem of finding a curve $\mathbf{x}(t)$ that has a specified velocity at each point through which it passes. In equation (1.11), this velocity depends only on the *position* $\mathbf{x}$. More generally it might depend on the *time* t, in which case we would have the equation

$$\mathbf{x}'(t) = \mathbf{v}(t, \mathbf{x}(t)) \tag{1.12}$$

Definition 3 (First order autonomous and non-autonomous equations for $\mathbb{R}^n$ valued functions)**.** *The differential equation (1.11) is the general* first order autonomous system of differential equations *in standard form. The more general differential equation (1.12) is the general* first order non-autonomous system of differential equations *in standard form. In either case, the curves* $\mathbf{x}(t)$ *satisfying the equations are called* integral curves *of the vector field.*

Example 4 (Integral curves of a simple vector field)**.** *Consider the vector field on* $\mathbb{R}^2$ *given by*

$$\mathbf{v}(x, y) = \big(-y, x\big) \ .$$

Then the differential equation

$$\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t)) \tag{1.13}$$

is an efficient way of writing the system of differential equations

$$\begin{aligned} x'(t) &= -y(t) \\ y'(t) &= x(t) \ . \end{aligned}$$

One way to solve this is to introduce the function $f(\mathbf{x}) = \|\mathbf{x}\|^2 = x^2 + y^2$*. Then by the multi-variable chain rule,*

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x}(t)) = \nabla f(\mathbf{x}(t)) \cdot \mathbf{x}'(t) \ . \tag{1.14}$$

Since $\nabla f(x, y) = \big(2x, 2y\big)$*, when* $\mathbf{x}(t)$ *satisfies the equation, (1.14) becomes*

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x}(t)) = \nabla f(\mathbf{x}(t)) \cdot \mathbf{v}(\mathbf{x}(t)) = 2\big(x(t), y(t)\big) \cdot \big(-y(t), x(t)\big) = 0 \ .$$

Therefore, $f(\mathbf{x}(t))$ *is* constant *along any integral curve of our vector field. This tells that the integral curves must be circles. Fix any time* t_0 *and define*

$$r = \sqrt{x^2(t_0) + y^2(t_0)}\ .$$

If $r = 0$*, then evidently* $\mathbf{x}(t) = \mathbf{0}$ *for all* t*, which is indeed a solution. To find the other solutions, let us suppose* $r > 0$*.*

Then for all t*,* $\mathbf{x}(t)$ *must lie on the centered circle of radius* r*, Therefore, for some function* $\theta(t)$*, to be determined, it is the case that*

$$\mathbf{x}(t) = r\big(\cos(\theta(t)), \sin(\theta(t))\big)\ .$$

Now differentiate to find

$$\mathbf{x}'(t) = \theta'(t)r\big(-\sin(\theta(t)), \cos(\theta(t))\big) = \theta'(t)\mathbf{v}(\mathbf{x}(t))\ .$$

Therefore, $\mathbf{x}(t)$ *satisfies (1.13) if and only if* $\theta'(t) = 1$*. By the Fundamental Theorem of Calculus, this is the case if and only if* $\theta(t) = t + c$*. Thus we have that the solution set of our system of equations is given by*

$$\mathbf{x}(t) = r\big((\cos(t+c), \sin(t+c)\big)$$

where $r \geq 0$*, and* c *is arbitrary. Note in the case* $r = 0$*,* $\mathbf{x}(t) = \mathbf{0}$ *for all* t*. This is a* fixed point *solution or, what is the same thing, a* steady state *solution. We shall be particularly interested in finding such solutions, if any, for other vector fields, and studying their* stability*, an important concept to be studied in the next chapter.*

The constant function $f(x, y) = x^2 + y^2$ *was the key to solving this differential equation. One might ask: Where did this come from? We shall develop several methods for finding such "constants of the motion" later in the course. In fact, this equation describes a very simple case of* Hamiltonian flow *which is important in physics, and for which there is always a constant* energy function*. This is the method behind our solution, and we shall come to that in due course. Right now, our main point is to make clear what is means to solve differential equations, and not so much to introduce techniques for doing so.*

Now let us return to our first example concerning Newton's Second Law. This involves the *second order* equation

$$\mathbf{x}''(t) = \frac{1}{m}\mathbf{F}(\mathbf{x}(t))\ .$$

This is second order because it involves second derivatives.

We can reduce this to a first order system of equations by the method of *reduction of order*. Define $\mathbf{y} = \mathbf{x}'$. Then, by definition, our equation is equivalent to the system

$$\begin{aligned} \mathbf{x}' &= \mathbf{y} \\ \mathbf{y}' &= \frac{1}{m}\mathbf{F}(\mathbf{x})\ . \end{aligned}$$

If we introduce the *phase space* variable $\mathbf{z} := (\mathbf{x}, \mathbf{y})$ in $\mathbb{R}^6$ (the vector whose first three entries come from $\mathbf{x}$, and whose second three entries come from $\mathbf{y}$) and the vector field

$$\mathbf{w}(\mathbf{z}) = (\mathbf{y}, m^{-1}\mathbf{F}(\mathbf{x}))\ ,$$

then the system can be written as

$$\mathbf{z}' = \mathbf{w}(\mathbf{z}) \ ,$$

and so we can write the equation coming from Newton's Second Law for a particle moving in $\mathbb{R}^3$ as a first order system, but in 6 variables instead of 3. Solving the second order equation is really the same problem as that of finding integral curves to the vector field $\mathbf{w}$, and it turns out that this is a very fruitful point of view.

This gives some hint of why the subject of finding integral curves of equations like $\mathbf{z}' = \mathbf{w}(\mathbf{z})$ is fundamental to the whole subject, even though many of the differential equations that arise directly in practice are second order. The method of reduction of order is useful also in equations where the highest derivative is not given as an explicit function of the lower derivatives, as we shall see later in this chapter.

We now return to the case of differential equations for real valued functions of a single variable. Though often it is natural to take the independent variable to be x and the dependent variable to be y, as in the brachistochrone problem, in the rest of this chapter we shall use the convention that the independent variable is t and the dependent variable is x. This will make our discussion as harmonious as possible with the higher dimensional case in which we seek integral curves $\mathbf{x}(t)$. It is also often useful to think of t as time and x as position, though that is not necessary.

1.2 Some classes of explicitly solvable equations

1.2.1 First order linear equations

A differential equation of the form

$$x'(t) = p(t)x(t) + q(t) \tag{1.15}$$

is called a *first order linear equation*: It is a first order equation in standard form, and the right hand side is of the form $px + q$, with coefficients that may depend on t. For each t, the graph of $y = px + q$ is a line in the x, y plane.

Suppose that $p(t)$ and $q(t)$ are continuous on some interval (a, b). Then the Fundamental Theorem of Calculus provided us with a continuously differentiable function $p(t)$ such that $P'(t) = p(t)$ also defined on (a, b).

We now compute,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x(t)e^{-P(t)}\right) = \left[x'(t) - p(t)x(t)\right]e^{-P(t)} \ .$$

Since $e^{-P(t)}$ is never zero,

$$x'(t) = p(t)x(t) + q(t) \qquad \Longleftrightarrow \qquad \left[x'(t) - p(t)x(t)\right]e^{-P(t)} = q(t)e^{-P(t)} \ .$$

But by the calculation made above, this is the same as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x(t)e^{-P(t)}\right) = q(t)e^{-P(t)} \ . \tag{1.16}$$

Then by the Fundamental Theorem of Calculus, for any t_0,

$$\left(x(t)e^{-P(t)}\right) - \left(x(t_0)e^{-P(t_0)}\right) = \int_{t_0}^{t} b(s)e^{-P(s)}\mathrm{d}s$$

Multiplying through by $e^{p(t)}$ and rearranging terms, we find

$$x(t) = e^{P(t)-P(t_0)}x(t_0) + \int_{t_0}^{t} e^{P(t)-P(s)}q(s)\mathrm{d}s \ .$$

This shows that for each given x_0 and $t_0 \in (a,b)$, there is exactly one function $x(t)$ defined on (a,b) such that

$$x'(t) = p(t)x(t) + q(t) \qquad \text{and} \qquad x(t_0) = x_0 \ . \tag{1.17}$$

The *initial value problem* for (1.15) is to solve (1.17), and we have just proved that there is always a unique solution, and have even found a formula for it.

If we simply want the *general solution* of (1.15) we may return to (1.16)and apply the Fundamental Theorem of Calculus in antiderivative (as oppose to definite integral) form to write

$$(xe^{-P}) = \int qe^{-P}\mathrm{d}t + c \ ,$$

and hence (since $P = \int p\mathrm{d}t$),

$$x = e^{\int p\mathrm{d}t}\left(\int qe^{-\int p\mathrm{d}t}\mathrm{d}t + c\right) \ . \tag{1.18}$$

We summarize:

Theorem 1 (Existence and uniqueness for first order linear equations)**.** *Let $p(t)$ and $q(t)$ be continuous functions on some interval $(a,b) \subset \mathbb{R}$, where $a = -\infty$ and $b = \infty$ are allowed. Then for every x_0 and $t_0 \in (a,b)$ there exists a unique solution to the initial value problem (1.17), and it is given explicitly by*

$$x(t) = e^{\int_{t_0}^{t} p(s)\mathrm{d}s}x(t_0) + \int_{t_0}^{t} e^{\int_s^t p(r)\mathrm{d}r}q(s)\mathrm{d}s$$

for all $t \in (a,b)$ The general solution of the equation $x' = px + q$ is given by (1.18) where c is an arbitrary constant.

The arbitrary constant c corresponds to the initial condition in the initial value problem in that for any given x_0 and t_0, there will be exactly one value of c for which $x(t_0) = x_0$. Thus, to solve the initial value problem, one may first find the general solution, and then chose the constant c, or one may proceed directly to the solution using definite integrals. We shall illustrate both approaches below.

The idea behind the formulas is easier to remember and apply than the formulas themselves. The point to remember is if you write the equation in the form $x' - px = q$, and multiply through by e^{-P}, where $P' = p$, the left hand side will be a total derivative, which allows x to be found by integration.

Example 5 (Solving a first order linear equation)**.** *Consider the equation*

$$x' = -\frac{1}{t}x + t$$

on the interval $t > 0$. Find the general solution, and the solution with $x(1) = 1$.

To do this, note that $\ln t$ *is an antiderivative of* $1/t$*. Thus, we should multiply*

$$\left(x' + \frac{1}{t}x\right) = t$$

through by $e^{\ln t} = t$ *to obtain the equivalent equation*

$$(x't + x) = t^2 \ ,$$

which is

$$\frac{\mathrm{d}}{\mathrm{d}t}(tx) = t^2 \ .$$

Integrating both sides,

$$tx = \frac{1}{3}t^3 + c \ .$$

Therefore, the general solution is

$$x(t) = \frac{t^2}{3} + \frac{c}{t} \ .$$

Let us check this result: differentiating,

$$x'(t) = \frac{2}{3}t - \frac{c}{t^2} \ ,$$

and

$$\frac{x(t)}{t} = \frac{t}{3} + \frac{c}{t^2} \ ,$$

so that indeed, $x' + x/t = t$ *for all choices of* c*.*

Now to match the initial condition $x(1) = 1$*, we evaluate the general solution at* $t = 1$*, and find*

$$x(1) = \frac{1}{3} + c \ .$$

We satisfy $x(1) = 1$ *is and only if* $c = 2/3$*. More generally, we se that* $c = x(1) - 1/3$*.*

It follows from (1.18) that the general solution of a first order linear equation is a family of curves of the form

$$x(t) = cf(t) + g(t)$$

where c is an arbitrary constant and $f(t) \neq 0$ for any $t \in (a, b)$. Thus, the curves $x(t)$ for different values of c never cross, and through each point in the vertical strip in the (t, x) plane given by $(a, b) \times \mathbb{R}$, there is exactly one such curve.

Conversely, given any such family of curves we combine

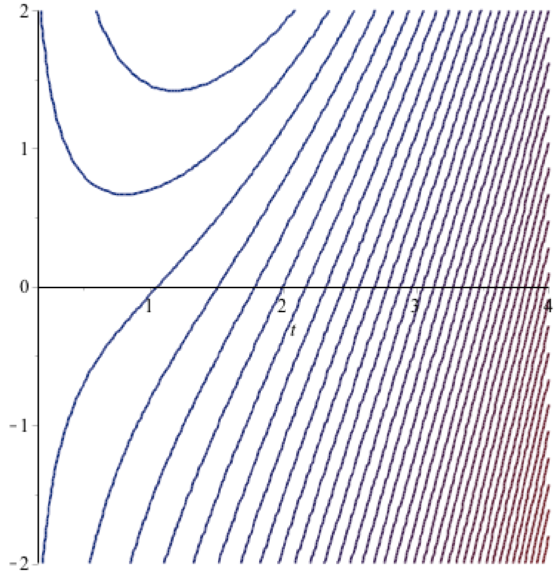
$$x'(t) = cf'(t) + g'(t) \qquad \text{and} \qquad c = \frac{1}{f(t)}(x(t) - g(t))$$

to deduce

$$x'(t) = \frac{f'(t)}{f(t)}(x(t) - g(t)) + g'(t) = \left[\frac{f'(t)}{f(t)}\right] x(t) + \left[g'(t) - \frac{f'(t)}{f(t)}g(t)\right] \ ,$$

so that the curves $x(t)$ are the solution curves of a first order linear equation.

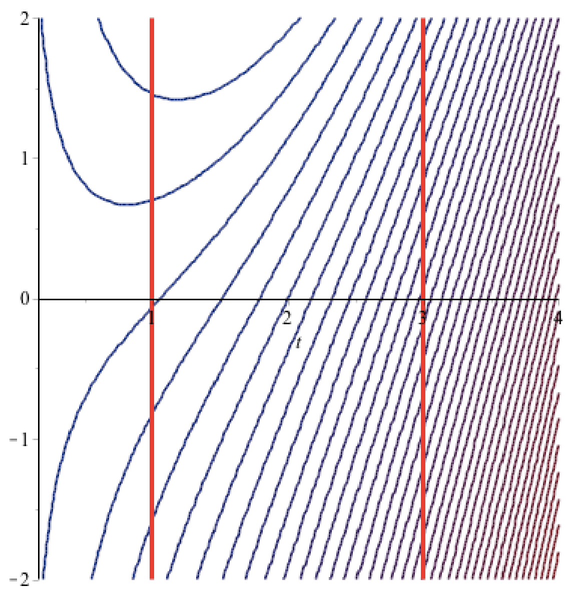
The next figure shows the family of curves in the t, x plane specified by the equation from Example 5 for $0 < t < 4$ and $-2 < x < 2$.



This field of flow lines extends over the entire right half of the t, x plane, though of course we can only graph a fraction of it.

If we think of the equation as describing the motion of a point on the line, and the point is at x_0 at time t_0, then by following the unique curve through (t_0, x_0), we can see where the particle is at every other $t > 0$.

There is a convenient way to think about the information contained in this family of curves. Consider any $0 < t_0, t_1$. Consider the vertical lines $t = t_0$ and $t = t_1$. In the next figure we graph them for $t_0 = 1$ and $t_1 = 3$.



The red lines are vertical, but an optical illusion may obscure this. In any case, since through each point in the right half plane there is exactly one flow curve, we define a function Φ_{t_1,t_0} from $\mathbb{R}$

to $\mathbb{R}$ as follows: $\Phi_{t_1,t_0}(x)$ is the intersection of the flow curve passing through (t_0, x) with the line $t = t_1$. In our graph, locate the point at height x in the first vertical line. Follow the flow line through this point until it intersects the line $t = t_1$. The the height at this point – $x(t_1)$ on this curve – is $\Phi_{t_1,t_0}(x)$. Notice that the *final time* t_1 is on the left in the subscript, and the *initial time* is on the right. We will soon see the utility of this. In any case, the definition does not require that $t_1 > t_0$; one can follow flow lines to the left as well as to the right. We formalize the deifinition below:

Definition 4 (Flow transformation of a linear first order equation)**.** *Consider a first order linear equation $x' = px + p$ with p and q continuous on (a, b), For each $t_0 \neq t_1$ in (a, b) define*

$$\Phi_{t_1,t_0}(x) = x(t_1)$$

where $x(t)$ is the unique solution to

$$x' = px + q \qquad \text{with} \quad x(t_0) = x \ .$$

In practical terms, computing a flow transformation amounts to solving the initial value problem "in general". We can, and will, work with flow transformations to other equations. We have specialized to first order linear equations at this point because this is the only class of equations for which we have so far proved the existence and uniqueness of a solutions curve for arbitrary initial data.

Example 6 (Computing a flow transformation)**.** *Consider the equation*

$$x' = -\frac{1}{t}x + t$$

on the interval $t > 0$ that we considered in Example 5. There we found that the general solution to this equation is

$$x(t) = \frac{t^2}{3} + \frac{c}{t} \ .$$

where c is an arbitrary constant. We must choose c so that $x(t_0) = x$. Evaluating at $t = t_0$, we find

$$x = \frac{t_0^2}{3} + \frac{c}{t_0} \quad \text{which means} \qquad c = t_0 x - \frac{t_0^3}{3} \ .$$

Therefore, the solution curve passing through x at $t = t_0$ is

$$x(t) = \frac{t^2}{3} + \frac{1}{t}\left[t_0 x - \frac{t_0^3}{3}\right] \ .$$

Evaluating this at $t = t_1$, we find

$$\Phi_{t_1,t_0}(x) = \frac{t_0}{t_1}x + \frac{1}{3}\left(t_1^2 - \frac{t_0^3}{t_1}\right) \ .$$

Notice that the flow transformation is a very simple *function of x, the starting point. For all t_0, t_1 it is an affine function in x; i.e., its graph is a straight line. The slope and intercept of the line do depend on t_0 and t_1, but still, the graph of $x \mapsto \Phi_{t_1,t_0}(x)$ is a line.*

Next, consider any $t_2 > 0$. A Simple computation shows that for all x,

$$\Phi_{t_2,t_1}\left(\Phi_{t_1,t_0}(x)\right) = \Phi_{t_2,t_0}(x) \ . \tag{1.19}$$

The identity (1.19) that we derived in the previous example has a natural interpretation which show that it is not an accident: If you start at height x at $t = t_0$, an then follow the unique flow-line to $t = t_1$, and then follow the unique, and therefore same, flow-line from there to $t = t_2$ you arrive at the same height you would have by simply following the flow line directly from $t = t_0$ to $t = t_2$. Thus, the identity (1.19) *always* is true as a consequence of the existence and uniqueness of soluion curve given by Theorem 1.

Neither was it a coincidence that the flow transformations were linear for all t_0, t_1 in the last example. By Theorem 1, the unique solution to $x' = px + q$ passing through x at $t = t_0$ is

$$x(t) = e^{\int_{t_0}^{t} p(s)\mathrm{d}s} x + \int_{t_0}^{t} e^{\int_s^t p(r)\mathrm{d}r} q(s)\mathrm{d}s\ .$$

Therefore, in general

$$\Phi_{t_1,t_0}(x) = \left[e^{\int_{t_0}^{t_1} p(s)\mathrm{d}s}\right] x + \left[\int_{t_0}^{t} e^{\int_s^{t_1} p(r)\mathrm{d}r} q(s)\mathrm{d}s\right]\ .$$

The flow map is important partly for what it tells us about the *sensitivity to initial data*. If solve our equation with initial data $x(t_0) = x_0$, then at time t_1, the value of the solutions is $\Phi_{t_1,t_0}(x)$, by the definition of the flow transform. But if instead we solve our equation with initial data $x(t_0) = x + h$, then at time t_1, the value of the solutions is $\Phi_{t_1,t_0}(x + h)$. In the case of a first order linear equation,

$$\Phi_{t_1,t_0}(x + h) - \Phi_{t_1,t_0}(x) = \left[e^{\int_{t_0}^{t_1} p(s)\mathrm{d}s}\right] h\ .$$

The right hand side does not depend on x, and it does not even depend on coefficient $q(t)$. Provided h is small compared to $\left[e^{\int_{t_0}^{t_1} p(s)\mathrm{d}s}\right]^{-1}$, the left hand side will be small. In this sense, a small change in the initial value does not lead to a large change in the solution, however if $\left[e^{\int_{t_0}^{t_1} p(s)\mathrm{d}s}\right]$ becomes large for some t_1, then the solution will indeed be sensitive to the initial data. The derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\Phi_{t_1,t_0}(x)$$

gives a measure of the sensititvity of the dependence of the initial data for the flow from t_0 to t_1 which we regard as fixed. In this case, the *sensitivity function*

$$\frac{\mathrm{d}}{\mathrm{d}x}\Phi_{t_1,t_0}(x) = e^{\int_{t_0}^{t_1} p(s)\mathrm{d}s}\ ,$$

independent of x.

1.2.2 Bernoulli equations

A differential equation of the form

$$x'(t) = p(t)x(t) + q(t)x^n(t) \tag{1.20}$$

with $n \neq 0, 1$ is called a *Bernoulli equation*. If $n = 0, 1$, then (1.20) is a first order linear equation, and we already know how to solve it.

To solve a Bernoulli equation, introduce a new dependent variable $y(t)$ defined by

$$y = x^{1-n} . \tag{1.21}$$

where we must have $x \neq 0$ for $n > 1$. (We do not assume that n is an integer, though this is suggestion by this notation, which is traditional.) Then

$$x = y^{1/(1-n)} \qquad \text{and} \qquad x' = \frac{1}{1-n} y^{n/(1-n)} y' .$$

Therefore, in the new variable, (1.20) becomes

$$\frac{1}{1-n} y^{n/(1-n)} y' = p y^{1/(1-n)} + q y^{n/(1-n)} .$$

Dividing by $(1-n)^{-1} y^{n/(1-n)}$, we obtain the first order linear equation

$$y'(t) = (1-n)p(t)y(t) + (1-n)q(t) \tag{1.22}$$

Any solution of (1.20) with $\mathbf{x}(t_0) = x_0 \neq 0$ must satisfy $x(t) \neq 0$ on some open interval about $t = t_0$ since $x(t)$ is continuous. Then the calculations above show that $y(t)$ is well defined on this interval, and that $y(t)$ must satisfy (1.22) there with $y(t_0) = x_0^{1-n}$. By Theorem 1, there exists a unique solution to this initial value problem. Now let (a, b) be the maximal open interval about t_0 on which $y(t) \neq 0$. Then everywhere on this interval $x = y^{1/(1-n)}$ is well defined and non-zero. By what we have noted above, it must be the unique solution to (1.20) with $x(t_0) = x_0$. Outside this interval, if $n > 1$, the solution may not exist, as we shall see in examples.

When applying the method, it is usually simplest to simply remember that the change of variables $y = z^{1-n}$ will lead to a linear equation than it is to work with the formula we have derived for this linear equation.

Example 7 (Solving a Bernoulli equation)**.** *Consider the equation*

$$tx^2x' + x^3 = t\cos t , \tag{1.23}$$

for $t > 0$. Dividing through by tx^2 we see that, when we are not dividing by zero, our equation becomes

$$x' = -\frac{x}{t} + x^{-2}\cos t$$

This is a Bernoulli equation with $n = -2$. Thus, we shall make the change of variable $y = x^3$, so that $x = y^{1/3}$. Now let us use this in the original equation*, so we do not introduce trouble be dividing by zero.*

Since $x^2x' = \frac{1}{3}y'$, this equation becomes

$$y' + \frac{3}{t}y = 3\cos t .$$

Since the antiderivative of $3/t$ is $3\ln t$, we multiply through by $e^{3\ln t} = t^3$ to obtain the equivalent equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(t^3 y) = 3t^3\cos t .$$

Integrating both sides, we find

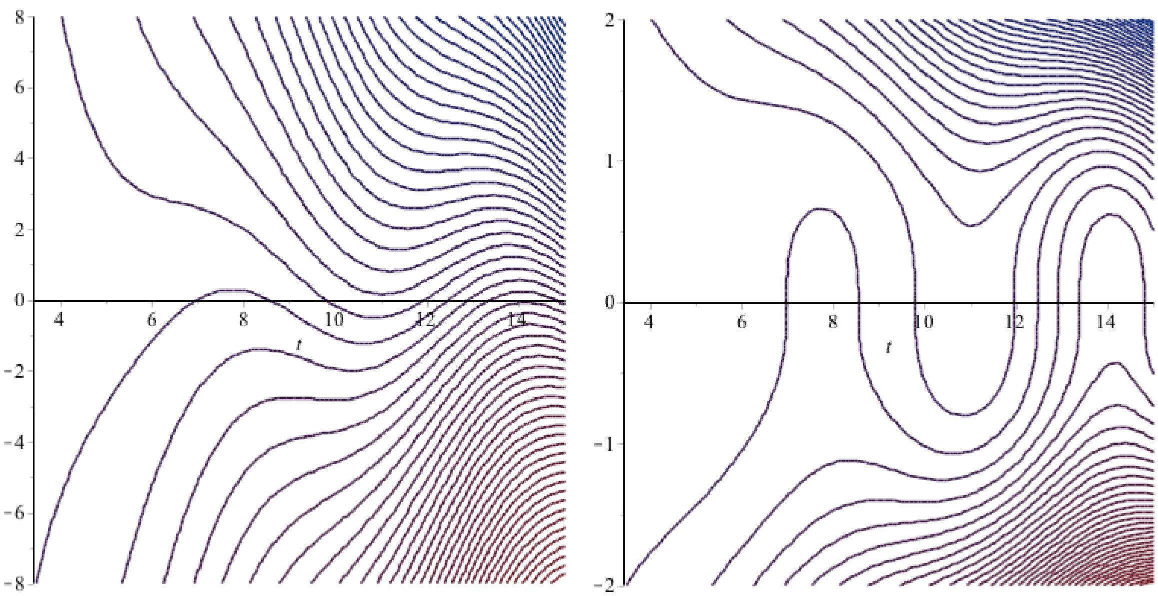
$$y(t) = \sin t + 3t^{-1}\cos t - 6t^{-2}\sin t - 6t^{-3}\cos t + ct^{-3} \ . \tag{1.24}$$

We have just proved that when $x(t)$ solves (1.23) on $(0,t)$, then $y(t) = x^3(t)$ is given by (1.24) for some constant c: At no point did we run any risk of dividing by zero.

Thus, we learn that the general solution of (1.23) is given by

$$x(t) = \left(\sin t + 3t^{-1}\cos t - 6t^{-2}\sin t - 6t^{-3}\cos t + ct^{-3}\right)^{1/3} \ . \tag{1.25}$$

It is interesting to look at the solution curves of both (1.24) and (1.25), which we plot below. On the left, we show that solution curves for (1.24) with $0 < t < 15$, $-8 < y < 8$, and on the right for (1.25) with $0 < t < 15$, $-2 < x < 2$, which is the corresponding region since $y = x^3$.



Whenever $y(t)$ crosses through zero, so does $x(t)$, but with an infinite derivative since $y \mapsto y^{1/3}$ has an infinite derivative at $y = 0$. Indeed, we see the flow curves on the right always cross the horizontal axis with an infinite slope. Still, the flows curves are continuous with a well defined tangent everywhere.

Example 8 (The flow transformation for a Bernoulli equation)**.** *Since the Bernouli equation of the previous example has a unique solution curve passing through each pointin the right half plane, for all $t_0, t_1 > 0$, we have a well defined flow transformation Φ_{t_1,t_0}: For any $x \in \mathbb{R}$, $\Phi_{t_1,t_0}(x)$ is the height at which the soltiion curve with $x(t_0) = x$ intersects the line $t = t_1$. We can compute a formula for it. To simplify the formula, let $g(t)$ denote the function*

$$g(t) := \sin t + 3t^{-1}\cos t - 6t^{-2}\sin t - 6t^{-3}\cos t \ .$$

By (1.25), the solution with $x(t_0) = x$ is

$$x(t) = \left(g(t) + [x^3 - g(t_0)]\frac{t_0^3}{t^3}\right)^{1/3} \ .$$

Therefore,

$$\Phi_{t_1,t_0}(x) = \left(g(t_1) + [x^3 - g(t_0)]\frac{t_0^3}{t_1^3} \right)^{1/3} .$$

This is no longer al linear funtion of x*, but notice that when* t_1 *is large compared to* $|x|$ *and* t_0*,*

$$\Phi_{t_1,t_0}(x) = (g(t_1))^{1/3} ,$$

independent of x*. In this sense, all solutions look like* $x(t) = \sin^{1/3}(t)$ *for large* t*. There are no periodic solutions, but* all *solutions approach this particular periodic function when the time* t *becomes sufficiently large (depending on* c*).*

We will see many example of equations whose solutions do something very complicated over a short time interval, but then "settle down" to some simple sort of long-time behavior.

Example 9 (Solving another Bernoulli equation)**.** *Consider the equation*

$$x' = x^2 . \tag{1.26}$$

There are a number of ways to solve this, but it is a Bernoulli equation, so let us use the corresponding change of variables. Since $n = 2$*, we define*

$$y = x^{1-n} = x^{-1} .$$

This change of variables is singular if $x = 0$*, but note any solution of this equation is monotone increasing, so that any solution* $x(t)$ *with* $x(t_0) = x_0 > 0$ *satisfies* $x(t) > x_0 > 0$ *for all* $t > t_0$*. We compute:*

$$x' = -\frac{1}{y^2}y' \qquad \text{and} \qquad x^2 = \frac{1}{y^2} ,$$

so (1.26) becomes

$$\frac{1}{y^2}y' = -\frac{1}{y^2} .$$

Hence, as long as $y(t) \neq 0$*,* $y'(t) = -1$ *and so*

$$y(t) = x_0^{-1} - (t - t_0) ,$$

and then

$$x(t) = \frac{x_0}{1 - x_0(t - t_0)} .$$

Notice that when

$$t = t_0 + \frac{1}{x_0} ,$$

the denominator vanishes, and the solution curve develops a vertical asymptote, at which point the function is undefined.

While for linear first order equations with continuous coefficients defined on all of $\mathbb{R}$*, all of the solutions are defined for all times* t*. As we see with this example, this need not be the case for non-linear equations. Although* $v(x) = x^2$ *is defined and continuous on all of* $\mathbb{R}$*, there are* no *solutions of* $x' = x^2$ *that are defined for all times* $t \in \mathbb{R}$ *except the trivial solution* $x(t) = 0$ *for all* t*.*

Finally, let us compute the flow transformation $\Phi_{t_0,t_1}(x)$ *for this equation. We have computed that solution passing through* x *at* $t = t_0$ *is given by*

$$x(t) = \frac{x}{1 - x(t - t_0)} \ ,$$

which is defined on $(-\infty, t_0 + 1/x)$ *when* $x > 0$ *and on* $(t_0 + 1/x, \infty)$ *when* $x < 0$*. Thus,*

$$\Phi_{t_1,t_0}(x) = \frac{x}{1 - x(t_1 - t_0)} \ ,$$

with $t_1 \in (-\infty, t_0 + 1/x)$ *when* $x > 0$ *and* $t_1 \in (t_0 + 1/x, \infty)$ *when* $x < 0$*.*

A simple calculation of the sensitivity function gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\Phi_{t_1,t_0}(x) = \frac{1}{(1 - x(t_1 - t_0))^2} \ .$$

Notice that the sensitivty function depends on x *in this case, and it diverges as* t_1 *approches* $t_0 + 1/x$*. A samll change in initial data can result in a large change in the position at a later time for this equation – or even lead to blow-up before time* t_1*.*

Because of the uniqueness, it is still true that as long as both t_1, t_2 *belong to (the same) one of the intervals,*

$$\Phi_{t_2,t_1}\left(\Phi_{t_1,t_0}(x)\right) = \Phi_{t_2,t_0}(x) \ .$$

1.2.3 Ricatti equations

A differential equation of the form

$$x'(t) = p(t) + q(t)x(t) + r(t)x^2(t) \tag{1.27}$$

is called a *Ricatti equation.*

This may appear only mildly more complicated than the first order linear equation since the right hand side is a quadratic, instead of linear, functions of $x(t)$. Unfortunately, there is no general method for solving Ricatti equations.

However, there a general method for deducing the general solution of a Ricatti equation from any one particular solution. Thus, if one can somehow find one solution, one can find the general solution – or at least express it in terms of explicit integrals.

Here is how this works. Suppose that $x_1(t)$ is some solution of (1.27). We seek the general solution in the form $x = x_1 + u$. If both x and x_1 satisfy (1.27) then

$$(x_1 + u)' - p - q(x_1 + u) - r(x_1 + u)^2 = 0 \ .$$

Both the left hand side equals

$$[x_1' - p - qx_1 - rx_1^2] + [u' - (q + 2rx_1)u - ru^2] \ .$$

Since x_1 solves (1.27), we are left with

$$u' - (q + 2rx_1)u - ru^2 = 0 \ .$$

In summary, if x and x_1 are any two solution of (1.27), the $u = x - x_1$ is a solution of

$$u'(t) = [q(t) + 2r(t)x_1(t)]u(t) + r(t)u^2(t) \tag{1.28}$$

which is a Bernoulli equation with $n = 2$, and hence solvable with the substitution $z = u^{-1}$.

Thus, the general solution of (1.27) will be

$$x_1(t) + u(t)$$

where x_1 is our particular solution of (1.27) and u is the general solution of the Bernoulli equation (1.28).

Of course this is only useful if one can find a particular solution. In practice this is often possible. Consider the equation

$$x'(t) = -\frac{1}{t}x(t) + \frac{1}{t^3}x^2(t) + 2t\ . \tag{1.29}$$

Since all of the coefficients are powers of t, it is natural to see of there is a solution of the form

$$x_1 = ct^\alpha$$

for some constants c and α. Inserting this into the equation, we find

$$\alpha ct^{\alpha-1} = -ct^{\alpha-1} + c^2t^{2\alpha-3} + 2t\ .$$

All of the powers of t will be equal if

$$\alpha - 1 = 2\alpha - 3 = 1\ ,$$

which is satisfied for $\alpha = 2$. The equation then reduces to $2c = -c + c^2 + 2$, which is the same as $(c-2)(c-1) = 0$. Hence both t^2 and $2t^2$ are solutions. We can use either one to find the general solution. In the next example, we carry this out.

Example 10 (Solving a Ricatti equation)**.** *We will now find the general solution to (1.29). As we have explained above,*

$$x_1 = t^2$$

is a particular solution. In this example, we have

$$p = 2t\ , \quad q = -\frac{1}{t}\ , \quad r = \frac{1}{t^3}\ .$$

Thus

$$q + 2rx_1 = -\frac{1}{t} + 2\frac{1}{t^3}t^3 = \frac{1}{t}\ ,$$

and (1.28) becomes

$$u' = \frac{1}{t}u + \frac{1}{t^3}u^2\ .$$

Then with $z = u^{-1}$, so that $u' = -z^{-2}z'$ this becomes

$$-z^{-2}z' = \frac{1}{t}z^{-1} + \frac{1}{t^3}z^{-2}\ ,$$

and multiplying through by $-z^2$ we obtain the linear equation

$$z' + \frac{1}{t}z = -\frac{1}{t^3} \ .$$

Since $1/t$ has the antiderivative $\ln t$, we multiply through by $e^{\ln t} = t$ to obtain

$$(tz)' = -\frac{1}{t^2} \ .$$

Integrating both sides, we find

$$tz = \frac{1}{t} + c \ .$$

Therefore, since $z = u^{-1}$,

$$u = \left(\frac{1}{t^2} + \frac{c}{t}\right)^{-1} = \frac{t^2}{ct+1} \ ,$$

and the general solution is

$$x(t) = x_1(t) + u(t) = t^2 + \frac{t^2}{ct+1} = \frac{ct^3 + 2t^2}{ct+1} \ .$$

It is interesting to note that $c = 0$ give the other particular solution that we found, while the solution x_1 is obtain only in the limit $c \to infty$, which brings out an important point about general solutions: It may be necessary to consider limiting values of the arbitrary constants.

The only part of this method that requires any ingenuity – and also some luck – is finding a particular solution. Consider another example

$$x' = \frac{2\cos^2 t - \sin^2 t + x^2}{2\cos t} \ .$$

is a Ricatti equation since the right hand side is quadratic in x, but it is not a Bernoulli equation.

Since the coefficients are powers of $\sin t$ and $\cos t$, it is natural to try a simple expression composed of these. Notice that with the choice $x = \sin t$, the right hand side simplifies to $\cos t$, and so $x_1(t) = \sin t$ is a particular solution.

The general idea is to try something simple for x_1 that is "built out of" the coefficients. As in the main example, it is often good to leave powers and multiples unspecified, and use the equation to determine possible values for them

1.2.4 Reduction of order

Some second order differential equations can be reduced to first order equations that we know how to solve. The general second order equation for a real valued function x of a real variable t is of the form

$$F(t, x, x', x'') = 0$$

for some function F on (some subset of) $\mathbb{R}^4$.

There are two cases in which one may reduce this to a first order equation by standard methods:

(1) The case in which F does not depend on x; i.e., when the equation has the form

$$F(t, x', x'') = 0 \ .$$

(2) The case in which F does not depend on t; i.e., when the equation has the form

$$F(x, x', x'') = 0 \ .$$

The first case is the simplest. Introduce $y = x'$ and then the equation becomes

$$F(t, y, y') = 0$$

which is first order. If it is a first order equation that we know how to solve, we do this to find $y(t) = x'(t)$. Then we integrate to find $x(t)$.

Example 11 (Reduction of order (depdentent varialbe x not present))**.** *Consider the equation*

$$tx'' - x' = t^3 \ .$$

Substituting in $y = x'$ it becomes

$$y' - \frac{1}{t}y = t^3 \ .$$

This is a linear first order equation. Multiplying through by t, we obtain

$$(ty)' = t^3 \ ,$$

and so

$$x' = y = \frac{1}{4}t^3 + \frac{c_1}{t} \ .$$

Integrating once more,

$$x = \frac{1}{16}t^4 + c_1 \ln t + c_2 \ .$$

Notice that we have two arbitrary constant in our general solution of this second order equation.

When the independent variable t is missing, the idea is a little different: We introduce $y = x'$ as before, but now we think of $y = x'$ as a function of x. By the chain rule

$$x'' = \frac{\mathrm{d}}{\mathrm{d}t}x' = \frac{\mathrm{d}}{\mathrm{d}t}y = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = y\frac{\mathrm{d}y}{\mathrm{d}x} \ .$$

Then $F(x, x', x'') = 0$ becomes

$$F\left(x, y, y\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0 \ .$$

which can be written in the form

$$G\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0 \ .$$

This is a first order equation for y as a function of x. If we can solve this, we then have an explicit equation for y as a function of x, and since $y = x'$, this is a first order equation for x.

Example 12 (Reduction of order (indepdentent varialbe t not present))**.** *Consider the equation*

$$x'' + \omega^2 x = 0 \tag{1.30}$$

where $\omega > 0$ is a parameter. Neither t nor x' appear explicitly in this equation. It is the fact that t does not appear that we use here.

Let $y = x'$ so that

$$x'' = y\frac{\mathrm{d}y}{\mathrm{d}x} \ .$$

Then our equation becomes

$$y\frac{\mathrm{d}y}{\mathrm{d}x} = -\omega^2 x \ .$$

This can be written as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y^2}{2}\right) = -\omega^2 x \ .$$

Integrating both sides, we find

$$\frac{y^2}{2} = -\omega^2\frac{x^2}{2} + C \ ,$$

and so

$$y^2 + \omega^2 x^2 = 2C \ ,$$

where evidently the constant of integration must be non-negative. The case $C = 0$ is trivial, and so let us write $2C = r^2$ for $r > 0$

Remembering that $y = x'$, we arrive at the first order equation

$$\omega^2 x^2 + (x')^2 = r^2 \ .$$

It follows that $(\omega x(t), x'(t))$ lies on the circle of radius r for all t. Therefore, for some function $\theta(t)$,

$$(\omega x(t), x'(t)) = r(\sin\theta(t), \cos\theta(t)) \ .$$

Since $(\omega x(t))' = \omega x'(t)$, it follows that

$$(\sin\theta(t))' = \omega\cos(\theta(t)) \ ,$$

and this means that $\theta'(t) = \omega$, so that $\theta(t) = \omega t + \theta_0$ for some constant θ_0. Altogether,

$$x(t) = r\sin(\omega t + \theta_0)$$

$r \geq 0$, $\theta_0 \in [0, 2\pi)$ is the general solution of the eqaution.

The equation(1.30) is very important, and there are many ways to solve it. Some of these are simpler than the present method, but we have chosen to illustrate the method on a simple and important example. You may have been able to see right away that $r\sin(\omega t + \theta_0)$ is a solution. It is important ot understnad how the analysis above shows that this is the general solution.

1.3 Exercises

1.1 Find the general solution of the differential equation

$$tx' = 3x + t^4$$

for $t > 0$. Find the correspoding flow transformation, and the particular solution with $x(1) = 2$.

1.2 Find the general solution of the differential equation

$$(1+t^2)x' + 2tx = \cot t$$

for $0 < t < \pi$. Find the correspoding flow transformation, and the particular solution with $x(\pi/2) = 2$.

1.3 The equation $(e^x - 2tx)x' = x^2$ is not linear, but think of t as a function of x, and recall that

$$\frac{\mathrm{d}}{\mathrm{d}x}t(x) = \frac{1}{x'(t(x))} \ .$$

Use this to rewrite the equation as a linear first order eqaution for $t(x)$, and solve this.

1.4 Use the method of the previous exercise to solve $x - tx' = x'x^2e^x$.

1.5 Find the general solution of $tx' + x = t^3x^3$.

1.6 Find the general solution of $x' = \frac{1}{3}x + e^{-2t}x^{-2}$. Also find the corresponding flow transformation, and the particular soluiton with $x(0) = 2$.

1.7 Find the general solution of $x' + \frac{4}{t}x = t^3x^2$, $t > 0$. Also find the corresponding flow transformation $\Phi_{t_1,t_0}(x)$ for theose pairs of t_0 and t_1 for which it is defined. and the particular soluiton with $x(1) = 2$.

1.8 For $0 < c < 1/4$, and $x_0 > 0$, find the solution to

$$x' = x(1-x) - c\ , \quad x(0) = x_0\ .$$

Show that for all $x_0 \geq \dfrac{1}{2} - \sqrt{\dfrac{1}{4} - c}$, the solution exists for all t, and compute $\lim_{t\to\infty} x(t)$ for such x_0. What happens for smaller (positive) values of x_0?

1.9 Find the solution of

$$x'(t) = tx\frac{4-x}{1+t} \qquad x(0) = x_0 > 0\ .$$

Also compute $\lim_{t\to\infty} x(t)$ for each x_0.

1.10 Find the general solution of the Ricatti eqaution

$$x' = -\frac{2}{t}x + t^3x^2 + t^{-5}\ .$$

1.11 Find the general solution of the Ricatti eqaution

$$x' = \frac{2\cos^2 t - \sin^2 t + x^2}{2\cos t}\ .$$

1.12 Find the general solution of the eqaution $xx'' + (x')^2 = 0$.

1.13 Find the general solution of the eqaution $x'' = 1 + (x')^2$.

1.14 Find the general solution of the eqaution $tx'' = x' + (x')^3$.

1.15 Find the general solution of the eqaution $t^2x'' = 2tx' + (x')^2$.

Chapter 2

FLOWS ON THE REAL LINE

2.1 Vector fields on the real line

2.1.1 Monotonicity on maximal intervals

We have already introduced the differential equation

$$\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t))$$

in which $\mathbf{v}$ is a continuous vector field on $\mathbb{R}^n$, and for which we seek to find all of the curves $\mathbf{x}(t)$ in $\mathbb{R}^n$ that satisfy this equation. We have claimed that this equation is fundamental to the whole theory of differential equations. This claim will be justified later. Our main goal now is to build an understanding of this equation in the simplest case, that in which $n = 1$.

When $n = 1$, our equation reduces to

$$x'(t) = v(x(t)) \ . \tag{2.1}$$

We now begin a thorough investigation of this equation in the case in which $v(x)$ is a continuous function of x. The first observation to make is that if for any x_0, $v(x_0) = 0$, then $x(t) := x_0$ for all t is a solution of (2.1) for all t.

Definition 5 (Equilibrium point and steady-state solutions)**.** *If v is a continuous vector field on $\mathbb{R}$, and $v(x_0) = 0$, then x_0 is called an* equilibrium point *for v. For any $t_0 \in \mathbb{R}$, the function $x(t) = x_0$ for all t is a solution of $x'(t) = v(x(t))$ with $x(t_0) = x_0$. Such a constant solution, where the constant value is necessarily an equilibrium point, is called a* steady-state solution *of $x'(t) = v(x(t))$.*

In most cases we consider, the equilibrium points will be isolated (and usually finite in number), and they divide the line into a class of intervals that we now introduce, in which motion necessarily takes place.

Definition 6 (Maximal interval for v)**.** *We say that an interval (a, b) is a* maximal interval for v *if $v(x) \neq 0$ for any $x \in (a, b)$, and if either $a = -\infty$ or $v(a) = 0$ and either $v(b) = 0$ or $b = \infty$.*

For example, if $v(x) = x(1-x)$, then there are three maximal intervals: $(-\infty, 0)$, $(0,1)$ and $(1,\infty)$. For $v(x) = 1 + x^2$, there is only one maximal interval, namely $(-\infty,\infty)$. Since we suppose that v is continuous, each x such that $v(x) \neq 0$ belongs to a unique maximal interval for v.

Suppose that (a,b) is a maximal interval for v and that $x_0 \in (a,b)$. Then $v(x_0) \neq 0$. Let us suppose, to be concrete, that $v(x_0) > 0$. Then since v is continuous and $v(x) \neq 0$ for any $x \in (a,b)$, it follows from the Intermediate Value Theorem that $v(x) > 0$ for all $x \in (a,b)$.

Therefore, if $x(t)$ is any solution of $x'(t) = v(x(t))$ with $x(t_0) = x_0$, then $x'(t) > 0$, which means that $x(t)$ is strictly monotone increasing, as long as $x(t)$ stays in the interval (a,b). On any interval on which $x(t)$ is strictly monotone (increasing or decreasing), it is one to one and hence an invertible functions onto its range. Therefore, there exists the inverse function $t(x)$ defined on (a,b).

Thus, under the assumption that that v is positive on (a,b) $x(t)$ and $t(x)$ both exists and are both strictly increasing functions. Define

$$T_a = \lim_{x\to a} t(x) \qquad \text{and} \quad T_b = \lim_{x\to b} t(x) \ .$$

Then $x(t)$ is an invertible function from (T_a, T_b) onto (a,b), and $t(x)$ is an invertible function from (a,b) onto (T_a,T_b).

In case v is negative on (a,b), both functions are decreasing, so that with T_a and T_b defined as above, $T_b < T_a$, and $x(t)$ is an invertible function from (T_b,T_a) onto (a,b), and $t(x)$ is an invertible function from (a,b) onto (T_b,T_a).

All of the information about the solution $x(t)$ in the time interval (T_a,T_b) (or, if v is negative on (a,b), (T_b,T_a)), is contained in the inverse function $t(x)$. It turns out that there is a simple integral formula for $t(x)$ that can be used to study the issue of existence and uniqueness for the equation $x' = v(x)$.

First, let us consider an example in which is it easy to find $x(t)$ and $t(x)$.

Example 13 (Solution on a maximal interval)**.** *Consider the differential equation*

$$x'(t) = x(1-x) \qquad \text{and} \qquad x(t_0) = x_0 \quad \text{with} \quad x(t_0) \in (0,1) \ , \tag{2.2}$$

so that $v(x) = x(1-x)$*. As we have noted above,* $(0,1)$ *is a maximal interval for* v*.*

Our equation is separable: Since on $(0,1)$

$$\frac{1}{x(1-x)} = \frac{1}{1-x} + \frac{1}{x} = \frac{\mathrm{d}}{\mathrm{d}x}F(x) \qquad \text{where} \qquad F(x) = \ln\left(\frac{x}{1-x}\right) \ ,$$

our differential equation can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}F(x(t)) = 1 \ .$$

Integrating both sides, we find $F(x(t)) - F(x(t_0)) = t - t_0$*. Then since* $x(t_0) = x_0$*, we have*

$$F(x(t)) = F(x_0) + t - t_0 \ . \tag{2.3}$$

We now solve this for $x(t)$*. Exponentiating both sides of* $F(x) = y$*, we find*

$$\frac{x}{1-x} = e^y \qquad \text{and hence} \qquad x = \frac{e^y}{1+e^y} \ .$$

Therefore, taking $y = F(x_0) + t - t_0$, $e^y = \frac{x_0}{1-x_0}e^{t-t_0}$.

$$x(t) = \frac{x_0 e^{t-t_0}}{(1-x_0) + x_0 e^{t-t_0}} \ . \tag{2.4}$$

You should verify that this is a solution, and that $x(t_0) = x_0$. *Next, note that* $x(t)$ *is defined for all* t, *and*

$$\lim_{t\to-\infty} x(t) = 0 \qquad \text{and} \qquad \lim_{t\to\infty} x(t) = 1 \ .$$

Therefore, $x(t)$ *is a strictly monotone increasing function from* $(-\infty,\infty)$ *onto* $(0,1)$, *and hence is a one-to-one function from* $(-\infty,\infty)$ *onto* $(0,1)$. *Therefore, it has a well defined inverse function* $t(x)$. *Going back to (2.3) we see that* $t(x) = t_0 + F(x) - F(x_0)$. *Then since* $F(x) - F(x_0) = \int_{x_0}^x \frac{1}{v(z)}\mathrm{d}z$,

$$t(x) = t_0 + \int_{x_0}^x \frac{1}{v(z)}\mathrm{d}z \ . \tag{2.5}$$

Note that

$$\int_{x_0}^1 \frac{1}{v(z)}\mathrm{d}z = \infty \qquad \text{and} \qquad \int_{x_0}^0 \frac{1}{v(z)}\mathrm{d}z = -\infty$$

for any $x_0 \in (0,1)$, *so* $T_0 = -\infty$ *and* $T_1 = \infty$, *as we found above working directly with* $x(t)$.

A brief summary of the content of this subsection is that a continuous vector field on the line may have one or more equilibrium points, and these divide the line into maximal intervals (if there is at least one equilibrium point). At each equilibrium point, there is at least one solution, the corresponding steady-state solution, but there may be others. As long as any solution $x(t)$ stays inside a maximum interval, it is strictly monotone – increasing or decreasing, depending on the sign of v in that interval – and hence has a well defined inverse function $t(s)$. In the next section we shall see how to find $t(x)$ and then $x(t)$.

2.1.2 Barrow's formula

The formula (2.5) we derived at the end of the last example gives an explicit integral expression, not for $x(t)$, but instead, for $t(x)$. But then if the integral can be done explicitly, one can invert to recover $x(t)$. The formula (2.5) is known as *Barrow's formula.*

Theorem 2 (Barrow's Theorem)**.** *Let* v *be continuous and let* (a,b) *be a maximal interval for* v *and let* $x_0 \in (a,b)$. *Fix any* $t_0 \in \mathbb{R}$ *and define the function* $t(x)$ *on* (a,b) *by Barrow's formula (2.5). Then* $t(x)$ *is a strictly monotone function on* (a,b), *so that*

$$T_a = \lim_{x\to a} t(x) \qquad \text{and} \quad T_b = \lim_{x\to b} t(x) \tag{2.6}$$

both exist.

If v *is positive on* (a,b), *then* $T_a < t_0 < T_b$, *and if* $x(t)$ *is the inverse function to* $t(x)$, *then* $x(t)$ *is a solution of*

$$x'(t) = v(x(t)) \qquad \text{and} \qquad x(t_0) = x_0 \ . \tag{2.7}$$

Moreover, every solution to (2.7) that is defined on any subinterval of (T_a, T_b) *containing* t_0 *equals the restriction of* $x(t)$ *to this subinterval. In particular, there is a unique solution of (2.7) defined on* (T_a, T_b). *If* v *is negative on* (a,b), *the same conclusion is valid provided we interchange* T_a *and* T_b.

Proof. We suppose first that v is positive on (a,b), and define the function $t(x)$ on (a,b) by Barrow's formula. For any $x \in (a,b)$ other than x_0, let J be the closed interval with endpoints x_0 and x, which is either $[x_0, x]$ in case $x_0 < x < b$, or else $[x, x_0]$ in case $a < x < x_0$.

Since v is continuous on J, it has a minimum value attained somewhere in J, and since v is positive everywhere on (a,b), and hence everywhere on J, there is a $c > 0$ so that $v(x) \geq c$ for all $x \in J$. Therefore, $1/v(x)$ is continuous and bounded on J. It follows that

$$\int_{x_0}^{x} \frac{1}{v(z)} \mathrm{d}z$$

is a proper integral for each $x \in (a,b)$. Thus,

$$t(x) = t_0 + \int_{x_0}^{x} \frac{1}{v(z)} \mathrm{d}z$$

does define a function on (a,b). By the Fundamental Theorem of Calculus, this function is differentiable, and

$$\frac{\mathrm{d}}{\mathrm{d}x} t(x) = \frac{1}{v(x)} \ . \tag{2.8}$$

Since the right hand side is continuous, $t(x)$ is continuously differentiable on (a,b). Since $v(x) > 0$ on (a,b), $t(x)$ is strictly monotone increasing on (a,b). Therefore, the limits defining T_a and T_b in (2.6) both exist with $T_a = -\infty$ and $T_b = \infty$ allowed, and $t(x)$ is an invertible function from (a,b) onto (T_a, T_b).

Since $t(x)$ is differentiable, by the Inverse Function Theorem of single variable calculus, $x(t)$ is differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \left(\frac{\mathrm{d}}{\mathrm{d}x} t(x) \right)^{-1} \Bigg|_{x=x(t)} \ .$$

By (2.8),

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} t(x) \right)^{-1} \Bigg|_{x=x(t)} = v(x(t)) \ ,$$

which shows that $x'(t) = v(x(t))$, and clearly $t(x_0) = t_0$, so $x(t_0) = x_0$.

Now suppose that $y(t)$ is a continuously differentiable function defined on some interval (S,T) with $t_0 \in (S,T)$. Suppose also that

$$y'(t) = v(y(t)) \qquad \text{and} \qquad y(t_0) = x_0 \ ,$$

and that $y(t) \in (a,b)$ for all $t \in (S,T)$.

Then for all $t \in (S,T)$, $y'(t) = v(y(t)) > 0$, and so $y(t)$ is strictly monotone and hence it is an invertible function from (S,T) onto its range. Let $t(y)$ be the inverse function. By the Inverse Function Theorem of single variable calculus, $t(y)$ is differentiable, and

$$\frac{\mathrm{d}}{\mathrm{d}y} t(y) = \left(\frac{\mathrm{d}}{\mathrm{d}t} y(t) \right)^{-1} \Bigg|_{t=y(t)} = \frac{1}{v(y)} \ .$$

Then by the Fundamental Theorem of Calculus,

$$t(y) - t(x_0) = \int_{x_0}^{y} \frac{1}{v(z)} \mathrm{d}z$$

for all y such that $y = y(t)$ for some $t \in (S, T)$. That is, $t(y)$ is given by Barrow's formula, and hence $y(t)$ is the inverse of the function defined by Barrow's formula. In other words, $y(t) = x(t)$ on its domain of definition (S, T). This proves the uniqueness of the solution – for as long as it stays inside (a, b). □

The theorem we have just proved shows that when v is continuous and $v(x_0) \neq 0$, the initial value problem

$$x'(t) = v(x(t)) \qquad \text{and} \qquad x(t_0) = x_0\ . \tag{2.9}$$

has a solution that is unique for as long as $x(t)$ stays inside the maximal interval (a, b) that contains x_0. In particular, if $T_a = -\infty$ and $T_b = \infty$ (or the other way around) so that $x(t)$ *never* leaves (a, b), then there is a unique solution, period, defined for all $t \in \mathbb{R}$.

However, if a solution of (2.9) reaches either a or b in finite time, then there will be non-uniqueness of solutions.

Example 14. *Consider the vector field in the real line given by*

$$v(x) = \begin{cases} \sqrt{x^2 - 1} & x < -1 \\ \sqrt{1 - x^2} & 1 < x < 1 \\ \sqrt{x^2 - 1} & x > 1 \end{cases}$$

It is easy to see that $v(x)$ is continuous, and that $(-\infty, 0)$, $(-1, 1)$ and $(1, \infty)$ are maximal intervals for $v(x)$.

Consider the initial value problem

$$x'(t) = v(x)\ , \qquad x(0) = 0\ . \tag{2.10}$$

By Barrow's formula,

$$t(x) = \int_0^x \frac{1}{\sqrt{1 - z^2}} \mathrm{d}x = \arcsin(x)$$

for all $x \in (-1, 1)$. Therefore,

$$x(t) = \sin(t)\ .$$

One easily checks that this does satisfy both the equation and the initial condition. However,

$$\lim_{t \to \pi/2} x(t) = 1 \qquad \text{and} \qquad \lim_{t \to -\pi/2} x(t) = -1\ .$$

That is, at times $t = \pm\pi/2$, our solution reaches the boundary of the maximal interval. In other words, we have that $T_{-1} = \pi/2$ and $T_1 = -\pi/2$.

On the interval $(T_{-1}, T_1) = (-\pi/2, \pi/2)$, the solution is unique, however, there are infinitely many choices for how to continue the solution outside this interval*, as we now explain.*

Since $v(\pm 1) = 0$, the solution is instantaneously at rest when it reaches $x = \pm 1$. One solution is to let it stay at rest: That is, define

$$x(t) = \begin{cases} -1 & t < \pi/2 \\ \sin(t) & -\pi/2 \le t \le \pi/2 \\ 1 & t > \pi/2\ . \end{cases}$$

However, since for $x > 1$, we can integrate $(x^2-1)^{-1/2}$, we can use Barrow's formula to define a solution of

$$x' = v(x)\ , \qquad x(\pi/2) = 1 \tag{2.11}$$

for $t > \pi/2$ by

$$t(x) - \pi/2 = \int_1^x \frac{1}{\sqrt{z^2-1}}\mathrm{d}z = \ln\left(x + \sqrt{x^2-1}\right)\ ,$$

and hence

$$x + \sqrt{x^2-1} = e^{t-\pi/2}\ .$$

We can solve this for x. To simplify the notation during the computations, let $a := e^{t-\pi/2}$. Then

$$\sqrt{x^2-1} = a - x\ .$$

Squaring both sides, $x^2 - 1 = a^2 - 2ax + x^2$, so that $x = (a^2+1)/2a$, and hence

$$x(t) = \cosh(t - \pi/2)\ . \tag{2.12}$$

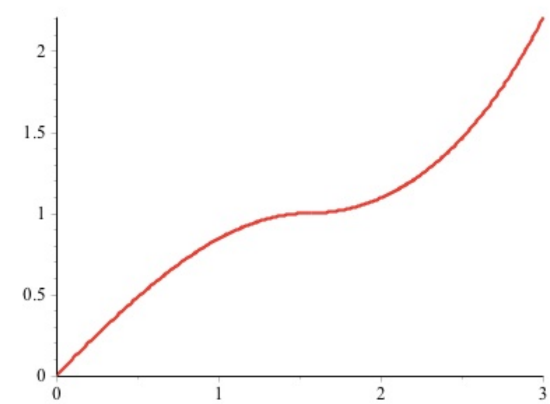
We can easily check that (2.12) does define a solution of (2.11): $x'(t) = \sinh(t-\pi/2) = \sqrt{\cosh^2(t-\pi/2) - 1}$, and $x(\pi/2) = 1$.

Therefore,

$$x(t) = \begin{cases} -1 & t < \pi/2 \\ \sin(t) & -\pi/2 \le t \le \pi/2 \\ \cosh(t-\pi/2) & t > \pi/2\ . \end{cases}$$

is also a solution of (2.10).

Here is a plot of this solution, which is strictly monotone increasing, but takes an "instantaneous rest" at $t = \pi/2$ before continuing upwards:

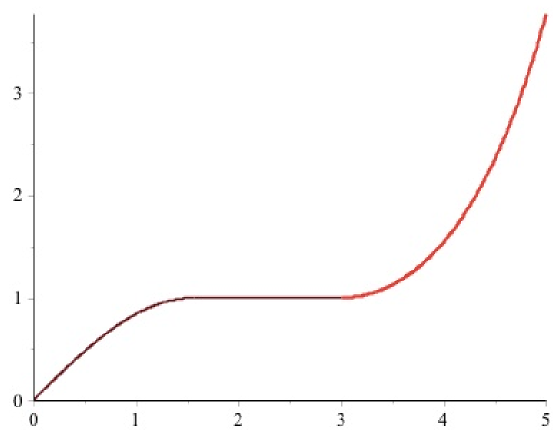


In fact, the same sort of computations we have made just above show that $x(t) := \cosh(t - T)$ solves $x'(t) = \sqrt{x^2(t) - 1}$ with $x(T) = 1$ for all $t \ge T$. (We deduced this above for the specific choice *of $T = \pi/2$. But then for any $T > \pi/2$,*

$$x(t) = \begin{cases} -1 & t < \pi/2 \\ \sin(t) & -\pi/2 \le t \le \pi/2 \\ 1 & \pi/2 \le t \le T \\ \cosh(t - \pi/2) & t > T\ . \end{cases}$$

also gives a solution of (2.10): This solution proceed from $x = 0$ to $x = 1$ in time $t = \pi/2$. It then "takes a rest", staying at $x = 1$ until $t = T$, at which time it resumes motion to the right. Since $T > \pi/2$ is arbitrary, there are infinitely many such solutions.

Here is a plot of such a solution with $T = 3$:



The previous example shows that even when $v(x)$ is continuous on the whole real line, it may not be the case that through each point in the t, x plane there is exactly one solution curve, defined for all t. In this example, we have more than one solution curve passing through each point of the form $(t, 1)$. We can describe this situation as "branching": when the solution reaches $x = 1$, there are two branches it may follow, and as long as it stays at $x = 1$, there is always the option of continuing to rest, or "branching away" upwards on the alternate solution curve.

As we have proved in the previous chapter, this never happens when $v(x) = px + q$. (In fact, we showed that it never happens even when p and q depend continuously on t.) For linear equations, there is no branching.

There is one more thing that might go wrong when we only assume that $v(x)$ is continuous, but which never happens in the linear case: The solutions curves might not be defined for all t. This occurs with the example $v(x) = x^2$ that we have treated as a separable equation in the previous chapter. In the next example we solve this equation using Barrow's formula.

Example 15. *Consider the equation*

$$x'(t) = x^2(t)\ , \qquad x(t_0) = x_0 \tag{2.13}$$

where (t_0, x_0) is an arbitrary point in the t, x plane.

By Barrow's formula,

$$t(x) = t_0 + \int_{x_0}^{x} \frac{1}{z^2}\mathrm{d}z = t_0 + \frac{1}{x_0} - \frac{1}{x} \ .$$

Solving for x in terms of t, we find

$$\frac{1}{x} = \frac{(x_0)(t_0 - t) + 1}{x_0} \ ,$$

so that

$$x(t) = \frac{x_0}{(x_0)(t_0 - t) + 1} \ . \tag{2.14}$$

By Theorem 2, this is the unique solution passing through (t_0, x_0) for all times t until this solution leaves the maximal interval containing x_0.

Since $v(x) = 0$ only for $x = 0$, there are two maximal intervals for v, namely $(-\infty, 0)$ and $(0, \infty)$.

Let us consider first $x_0 > 0$ so that our solution starts in the maximal interval $(0, \infty)$. It turns out that the solution will exit this maximal interval in a finite time – but not by reaching zero for any $t < t_0$. Instead, the solution "blows up" to infinity for a finite time $t > t_0$.

First, we compute the time at which the solution first reaches $x = \epsilon$ where $0 < \epsilon < x_0$. Barrow's formula says that

$$t(\epsilon) = t_0 + \int_{x_0}^{\epsilon} \frac{1}{z^2}\mathrm{d}z = \left(t_0 + \frac{1}{x_0}\right) - \frac{1}{\epsilon} \ .$$

Evidently,

$$\lim_{\epsilon \to 0} t(\epsilon) = -\infty \ ;$$

it takes all (past) eternity for the solution to reach $x = 0$.

On the other hand, at $t = t_0 + 1/x_0$, the formula (2.14) for $x(t)$ breaks down due to division by zero, and indeed,

$$\lim_{t \to t_0 + 1/x_0} x(t) = \infty \ .$$

Therefore, the solution is only defined for t in$(-\infty, t_0 + 1/x_0)$, and at $t = t_0 + 1/x_0$ it "blows up", exiting exiting the maximal interval in which it started. A similar analysis of the case $x_0 < 0$ leads to similar conclusions.

Thus, we have seen examples in which solutions of $x'(t) = v(x(t))$, with $v(x)$ continuous on the whole real line, were non-unique due to "branching" or failed to be defined for all times t due to "blow-up". The branching and blow-up occurred when a solution reached the boundary of a maximal interval in finite time. As long as this dies not happen, Theorem 2 tells us that the solutions are well-behaved.

2.2 Global existence and uniqueness for Lipschitz vector fields

2.2.1 Lipschitz continuity

There is a simple condition on v that guarantees solutions of $x' = v(x)$ do not reach the boundaries of maximal intervals in a finite time, and are therefore unique for all times.

Definition 7 (Lipschitz function). *A function $v(x)$ is* Lifshitz on the interval (a,b) *in case there is a constant $L < \infty$ such that*

$$|v(y) - v(x)| \leq L|y - x| \tag{2.15}$$

for all $x, y \in (a,b)$. Then L is a Lipschitz constant *for v on (a,b), The cases $a = \infty$ and $b = \infty$ are allowed. If v is Lipschitz on $(-\infty, \infty)$, then we say v is* globally Lipschitz.

If v is continuously differentiable, the Fundamental Theorem of Calculus says that

$$v(y) - v(x) = \int_x^y v'(z)\mathrm{d}z \ .$$

Therefore, if $|v'(z)| \leq L$ for all $z \ in (a,b)$, then for all $x \leq y \in (a,b)$,

$$|v(y) - v(x)| \leq \int_x^y |v'(z)|\mathrm{d}z \leq \int_x^y L\mathrm{d}z = L|y - x| \ .$$

This shows that v is Lipschitz on (a,b) with Lipschitz constant L.

On the other hand, suppose that x_0 is a boundary point of (a,b), so $x_0 = a$ or $x_0 = b$, and that

$$\lim_{x \to x_0} |v'(x)| = \infty \tag{2.16}$$

For example, consider $v(x) = x^{1/3}$. Then v is continuously differentiable except at $x = 0$, with $v'(x) = \frac{1}{3}x^{-2/3}$. Then with $x_0 = 0$, which is a boundary point of both maximal intervals $(-\infty, 0)$ and $(0, \infty)$, (2.16) is satisfied.

In this case, v is not Lipschitz on (a,b). Indeed, whenever (2.16) is satisfied, for any positive integer N, there will be an interval (x,y) inside (a,b) but close to x_0 such that $|v'(z)| > N$ for all $z \in (x,y)$. Then, since v' does not change sign on (a,b),

$$|v(y) - v(x)| = \left|\int_x^y v'(z)\mathrm{d}z\right| = \int_x^y |v'(z)|\mathrm{d}z \geq N|y - x| \ .$$

Therefore,

$$|v(y) - v(x)| \geq N|y - x| \ ,$$

and since N can be arbitrarily large, v cannot be Lipschitz.

We summarize these useful observations in a Lemma.

Lemma 1 (Bounded derivatives and the Lipschitz property). *Let v be continuously differentiable on an open interval (a,b), and continuous on its closure in case either a or b are finite. Then if*

$$|v'(z)| \leq L < \infty \qquad \text{for all} \qquad z \in (a,b) \ ,$$

v is Lipschitz on (a,b) with Lipschitz constant L.

On the other hand, (2.16) is satisfied for either $x_0 = a$ or $x_0 = b$, then v is not Lipschitz on (a,b).

Usually when we prove that a function is Lipschitz, or is not, we will do so using Lemma 1. Lipschitz functions need not be differentiable everywhere. For example, $v(x) = |x|$ is globally Lipschitz with Lipschitz constant 1, but it is not differentiable at $x = 0$. It may be shown, however, that Lipschitz functions are differentiable "almost everywhere". We shall not need this more refined result here.

In the next subsection we give an important application of Lipschitz continuity.

2.2.2 Existence and uniqueness within maximal intervals

Theorem 3 (Lipschitz continuity implies existence and uniqueness within maximal intervals)**.** *Let (a, b) be a maximal interval for a continuous vector field v on the real line. Suppose that v is Lipschitz continuous on (a, b). Then for all $x_0 \in (a, b)$, the solution of*

$$x'(t) = v(x(t)) \ , \qquad x(t_0) = x_0 \tag{2.17}$$

exists and is well defined for all $t \in \mathbb{R}$, and stays inside (a, b) for all $t \in \mathbb{R}$. In particular, for each (t_0, x_0) with $t_0 \in \mathbb{R}$ and $x_0 \in (a, b)$, there is a unique solution curve of (2.17) passing through (t_0, x_0).

Proof. For the sake of being concrete, let us suppose that $v(x) > 0$ on (a, b), so that the solution of (2.17) is monotone increasing. We will now show that $x(t) < b$ for all $t \in \mathbb{R}$. Since $x(t)$ is monotone increasing, we need only consider $t > t_0$, since otherwise $x(t) \le x_0 < b$.

Therefore are two cases: $b = \infty$ and $b < \infty$. Suppose that $b = \infty$. Then since v is Lipschitz with Lipschitz constant L, for all $z > x_0$,

$$v(z) - v(x_0) \le |v(z) - v(x_0)| \le L|z - x_0| = L(z - x_0) \ .$$

Therefore

$$v(z) \le v(x_0) + L(z - x_0) \ ,$$

and hence, by Barrow's formula, for any $x > x_0$,

$$\begin{aligned} t(x) &= t_0 + \int_{x_0}^{x} \frac{1}{v(z)} \mathrm{d}z \\ &\ge t_0 + \int_{x_0}^{x} \frac{1}{v(x_0) + L(z - x_0)} \mathrm{d}z \\ &= t_0 + \frac{1}{L} \ln\left(1 + \frac{L(x - x_0)}{v(x_0)}\right) \ . \end{aligned}$$

Since

$$\lim_{x\to\infty} \frac{1}{L} \ln\left(1 + \frac{L(x - x_0)}{v(x_0)}\right) = \infty \ ,$$

it follows that

$$\lim_{x\to\infty} t(x) = \infty \ .$$

Therefore, $x(t) < b = \infty$ for all t.

Next, we consider the case in which $b < \infty$. We know that $v(b) = 0$. Also, since v is continuous, $\lim_{x\to b} v(x) = 0$. That is, as $x(t)$ approaches b, the velocity approaches zero. We want to show that this "slowing down" is strong enough that the solution never reaches b.

To do this, we need an upper bound on the velocity $v(z)$ for z near b. The Lipschitz condition provides this:

$$v(z) = v(z) - v(b) = |v(z) - v(b)| \le L|z - b| = L(b - z) \ .$$

Therefore

$$v(z) \le L(b - z) \ ,$$

and hence, by Barrow's formula, for any $x_0 < x < b$,

$$\begin{aligned} t(x) &= t_0 + \int_{x_0}^{x} \frac{1}{v(z)} \mathrm{d}z \\ &\geq t_0 + \int_{x_0}^{x} \frac{1}{L(b-z)} \mathrm{d}z \\ &= t_0 + \frac{1}{L} \ln\left(\frac{b - x_0}{b - x}\right) . \end{aligned}$$

Since

$$\lim_{x \to b} \frac{1}{L} \ln\left(\frac{b - x_0}{b - x}\right) = \infty ,$$

it follows that

$$\lim_{x \to b} t(x) = \infty .$$

Therefore, $x(t) < b$ for all t.

Exactly the same sort of reasoning shows that $x(t) > a$ for all t, regardless of whether $a = -\infty$ or $a > -\infty$.

□

Example 16. *Consider the vector field $v(x)$ defined by*

$$v(x) = x^{1/3}$$

for all x. The maximal intervals for this vector field are $(0, \infty)$ and $(-\infty, 0)$. By Lemma 1, this vector field is not Lipschitz.

Let us use Barrow's formula to compute the solution of $x'(t) = v(x(t))$ with $x(0) = 1$:

$$t(x) = \int_{1}^{x} \frac{1}{z^{1/3}} \mathrm{dz} = \frac{3}{2}(x^{2/3} - 1) .$$

The solutions is

$$x(t) = \left(1 + \frac{2}{3}t\right)^{3/2} \qquad \text{for} \qquad t \geq -\frac{3}{2} .$$

Now let us "shift the starting time" of the solution, by introducing $y(s)$ through

$$y(s) = x(-3/2 + s) .$$

That is, $y(s)$ is simply a repparameterized version of the function $x(t)$, where $s = 0$ corresponds to $t = -3/2$. Thus $y(0) = x(-3/2) = 0$.

Now we readily compute that

$$y'(s) = x'(-3/2 + s) = v(x(-3/2 + s) = v(y(s))$$

and so the function $y(s)$ is a solution of

$$y'(s) = v(y(s)) , \qquad y(0) = 0 .$$

But we have a second solution to this equation, namely $y_2(s) = 0$ for all s: It is clear that

$$y_2'(s) = v(y_2(s)) , \qquad y_2(0) = 0 .$$

Hence, for this non-Lipschitz vector field, we have at least two solution curves passing through $(0, 0)$, and we see that this arrises because solution starting inside $(0, \infty)$ reach the origin in a finite time.

2.2.3 Uniqueness of steady-state solutions

We have seen in the previous example, that if the initial point is an equilibrium point x_0, the constant solution $x(t) = x_0$ for all t may not be the only solution; there may also be non-constant solutions. But the example in which this occurred, the vector field was not Lipschitz on any open interval about the equilibrium point. Our next theorem says that this non-uniqueness cannot occur is the vector field is Lipschitz on an open interval about the equilibrium point.

Theorem 4 (Uniqueness of steady state solutions)**.** *Let v be a continuous vector field on the line and suppose that $v(x_0) = 0$. Suppose that v is Lipschitz on every bounded open interval containing x_0. Then the only solution to*

$$x'(t) = v(x(t)) \ , \qquad x(t_0) = x_0$$

defined on any open interval about t_0 is the constant solution $x(t) = x_0$ for all t.

Proof. Suppose that there exists a non-constant solution $x(t)$. Then $v(x(t))$ is also non-constant since otherwise $v(x(t)) = v(x(t_0)) = 0$ for all t, and then $x'(t) = v(x(t))$ implies that $x(t)$ is constant.

Hence there exists t_1 such that and $v(x(t_1)) \neq 0$, and then necessarily $x(t_1) \neq x(t_0)$. Let us suppose that $t_1 > t_0$; the other case may be treated in the same way.

Let t_2 be the largest value of $t < t_1$ such that $v(x(t)) = 0$. It might be that $t_2 = t_0$, but it might also be that $t_0 < t_2 < t_1$. In any case, for all z in between $x(t_2)$ and $x(t_1)$, $v(z) \neq 0$, and we can apply Barrow's Formula. We have

$$t(x(t_2)) - t(x(t_1)) = \int_{x(t_1)}^{x(t_2)} \frac{1}{v(z)} \mathrm{d}z \ . \tag{2.18}$$

Let $a = \min\{x(t_1), x(t_2)\}$ and let $b = \max\{x(t_1), x(t_2)\}$. Let L be the Lipschitz constant of v on any bounded open interval containing $[a, b]$. Then since $v(x(t_2)) = 0$, for any $z \in [a, b]$,

$$|v(z)| = |v(z) - v(x(t_2))| \leq L|z - x(t_2)|$$

and therefore,

$$\frac{1}{|v(z)|} \geq \frac{1}{L} \frac{1}{|z - x(t_2)|} \ .$$

Then since $t(x(t_1)) = t_1$ and $t(x(t_2)) = t_2$,

$$|t_1 - t_2| = \int_a^b \frac{1}{|v(z)|} \mathrm{d}z \geq \frac{1}{L} \int \frac{1}{|z - x(t_2)|} \mathrm{d}z = \infty$$

since either $x(t_2) = a$ or $x(t_2) = b$ and neither $|z - a|^{-1}$ nor $|z - b|^{-1}$ are integrable on $[a, b]$: In either case, there is a logarithmic divergence in the improper integral as the singular limit is approached.

However, since by hypothesis t_1 is finite and t_0 is finite, and since t_2 lies between t_0 and t_1, it is impossible to have $|t_1 - t_2| = \infty$. Hence, no non-constant solution exists. □

Example 17. *Consider the vector field*

$$v(x) = x^3 \ .$$

Are there any solutions to

$$x'(t) = v(x(t)) \ , \qquad x(0) = 0$$

other than the obvious solutions $x(t) = 0$ *for all* t*?*

According to Theorem 4, the answer will be "No" in case v *is Lipschitz on any bounded interval containing the origin. But* $v'(x) = x^2$ *is continuous. By what we have noted above, continuously differentiable functions are Lipschitz on every bounded open interval. Hence,* v *is Lipschitz on any bounded open interval containing* 0*.*

2.3 The flow transformation of a Lipschitz vector field on ℝ

2.3.1 Time-shift invariance for time independent vector fields

Let v be a continuous vector field on $\mathbb{R}$. Let (a, b) be a maximal interval for v, and suppose that v is Lipschitz on (a, b) with Lipschitz constant L. Then by Theorem 2, for each $r \in \mathbb{R}$, and each $x \in (a, b)$, there exists a unique solution of $x'(t) = v(x(t))$ with $x(r) = x$, and this solution is defined for all $t \in R$, and this is the only solution passing through x at time $t = r$.

Therefore, for any $r, s \in \mathbb{R}$, we may define a function $\Phi_{s,r}$ from (a, b) to (a, b) by

$$\Phi_{s,r}(x) = x(s) \tag{2.19}$$

where $x(t)$ is the unique solution to $x'(t) = v(x(t))$ with $x(r) = x$.

The function $\Phi_{s,r}$ tells us what the value of $x(s)$ is when $x(t)$ is the solution to $x'(t) = v(x(t))$ with $x(r) = x$. It allows us to study how $x(s)$ depends on this initial data, or "starting point" x.

There is a significant simplification that arises in the flow transformation for an autonomous equation such as $x' = v(x)$ in which the vector field depends only on x and not on t. In this case, $\Phi_{s,r}(x)$ depends only on $r - s$. That is, for all $x \in (a, b)$ and all $r, s \in \mathbb{R}$,

$$\Phi_{s+h,r+h}(x) = \Phi_{s,r}(x) \tag{2.20}$$

for all $h \in \mathbb{R}$.

To see this, let $x(t)$ be the solution of $x'(t) = v(x(t))$ with $x(r) = x_0$. Now define fix any $h \in \mathbb{R}$ and define $y(t) = x(t - h)$, Then

$$y'(t) = x'(t - h) = v(x(t - h)) = v(y(t))$$

so that $y(t)$ is a solution of our equation. Next, note that $y(r + h) = x(r + h - h) = x(r) = x_0$, and so $y(t)$ is the solution that passes through x_0 at time $t = r + h$.

That is, if you have found the solution of

$$x'(t) = v(x(t)) \ , \quad x(r) = x_0 \ , \tag{2.21}$$

you also know the solution of

$$x'(t) = v(x(t)) \ , \quad x(r + h) = x_0 \ . \tag{2.22}$$

By what we have computed above, if $x(t)$ is a solution of (2.21), then $x(t - h)$ is a solution of (2.22).

Example 18 (Time shifts)**.** *Consider the very simple equation $x' = v(x)$ with $v(x) = x$. The general solution of this equation is $x(t) = Ce^t$ and so the solution with $x(r) = x_0$ has $x_0 = Ce^r$ so that $C = x_0 e^{-r}$, and hence*

$$x(t) = x_0 e^{t-r} \ .$$

Then for all h, $x(t-h) = x_0 e^{t-(r+h)}$ and it is easy to check that this solves $x' = v(x)$ with $x(r+h) = x_0$.

By definition, $\Phi_{s,r}(x) = x(s)$ where $x(t)$ solves $x' = v(x)$ with $x(r) = x$. But then $x(t-h)$ solves $x' = v(x)$ with $x(r+h) = x$. Therefore

$$\Phi_{s+h,r+h}(x) = x((s+h)-h) = x(s) = \Phi_{s,r}(x) \ .$$

This proves (2.20).

Thus, in the autonomous case, it suffices to study the simpler function

$$\Psi_s(x) = \Phi_{s,0}(x)$$

since then we have for any $r, s \in \mathbb{R}$,

$$\Phi_{s,r}(x) = \Phi_{s-r,0}(x) = \Psi_{s-r}(x) \ .$$

Definition 8 (Flow transformation for an time-independent Lipschitz vector field)**.** *Let v be a Lipschitz vector field on the maximal interval (a,b). The function $\Psi_s(x)$ defined for s in$\mathbb{R}$ and $x \in (a,b)$ with values in (a,b) is defined by*

$$\Psi_s(x) = x(s)$$

where $x(t)$ is the solution of $x' = v(x)$ with $x(0) = x$.

The flow function allows us to study how the solution of $x' = v(x)$ with $x(0) = x$ depends on s and on the initial data, or "starting point" x. Though the notation is a bit asymmetric, we may think of it as a functions of the two variables s and x.

Example 19 (Flow transformation for a vector field on $\mathbb{R}$)**.** *Consider the once more the logistic equation $x' = v(x)$ with $v(x) = x(1-x)$. In Example 13 (see (2.4)) we have computed that the solution of this equation with $x(0) = x$ in$(0,1)$ is given by*

$$x(t) = \frac{xe^t}{(1-x)+xe^t} \ . \tag{2.23}$$

Therefore, the flow transform for this equation on the maximal interval $(0,1)$ is given by

$$\Psi_s(x) = x(s) = \frac{xe^s}{(1-x)+xe^s} \ .$$

Notice that $\Psi_0(x) = x$, and $\lim_{s\to\infty} \Psi_s(x) = 1$ and $\lim_{s\to-\infty} \Psi_s(x) = 0$.

A simple computation shows that

$$\frac{\partial}{\partial x}\Psi_s(x) = \frac{e^s}{(1-x+xe^s)^2} \ . \tag{2.24}$$

The fact that this tends to zero as $s \to \pm\infty$ is consistent with our computation that $\lim_{s\to\infty} \Psi_s(x) = 1$ and $\lim_{s\to-\infty} \Psi_s(x) = 0$: No mater where you start, for large enough t, $x(t)$ will be close to 1, and for t negative enough, $x(t)$ will be close to 0 In this sense the long-time behavior is very insensitive to the starting point.

2.3.2 Properties of the flow transformation

One question of interest to us is "how sensitive" this dependence on the initial data is. For this reason it is of interest to compute the derivative

$$\frac{\partial}{\partial x}\Psi_s(x) \ .$$

In the previous example, we did this by direct computation for a particular vector field. It turns out there is a simple general formula for this derivative, and Barrow's Formula provides the means to determine it. For any $x, y \in (a, b)$, define the function

$$G(x, y) = \int_x^y \frac{1}{v(z)} \mathrm{d}z \ . \tag{2.25}$$

Then by Barrow's Formula, $y = \Psi_s(x)$ if and only if

$$G(x, y) = s. \tag{2.26}$$

Thus, if we define a curve $y(x)$ in the x, y plane by

$$y(x) = \Psi_s(x) \ , \tag{2.27}$$

then the graph of $y = y(x)$ is the solution set of the equation (2.26).

The gradient of G is well defined and continuous on $(a, b) \times (a, b)$ since v is never zero on (a, b). Indeed,

$$\nabla G(x, y) = \left(-\frac{1}{v(x)}, \frac{1}{v(y)} \right) \ ,$$

which is clearly continuous on $(a, b) \times (a, b)$, and never equal to $(0, 0)$.

Now, the Implicit Function Theorem implies that $y(x)$ is continuously differentiable. By (2.26), $G(x, y(x)) = s$, which is independent of x. Hence

$$\frac{\mathrm{d}}{\mathrm{d}x} G(x, y(x)) = 0 \ . \tag{2.28}$$

On the other hand, by the Chain Rule and the computations above,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} G(x, y(x)) &= \nabla G(x, y(x)) \cdot (1, y'(x)) \\ &= \left(-\frac{1}{v(x)}, \frac{1}{v(y)} \right) \cdot (1, y'(x)) \\ &= -\frac{1}{v(x)} + \frac{y'(x)}{v(y)} \ . \end{aligned}$$

Comparing the last two equations, we see that

$$y'(x) = \frac{v(y)}{v(x)} \ .$$

Going back to (2.27), we see that this proves

$$\frac{\mathrm{d}}{\mathrm{d}x} \Psi_s(x) = \frac{v(\Psi_s(x))}{v(x)} \ . \tag{2.29}$$

In particular, whenever v is merely Lipschitz on (a,b), Ψ_s is differentiable and hence continuous. Therefore, the right hand side of (2.32) is the composition of two continuous functions, and it therefore continuous. It follows that whenever v is merely Lipschitz on (a,b), Ψ_s is *continuously differentiable* on (a,b).

But then if v itself is continuously differentiable on (a,b), the right hand side of (2.32) is continuously differentiable, and so $\Psi_s(x)$ is *twice continuously differentiable* in x, and

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Psi_s(x) = \frac{v'(\Psi_s(x))}{v(x)}\frac{v(\Psi_s(x))}{v(x)} - \frac{v(\Psi_s(x))v'(x)}{v^2(x)} \ . \tag{2.30}$$

The formula for the second derivative is not very enlightening, but the point to notice is that the flow transformation $\Psi_s(x)$ has whatever "smoothness" the vector field v has, and a bit more: The flow transformation is a very nice function. The formula for the first derivative is quite simple and useful.

The flow function has a number of useful properties that are summarized in the following theorem.

Theorem 5 (Flow transformations for Lipschitz vector fields on $\mathbb{R}$)**.** *Let v be a continuous vector field on $\mathbb{R}$. Let (a,b) be a maximal interval for v, and suppose that v is Lipschitz on (a,b). Let $\Psi_s(x)$ be the corresponding flow transformation considered as a function of $(s,x) \in \mathbb{R} \times (a,b)$ with values in (a,b). Then:*

(1) For all $r, s \in \mathbb{R}$,

$$\Psi_s \circ \Psi_r = \Psi_{s+r} \ . \tag{2.31}$$

(2) For all $s \in \mathbb{R}$, Ψ_r is an invertible transformation from (a,b) onto (a,b), and the inverse is Φ_{-s}

(3) The function $(s,x) \mapsto \Psi_s(x)$ defined on $\mathbb{R}\times(a,b)$ with values in (a,b) is continuously differentiable with:

$$\frac{\partial}{\partial x}\Psi_s(x) = \frac{v(\Psi_s(x))}{v(x)} \ . \tag{2.32}$$

and

$$\frac{\partial}{\partial s}\Psi_s(x) = v(\Psi_s(x)) \ . \tag{2.33}$$

The equation (2.33) says that for fixed x and r, $s \mapsto \Phi_{s,r}(x)$ is the solution of

$$x'(s) = v(x(s)) \ , \qquad x(r) = x \ .$$

The fact that for fixed s and r, $x \mapsto \Phi_{s,r}(x)$ is continuously differentiable in x means that the solutions changes smoothly when the initial data is changed.

Proof of Theorem 5. Let $x(t)$ be the solution of $x'(t) = v(x(t))$ with $x(0) = x$. Then by definition, $\Psi_r(x) = x(r)$, and therefore, $y(t) := x(t+r)$ solves. $y'(t) = v(y(t))$ with $y(0) = \Psi_r(x)$. Then by definition,

$$\Psi_s(\Psi_r(x)) = y(s) = x(s+r) = \Psi_{s+r}(x) \ .$$

This proves (2.31).

Also, by definition, for any r, and any $x \in (a,b)$,

$$\Psi_0(x) = x(r) = x$$

so that Ψ_0 is the *identity transformation* Id defined by $\mathrm{Id}(x) = x$. In particular,

$$\Psi_s \circ \Psi_{-s} = \mathrm{Id} \qquad \text{and} \qquad \Psi_{-s} \circ \Psi_s = \mathrm{Id} \ ,$$

so that each Ψ_s is invertible, and its inverse is Ψ_{-s}.

In the computations leading to (2.29), we have already found

$$\frac{\partial}{\partial x}\Psi_s(x) = \frac{v(\Psi_s(x))}{v(x)} \ . \tag{2.34}$$

Finally it is very easy to compute $\dfrac{\partial}{\partial s}\Psi_s(x)$. By definition, $\Phi_s(x)$, regarded as a function of s with x held fixed, is the solution of $x' = v(x)$ with $x(r) = x$. Hence the s partial derivative is simply $x'(s) = v(x(s))$. That is,

$$\frac{\partial}{\partial s}\Psi_s(x) = v(\Psi_s(x)) \ . \tag{2.35}$$

□

Example 20. *In Example 19 we have computed that for $v(x) = x(1-x)$, the flow transformation on $(0, 1)$ is given by*

$$\Psi_s(x) = \frac{xe^s}{(1-x) + xe^s} \ .$$

By direct computation, we found in (2.24) that

$$\frac{\partial}{\partial x}\Psi_s(x) = \frac{e^s}{(1-x+xe^s)^2} \ . \tag{2.36}$$

On the other hand we have the general formula (2.32):

$$\frac{\partial}{\partial x}\Psi_s(x) = \frac{v(\Psi_s(x))}{v(x)} \ .$$

In this case, the right hans side is

$$\frac{\Psi_s(x)(1-\Psi_s(x))}{x(1-x)} \ .$$

Since $(1-\Psi_s(x))\Psi_s(x) = \dfrac{(1-x)}{(1-x)+xe^s}\dfrac{xe^s}{(1-x)+xe^s}$, *we have*

$$\frac{\Psi_s(x)(1-\Psi_s(x))}{x(1-x)} = \frac{e^s}{(1-x+xe^s)^2} \ ,$$

and thus the general formula yields the same result as the direct computation.

2.4 Uniqueness for time dependent vector fields on the line

2.4.1 A second approach to uniqueness

In this section we study the *non-autonomous* equation $x'(t) = v(t, x(t))$ for a *time dependent* vector field $v(t, x)$. Suppose that $v(t, x)$ is a continuous function of $(t, x) \in \mathbb{R}^2$.

For example we might have $v(t, x) = tx - t^2$ in which case $x' = v(t, x)$ is the first order linear equation $x' = tx - t^2$. Or we might have $v(t, x) = t^2x + tx^2$ in which case $x' = v(t, x)$ is the Bernoulli equation $x' = t^2x + tx^2$.

In both of these cases, we know how to solve the equations, and find the general solution. However, when v depends on t as well as x, there is nothing like Barrow's formula that reduces solution of the differential equation to an integral.

Apart from the questions of whether solutions to $x'(t) = v(t, x(t))$ exist, and how to compute them, there is the question of uniqueness: Can there ever be more than one solution to

$$x'(t) = v(t, x(t)) \ , x(t_0) = x_0 \ , \tag{2.37}$$

or are solutions unique whenever the exist?

We have already seen that even when v does not depend on t, solutions may not be unique if v is not Lipschitz, but that when v is Lipschitz we have uniqueness. The next Theorem gives a version of this result that covers the time-dependent vector field case.

Theorem 6 (Uniqueness for Lipschitz vector fields)**.** *Let $v(t, x)$ be a continuous on $(c, d) \times (a, b)$ with values in $\mathbb{R}$. Suppose that for some $L < \infty$,*

$$|v(t, y) = v(t, x)| \leq L|y - x| \qquad \text{for all} \quad t \in (c, d) \quad \text{and} \quad x, y \in (a, b) \ . \tag{2.38}$$

Let $t_0 \in (c, d)$. Suppose that $x(t)$ solves $x'(t) = v(t, x(t))$ with $x(t_0) = x$ for all $t \in (c, d)$ and that $y(t)$ solves $y'(t) = v(t, y(t))$ with $y(t_0) = y$ for all $t \in (c, d)$. Then for all $t \in (c, d)$,

$$e^{-L|t-t_0|}|y - x| \leq |y(t) - x(t)| \leq e^{L|t-t_0|}|y - x| \ . \tag{2.39}$$

Proof. Define $z(t) = [y(t) - x(t)]^2$. Then

$$z'(t) = 2[y(t) - x(t)][y'(t) - x'(t)] = 2[y(t) - x(t)][v(t, y(t)) - v(t, x(t))| \ .$$

Therefore

$$|z'(t)| \leq 2|y(t) - x(t)||v(t, y(t)) - v(t, x(t))| \leq 2L|y(t) - x(t)|^2 = 2Lz(t) \ .$$

That is,

$$-2Lz(t) \leq z'(t) \leq 2Lz(t) \ . \tag{2.40}$$

The second inequality in (2.40) can be written as $z'(t) - 2Lz(t) \leq 0$. Multiplying both sides by e^{-t2L}, we have

$$\left(z(t)e^{-t2L}\right)' \leq 0 \ .$$

Hence $z(t)e^{-t2L}$ is a non-increasing function of t. It follows that:

$$z(t)e^{-t2L} \leq z(t_0)e^{-t_0 2L} \quad \text{for } t > t_0 \quad \text{and} \quad z(t)e^{-t2L} \geq z(t_0)e^{-t_0 2L} \quad \text{for } t < t_0 \ .$$

Since $z(t_0) = |y - x|^2$, this is the same as

$$|y(t) - x(t)| \leq |y - x|e^{(t-t_0)L} \quad \text{for } t > t_0 \quad \text{and} \quad |y(t) - x(t)| \geq |y - x|e^{(t-t_0)L} \quad \text{for } t < t_0 \ .$$

The first inequality in (2.40) can be written as $z'(t) + 2Lz(t) \geq 0$. Multiplying both sides by e^{t2L}, we have

$$\left(z(t)e^{t2L}\right)' \geq 0 \ .$$

Hence $z(t)e^{t2L}$ is a non-decreasing function of t. It follows that:

$$z(t)e^{t2L} \geq z(t_0)e^{t_02L} \quad \text{for } t > t_0 \quad \text{and} \quad z(t)e^{t2L} \leq z(t_0)e^{t_02L} \quad \text{for } t < t_0 \ .$$

Since $z(t_0) = |y - x|^2$, this is the same as

$$|y(t) - x(t)| \geq |y - x|e^{-(t-t_0)L} \quad \text{for } t > t_0 \quad \text{and} \quad |y(t) - x(t)| \leq |y - x|e^{-(t-t_0)L} \quad \text{for } t < t_0 \ .$$

The two lower bounds on $|y(t) - x(t)|$ can be summarized as $|y(t) - x(t)| \geq |y - x|e^{-|t-t_0|L}$ for all t and the two upper bounds on $|y(t) - x(t)|$ can be summarized as $|y(t) - x(t)| \leq |y - x|e^{|t-t_0|L}$ for all t. □

This theorem proves uniqueness, and more. First, suppose $y = x$. Then $y(t)$ and $x(t)$ are any two solution passing through x_0 at $t = t_0$. Then since $|y - x| = 0$, the right hand inequality says that $|y(t) - x(t)| = 0$ for all $t > t_0$, so the solutions must be equal for all $t > t_0$.

Now note that the theorem says that if $x \neq y$, the curves $x(t)$ and $y(t)$ *never meet*; for all t $x(t) \neq y(t)$. Indeed, for all t, $|x(t) - y(y)| \geq e^{-tL}|x - y| > 0$.

This theorem also applies in the case in which v is independent of t but Lipschitz. It is conceptually quite different from the earlier proofs we gave for this case that were based on Barrow's formula.

2.4.2 Existence for time dependent vector fields on the line

We have proved that when a vector field $v(t, x)$ is continuous on $(c, d) \times (a, b)$, and the Lipschitz condition (2.38) is satisfied, then for each $t_\in (c, d)$ and each $x_0 \in (a, b)$ there is *at most* one solution to the equation

$$x'(t) = v(t, x(t)) \ , x(t_0) = x_0 \ .$$

Later we shall prove the there always is a solution under the same conditions, and is is defined for all $t \in (c,)d$, where $c = -\infty$ and $d = \infty$ are allowed. However, the proof will take some time, and the same proof works for the multidimensional case $\mathbf{x}'(t) = \mathbf{v}(t, \mathbf{x}(t))$, and therefore we postpone the proof of existence until we come to it in the multidimensional case. The proof we have given in this subsection for uniqueness extends in a simple way to the multidimensional case, as we shall see, but it is short and the idea on which it rests is useful enough that it is worth repeating.

2.5 Exercises

1. Let $v(x) = \sin(x)$. For all $0 \leq x \leq \pi$, Find all solutions of

$$x'(t) = v(x(t)) \ , \qquad x(0) = x_0 \ .$$

For which values of t is each solution defined?
Hint: It will probably help to recall the identity

$$\frac{1 - \cos x}{\sin x} = \tan(x/2) \ .$$

2. Let $v(x) = \tan(x)$, which is continuous on $-\pi/2 < x < \pi/2$. For all x_0 in this interval, find all solutions of

$$x'(t) = v(x(t)) \ , \qquad x(0) = x_0 \ .$$

For which values of t is each solution defined?

3. **(a)** Let $v(x) = (1 - x^4)^{1/2}$ Consider the solution of $x'(t) = v(x(t))$ with $x(0) = 0$. Does this solution exist for all t and remain within the interval $(-1, 1)$ for all t? Justify your answer.

(b) Let $v(x) = (1 - x^4)^2$ Consider the solution of $x'(t) = v(x(t))$ with $x(0) = 0$, Does this solution exist for all t and remain within the interval $(-1, 1)$ for all t? Justify your answer.

4. Consider the two equation

$$(\mathbf{1}) \qquad (x')^2 + x^2 = 1 \qquad \text{and} \qquad (\mathbf{2}) \qquad (x')^2 - x^2 = 1 \ .$$

Let $-1 < x_0 < 1$.

(a) One of these two equations has a unique solution with $x(0) = x_0$, and the other has infinitely many such solutions. Which is which? Justify your answer.

(b) For each $-1 < x_0 < 1$, find infinitely many solutions of the equation for which there is no uniqueness.

5. Let $v(x)$ be Lipschitz and positive on the maximal interval (a, b). Let $a < x_1 < x_2 < b$. For $j = 1, 2$, let $x_j(t)$ be the solution to $x_j'(t) = v(x_j(t))$ with $x_j(0) = x_j$. Let $T > 0$ be the time at which $x_1(t)$ 'catches up' to where $x_2(t)$ started. That is

$$x_1(T) = x_2 = x_2(0) \ .$$

Show that for all positive integers k,

$$x_1((k+1)T) = x_2(kT) \ .$$

6. **(a)** Let $v(x) = \tanh(x)$. Show that $v(x)$ is Lipschitz on the whole real line. Then find the flow transformation $\Psi_t(x)$. Finally, by direct computation, verify the formulae

$$\frac{\mathrm{d}}{\mathrm{d}x}\Psi_t(x) = \frac{v(\Psi_t(x))}{v(x)} \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\Psi_t(x) = v(\Psi_t(x))$$

on each of the maximal intervals $(-\infty, 0)$ and $(0, \infty)$.

(b) Let $0 < x_1 < x_2$. For $j = 1, 2$, let $x_j(t)$ be the solution to $x_j'(t) = v(x_j(t))$ with $x_j(0) = x_j$. Show that

$$\lim_{t\to\infty} (x_2(t) - x_1(t)) = \int_{x_1}^{x_2} \frac{1}{v(x)} \mathrm{d}x \ ,$$

which is the time it takes $x_1(t)$ to reach the starting point of the second solution, $x_2(0) = x_2$.

(c) If we changed $v(x)$ to $2\tanh(x)$, and for $j = 1, 2$, let $x_j(t)$ be the solution to $x_j'(t) = v(x_j(t))$ with $x_j(0) = x_j$ for this new $v(x)$, what would

$$\lim_{t\to\infty} (x_2(t) - x_1(t))$$

be now?

7. **(a)** Let $v(x) = x^3/(1+x^2)$. Show that $v(x)$ is Lipschitz on the whole real line so that the corresponding flow transformation $\Psi_t(x)$ is defined for all t on each of the maximal intervals $(-\infty, 0)$ and $(0, \infty)$. Show that for all $x \neq 0$,

$$\lim_{t\to\infty} \left| \frac{\mathrm{d}}{\mathrm{d}x} \Psi_t(x) \right| = \infty \ ,$$

meaning that the the effect of a small change in the initial data becomes arbitrarily large at t becomes large.

(b) Let $0 < x_1 < x_2$. For $j = 1, 2$, let $x_j(t)$ be the solution to $x_j'(t) = v(x_j(t))$ with $x_j(0) = x_j$. Using the fact that

$$|x_2(t) - x_1(t)| = |\Psi_t(x_2) - \Psi_t(x_2)| = \int_{x_1}^{x_2} \frac{\mathrm{d}}{\mathrm{d}x} \Psi_t(x) \mathrm{d}x \ , \tag{2.41}$$

show that 'second solution runs away from the first'. That is show that

$$\lim_{t\to\infty} |x_2(t) - x_1(t)| = \infty \ .$$

8. **(a)** Let $v(x) = x^3/(1+x^4)$. Show that $v(x)$ is Lipschitz on the whole real line so that the corresponding flow transformation $\Psi_t(x)$ is defined for all t on each of the maximal intervals $(-\infty, 0)$ and $(0, \infty)$. Show that for all $x \neq 0$,

$$\lim_{t\to\infty} \left| \frac{\mathrm{d}}{\mathrm{d}x} \Psi_t(x) \right| = 0 \ .$$

(b) Let $0 < x_1 < x_2$. For $j = 1, 2$, let $x_j(t)$ be the solution to $x_j'(t) = v(x_j(t))$ with $x_j(0) = x_j$. Using (2.41), show that asymptotically, the 'second solution catches up with the first'. That is show that

$$\lim_{t\to\infty} |x_2(t) - x_1(t)| = 0 \ .$$

9. For $\alpha > 0$, let

$$v(x) = x|\ln|x||^\alpha$$

for $x \neq 0$, and $v(0) = 0$, so that v is continuous on $\mathbb{R}$. The interval $(0, 1)$ is a maximal interval for v since $v(0) = v(1) = 0$ and $v(x) > 0$ on $(0, 1)$.

(a) For all $\alpha > 0$, and all $x_0 \in (0, 1)$, and $t_0 \in \mathbb{R}$, find the solution of $x'(t) = v(t)$ for $x(t_0) = x_0$ for all t for which the solution stays in the interval $(0, 1)$. For which values of α does the solution remain in $(0, 1)$ for all $t > t_0$? For which values of α does the solution remain in $(0, 1)$ for all $t < t_0$?

(b) Note that $x = 0$ and $x = 1$ are both equilibrium points for v (as is $x = -1$). For which values of α is the steady state solution $x(t) = 0$ for all t the only solution of $x'(t) = v(x(t))$ with $x(0) = 0$? For which values of α is the steady state solution $x(t) = 1$ for all t the only solution of $x'(t) = v(x(t))$ with $x(0) = 1$?

(c) For which values of α is v Lipschitz on $(0, 1)$?

10. Consider the equation

$$x''(t) = F(x(t)) \qquad \text{where} \qquad F(x) = -\frac{\mathrm{d}}{\mathrm{d}x}V(x) \tag{2.42}$$

for some continuously differentiable function V.

(a) Define the function $H(x, y)$ by

$$H(x, y) = \frac{1}{2}y^2 + V(x) \ . \tag{2.43}$$

Show that if $x(t)$ is any solution of (2.42) defined on some open interval containing t_0, then

$$H(x'(t), x(t)) = H(x'(t_0), x(t_0))$$

for all t in the interval. Therefore, to solve (2.42) with $x(t_0) = x_0$ and $x'(t_0) = v_0$, we need only solve

$$x' = \pm\sqrt{H(v_0, x_0)) - V(x)} \ . \tag{2.44}$$

(b) Let $V(x) = \frac{1}{2}x^2$, and take $x_0 = 1$ and $v_0 = 0$. There will be infinitely many solutions of (2.44. Describe all of them (The description will involve arbitrary "rest periods" at equilibrium points.). Of these solutions, how many are twice continuously differentiable?

(c) How many solutions of

$$(x'(t))^2 + (x(t))^4 = 1$$

are there with $x(0) = 1$? How many of these are twice continuously differentiable?

11. (a) Let $V(x)$ be a continuously differentiable function on $\mathbb{R}$. Fix some number E, and suppose there is no x such that $V(x) = E$ and $v'(x) = 0$. Let x_0 be such that $V(x_0) < E$. Let $x(t)$ be any solution of

$$(x'(t))^2 + V(x) = E \qquad \text{with} \qquad x(0) = x_0 \ .$$

Show that there is a $0 < T < \infty$ such that $x'(T) = 0$. That is, the solution 'cones to rest', at least instantaneously in a finite time.

(b) Show that the conclusion of **f**(a) may be false if there is a point x with $V(x) = E$ and $V'(x) = 0$.

Chapter 3

INTRODUCTION TO FIRST ORDER SYSTEMS

3.1 Flows on $\mathbb{R}^n$

3.1.1 Vector Fields on $\mathbb{R}^n$ and their associated flows

Let U be an open subset of $\mathbb{R}^n$, and let $\mathbf{v}$ be a continuous function defined on U with values in $\mathbb{R}^n$. We shall call such a function a *vector field on* U. Given any $t_0 \in \mathbb{R}$ and any $x_0 \in U$, we seek to find all solutions, if any, of the first order differential equation

$$\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t)) \ , \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{3.1}$$

and the maximal interval of times t on which such a solution is defined.

Exactly as in the one dimensional case, we have the notion of *equilibrium points* and and *steady state solutions*.

Definition 9 (Equilibrium points and steady state solutions)**.** *Let* $\mathbf{v}$ *be a continuous vector field defined on* U*, an open subset of* $\mathbb{R}^n$*. Any* $\mathbf{x}_0 \in U$ *for which* $\mathbf{v}(\mathbf{x}_0) = \mathbf{0}$ *is an* equilibrium point *of* $\mathbf{v}$*. In this case, the function* $\mathbf{x}(t) = \mathbf{x}_0$ *for all* t *is a solution of (3.1). Such a solution is called a* steady-state solution.

As we have already seen, even when $n = 1$, when $\mathbf{x}_0$ is a an equilibrium point, the steady state solution of (3.1) may not be the *only* solution of (3.1). However, also as in the one dimensional case, there is a simple condition – Lipschitz continuity – that guarantees this uniqueness. It turns out that this same condition will also guarantee existence as well as uniqueness of solutions.

However, before we turn to general theorems, it is instructive to study some examples in which we can compute all of the solutions. This will be our focus in the next subsection, in which introduce a class of vector fields that can be treated using one-dimensional methods.

3.1.2 Uncoupled Systems

Let us begin with $n = 2$. We may write, using Cartesian coordinates,

$$\mathbf{x}(t) = \big(x(t), y(t)\big) \quad \text{and} \quad \mathbf{v}(x,y) = \big(f(x,y), g(x,y)\big) \ .$$

Then (3.1) is equivalent to

$$\begin{aligned} x'(t) &= f(x(t), y(t)) \ , \quad x(t_0) = x_0 \\ y'(t) &= g(x(t), y(t)) \ , \quad y(t_0) = y_0 \end{aligned}$$

This is an examples of a *system of first order equations.* The generalization to higher dimensions is clear.

The simplest case is that in which f depends only on x and g depends only on y. In this case, (3.2) reduces to

$$\begin{aligned} x'(t) &= f(x(t)) \ , \quad x(t_0) = x_0 \\ y'(t) &= g(y(t)) \ , \quad y(t_0) = y_0 \end{aligned}$$

There is no "coupling" between the two equations; the equation for the evolution of x does not involve y and *vice-versa*. Hence, to solve such a system, we need only solve the equations separately, and for this we can use our one dimensional methods.

Example 21 (An uncoupled system)**.** *Fix some number $\alpha \in \mathbb{R}$. Let*

$$\mathbf{v}(x,y) = \big(x, \alpha y\big) \ .$$

That is, with $\mathbf{v}(x,y) = \big(f(x,y), g(x,y)\big)$, $f(x,y) = x$ and $g(x,y) = \alpha y$. Then (3.2) becomes

$$\begin{aligned} x'(t) &= x(t) \ , \quad x(t_0) = x_0 \\ y'(t) &= \alpha y(t) \ , \quad y(t_0) = y_0 \end{aligned}$$

Solving these equations separately, we find

$$x(t) = x_0 e^{t-t_0} \qquad \text{and} \qquad y(t) = y_0 e^{\alpha(t-t_0)} \ .$$

In particular, if $x_0 \neq 0$,

$$e^{t-t_0} = \frac{1}{x_0} x(t) \ .$$

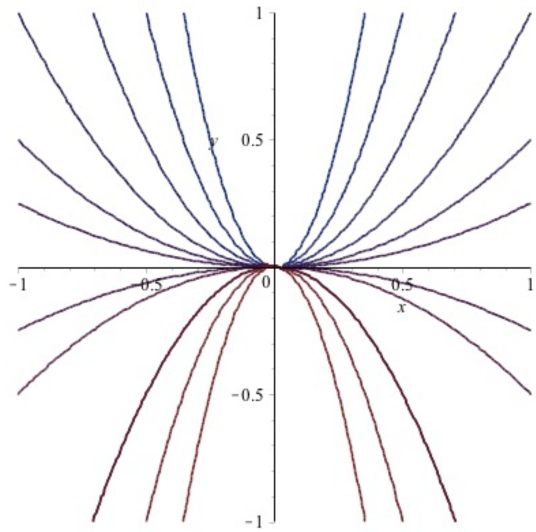
Therefore,

$$y(t) = \frac{y_0}{|x_0|^\alpha} |x(t)|^\alpha \ .$$

In particular, for $\alpha = 2$, each $\big(x(t), y(t)\big)$ lies on the curve

$$y = \frac{y_0}{x_0^2} x^2 \ , \tag{3.2}$$

which is a parabola. The equation (3.2) describes the family of all parabolas that pass through $(0,0)$ tangent to the x-axis. Here is a plot of some of these:



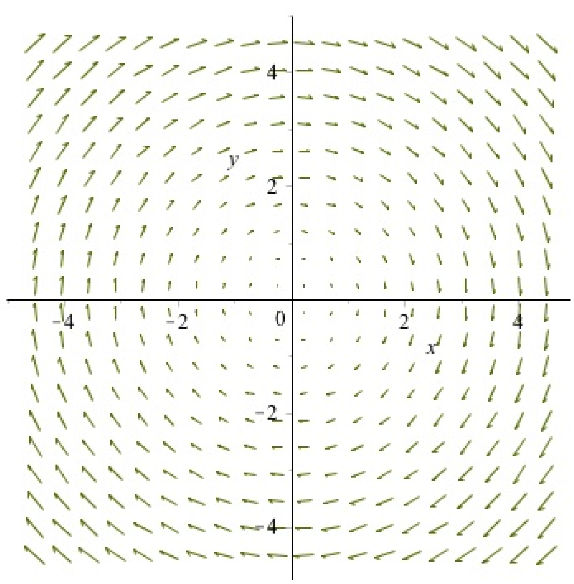
However, the solutions of (3.2) do not trace out any entire parabolas: All of the parabolas pass through $(0,0)$*, which is an equilibrium point, and we see from the explicit form of the solutions that* $x(t)$ *and* $y(t)$ *never change sign. If the solution is not the steady-state solution, then* $\mathbf{x}(t)$ *traces out one side or the other of one of these parabolas.*

This plot should not be confused with our earlier plots in the t, x *plane. This plot, unlike those, does not involve time, and does not give any information about the speed of the motion.*

Sometimes a system of equations can be decoupled by a judicious change of variables. For example, consider $\mathbf{v}(x,y) = (y, -x)$. The corresponding system of first order equations is

$$\begin{aligned} x'(t) &= \quad y(t) \ , \quad x(t_0) = x_0 \\ y'(t) &= -x(t) \ , \quad y(t_0) = y_0 \ . \end{aligned}$$

Here is a plot of the vector field, with the arrows shortened for better visibility of the pattern:



Looking at the "sweep" of the arrows, one can anticipate the circular nature of the flow, and this suggests that it may have a simpler description in polar coordinates. Introducing polar coordinates as usual, we have $x = r\cos\theta$ and $y - r\sin\theta$. That is,

$$x(t) = r(t)\cos\theta(t) \qquad \text{and} \qquad y(t) = r(t)\sin\theta(t) \ , \tag{3.3}$$

and

$$r(t) = \left(x^2(t) + y^2(y)\right)^{1/2} \ .$$

Therefore,

$$r'(t) = \frac{1}{r(t)}(x(t)x'(t) + y(t)y'(y)) = \frac{1}{r(t)}(x(t)y(t) - y(t)x(t) = 0 \ ,$$

which shows that r is constant. To obtain an equation for θ, it is simplest to differentiate $x(t) = r\cos\theta(t)$, using the fact that r is constant, to obtain

$$x'(t) = -\theta'(t)r\sin\theta(t) = -\theta'(t)y(t) \ .$$

From the equation $x' = y$ we deduce that $\theta'(t) = -1$ for all t.

Therefore, in the new variables, (3.3) becomes

$$\begin{aligned} r'(t) &= 0 \ , \quad r(0) = \sqrt{x_0^2 + y_0^2} \\ \theta'(t) &= -1 \ , \quad \theta(t_0) = \theta_0 \end{aligned}$$

where θ_0 is the unique angle in $[0, 2\pi)$ such that $r\cos\theta_0 = x_0$ and $r\sin\theta_0 = y_0$. (There is a formula for this; $\theta_0 = \pi/2$ if $x_0 = 0$ and $y_0 > 0$, $\theta_0 = -\pi/2$ if $x_0 = 0$ and $y_0 < 0$, $\theta_0 = \arctan(y_0/x_0)$ if $x_0 > 0$, and $\theta_0 = \arctan(y_0/x_0) + \pi/2$ if $x_0 < 0$. But we shall not use this formula below.)

The new system is decoupled in the obvious sense, and we have the unique solution $r(t) = r_0$ and $\theta(t) = \theta_0 - t$ for all t. Therefore

$$x(t) = r\cos(\theta_0 - t) = r[\cos\theta_0\cos t + \sin\theta_0\sin t] = x_0\cos t + y_0\sin t$$

and

$$y(t) = r\sin(\theta_0 - t) = r[-\cos(\theta_0)\sin(\theta) + \sin\theta_0\cos t] = -x_0\sin t + y_0\cos t \ .$$

We can write this in a very simple form by introducing the matrix

$$\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \ .$$

We then have

$$\mathbf{x}(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \mathbf{x}_0 \ . \tag{3.4}$$

Since for each $x \in \mathbb{R}^2$, there is a unique solution curve passing through x at time $t = 0$, we may define the flow transformation for this vector field $\Psi_t(\mathbf{x})$ by $\Psi_t(\mathbf{x}) = \mathbf{x}(t)$ where $\mathbf{x}(t)$ is this solution evaluated at time t. Our computation of the general solution gives us a formula for Ψ_t, namely

$$\Psi_t(\mathbf{x}) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \mathbf{x}_0 \ .$$

Since the solutions curves are unique, $\Psi_{t+s}(\mathbf{x})$ is what you get be starting at $\mathbf{x}$ and then following the solution curve through $\mathbf{x}$ to $\mathbf{x}(t)$, and from there along the same curve – by the uniqueness – on to $\mathbf{x}(t+s)$. Or you could follow this curve to $\mathbf{x}(s)$, and from there on to $\mathbf{x}(s+t)$. Either way, you see that

$$\Psi_{t+s} = \Psi_t \circ \Psi_s = \Psi_s \circ \Psi_t \ , \tag{3.5}$$

and it is clear the Ψ_0 is the identity transformation, since by definition, in this case one does not move along the curve at all.

In this case, our flow transformation is a linear transformation represented by a matrix. The composition of the linear transformations represented by two matrices is the linear transformation of their matrix product. Thus, (3.5) is equivalent to

$$\begin{aligned}\begin{bmatrix} \cos(t+s) & \sin(t+s) \\ -\sin(t+s) & \cos(t+s) \end{bmatrix} &= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\begin{bmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{bmatrix} \\ &= \begin{bmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{bmatrix}\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \ , \end{aligned} \tag{3.6}$$

which can be verified by multiplying out the matrices and using the angle addition formulas.

The formulas (3.5) and (3.6) are reminiscent of the addition formula for products of exponentials $e^{ta}e^{sa} = e^{(t+s)a}$ for all $a, t, s \in \mathbb{R}$. In fact, if we introduce the matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, we can write (3.3) in the form

$$\mathbf{x}'(t) = A\mathbf{x}(t) \ , \qquad \mathbf{x}(0) = \mathbf{x}_0 \ ,$$

which suggests that in some sense

$$e^{tA} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \ ,$$

and that then (3.6) simply says $e^{(t+s)A} = e^{tA}e^{sA} = e^{sA}e^{tA}$. We shall see that there is indeed a very useful extension of the exponential function to matrices for which this formula is indeed true. This is a very useful train of thought to which we shall return, but first, let us further develop the idea of decoupling equations by a change of variables.

Our next example illustrates the decoupling of a more interesting system using polar coordinates.

Example 22 (Decoupling by a change of variables)**.** *Consider the system*

$$\begin{aligned} x'(t) &= y(t) + x(t)[1 - x^2(t) - y^2(t)] \ , \quad x(0) = x_0 \\ y'(t) &= -x(t) + y(t)[1 - x^2(t) - y^2(t)] \ , \quad y(0) = y_0 \ . \end{aligned} \tag{3.7}$$

The linear part of the vector field $\mathbf{v}$ *associated to this system is the same as the one that we simplified by changing to polar coordinates. The nonlinear part involves* $x^2 + y^2$ *which simplifies in polar coordinates. Thus, it is natural to once again try polar coordinates.*

Since $r = \sqrt{x^2 + y^2}$, the chain rule yields

$$\begin{aligned} r'(t) &= \frac{1}{r(t)}(x(t)x'(t) + y(t)y'(t)) \\ &= \frac{1}{r(t)}(x^2(t) + y^2(t))[1 - x^2(t) + y^2(t)] \\ &= r(t)[1 - r^2(t)] \ . \end{aligned}$$

This is part of what we sought; the equation expressing $r'(t)$ does not involve $\theta(t)$.

Next, to get an equation for $\theta(t)$, we proceed as before, and differentiate both sides of $x(t) = r(t)\theta(t)$. We obtain

$$\begin{aligned} x'(t) &= r'(t)\cos\theta(t) - \theta'(t)r(t)\sin\theta(t) \\ &= [1 - r^2(t)]r(t)\cos(t) - \theta'(t)r(t)\sin\theta(t) \\ &= [1 - r^2(t)]x(t) - \theta'(t)y(t) \end{aligned}$$

Comparing with the first equation in (3.7) we see that $\theta'(t) = -1$. Therefore, in the new variables, our system becomes

$$\begin{aligned} r'(t) &= r(t)[1 - r^2(t)] \ , \quad r(0) = \sqrt{x_0^2 + y_0^2} \\ \theta'(t) &= -1 \ , \quad \theta(t_0) = \theta_0 \end{aligned}$$

This system is decoupled; the rate of change of r does not depend on θ, and the rate of change of θ does not depend on r. We may solve the two equations separately using our single variable methods.

The equation for θ is trivial; it yields $\theta = \theta_0 - (t - t_0)$. To solve the equation for $r(t)$ we use Barrow's formula

$$t(r) - t_0 = \int_{r_0}^{r} \frac{1}{z(1 - z^2)} \mathrm{d}z \ .$$

The vector field $v(r) = r[1 - r^2]$ has equilibrium points at $r = -1$, $r = 0$ and $r = 1$. Since r is non-negative by definition, we are only concerned with the maximal intervals $(0, 1)$ and $(1, \infty)$.

Consider $r_0 \in (0, 1)$. We have the partial fractions expansion

$$\frac{1}{z(1 - z^2)} = \frac{1}{z} + \frac{1}{2}\frac{1}{1 - z} - \frac{1}{2}\frac{1}{1 + z} \ , \tag{3.8}$$

Since z, $1 - z$ and $1 + z$ are all positive for $z \in (0, 1)$, we have for $r \in (0, 1)$,

$$t(r) - t_0 = \left[\ln(z) - \frac{1}{2}\ln(1 - z) - \frac{1}{2}\ln(1 + z)\right]\Bigg|_{z=r_0}^{z=r} = \ln\left(\frac{r}{\sqrt{1 - r^2}}\right) - \ln\left(\frac{r_0}{\sqrt{1 - r_0^2}}\right)$$

Thus, with $a = e^{t - t_0} r_0 / \sqrt{(1 - r_0^2)}$, we have $\dfrac{r}{\sqrt{1 - r^2}} = a$. Squaring and solving, we find $r = \dfrac{a}{\sqrt{1 + a^2}}$, and so

$$r(t) = \frac{r_0 e^{t - t_0}}{\sqrt{(1 - r_0^2) + r_0^2 e^{2(t - t_0)}}} \ . \tag{3.9}$$

Notice that $r(t_0) = r_0$, as it must, and $\lim_{t \to -\infty} r(t) = 0$ and $\lim_{t \to \infty} = 1$. Thus the solutions stay for all time in the maximal interval $(0, 1)$, as it must since $r(1 - r^2)$ is Lipschitz on this interval.

Since $r(1-r^2)$ *is Lipschitz on any bounded open interval about* $r=1$*, the steady state solution is the unique solution of* $r' = r(1-r^2)$ *with* $r(t_0)=1$*. (The same applies to the other equilibrium point* $r=0$*.)*

Next, we consider $r_0 > 1$*. Since* $1-z$ *is negative for* $z>1$*, and since we shall be taking logarithms, we rewrite our partial fractions expansion as*

$$\frac{1}{z(1-z^2)} = \frac{1}{z} - \frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{1}{1+z} ,$$

Then with $r_0, r \in (1,\infty)$*, Barrow's formula yields*

$$t(r) - t_0 = \ln\left(\frac{r}{\sqrt{r^2-1}}\right) - \ln\left(\frac{r_0}{\sqrt{r_0^2-1}}\right) ,$$

and solving as before we find

$$r(t) = \frac{r_0 e^{t-t_0}}{\sqrt{r_0^2 e^{2(t-t_0)} - (r_0^2-1)}} . \tag{3.10}$$

This is the same formula (written slightly differently) as (3.9), which also gives the steady state solutions for $r_0=0$ *and* $r_0=1$*. Therefore, (3.9) gives the unique solution for all* $r_0 \geq 0$*. However, the form (3.10) is convenient for studying the solution when* $r_0>1$*. Notice that in this case,* $r(t)$ *is well-defined for all* $t>t_0$*, and* $\lim_{t\to\infty} r(t) = 1$*, as we might expect since* $r(1-r^2)<0$ *for* $r>1$*, so* $r(t)$ *is always decreasing when* $r_0>1$*. However, when*

$$r_0^2 e^{2(t-t_0)} = (r_0^2-1) ,$$

there is division by zero in (3.10), and the solution "blows up". Solving for t*, this happens when*

$$t = t_0 - \frac{1}{2}\ln\left(\frac{r_0^2}{r_0^2-1}\right)$$

which is, of course, less than t_0*.*

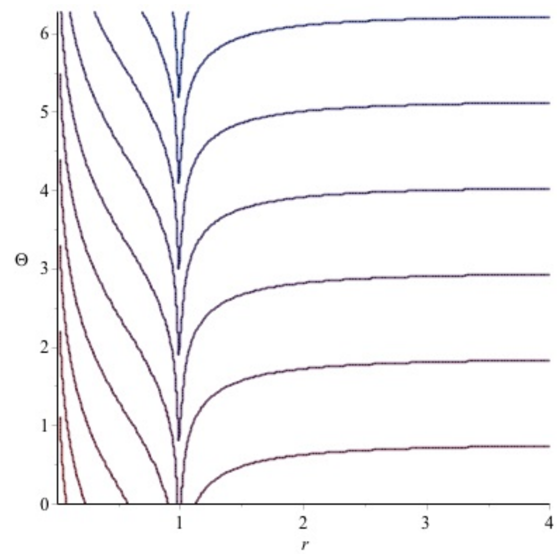
To plot the curves traced out in the r,θ *plane by the solutions, we note that*

$$\theta - \theta_0 = t - t_0 = \frac{1}{2}\ln\left(\frac{r^2}{|1-r^2|}\right) - \frac{1}{2}\ln\left(\frac{r_0^2}{|1-r_0^2|}\right) .$$

Therefore, the solution curves in the r,θ *plane are given by*

$$\theta + \frac{1}{2}\ln\left(\frac{r^2}{|1-r^2|}\right) = c$$

where c *is a constant. The next figure shows a contour plot of the function on the left:*



Finally, let us return to Cartesian coordinates in the x, y plane. Since $x = r\cos\theta$ and $y = r\sin\theta$, we obtain

$$\begin{aligned}(x(t), y(t)) &= \frac{\sqrt{x_0^2+y_0^2}e^{t-t_0}}{\sqrt{(1-(x_0^2+y_0^2)) + (x_0^2+y_0^2)e^{2(t-t_0)}}}\big(\cos(\theta_0-(t-t_0)), \sin(\theta_0-(t-t_0))\big)\\ &= \frac{e^{t-t_0}}{\sqrt{1-\|\mathbf{x}_0\|^2+\|\mathbf{x}_0\|^2e^{2(t-t_0)}}}\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}\mathbf{x}_0\ , \end{aligned} \tag{3.11}$$

where we have used the angle addition formulas and matrix notation to arrive at the final form as in (3.4).

The solution simplifies for large t: Since

$$\lim_{t\to\infty} r(t) = 1$$

for all $r_0 > 0$, we have for any starting point except *the equilibrium point $(0,0)$ that the solutions has*

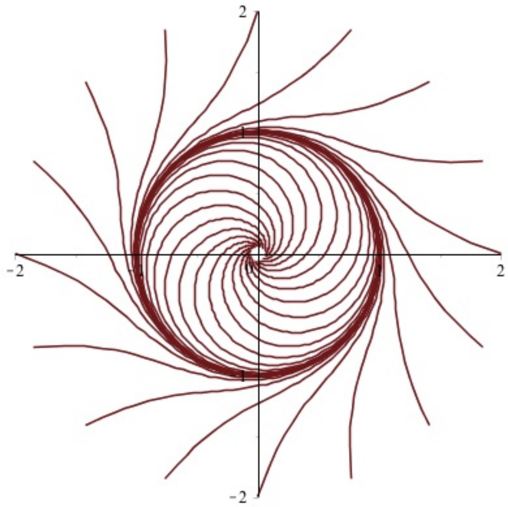
$$(x(t), y(t)) \approx \big(\cos(\theta_0-(t-t_0)), \sin(\theta_0-(t-t_0))\big)\ ,$$

or equivalently

$$\mathbf{x}(t) \approx \begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}\frac{\mathbf{x}_0}{\|\mathbf{x}_0\|}\ ,$$

for all largetimes t. The solution os periodic *if and only if $r_0 = \sqrt{x_0^2+y_0^2} = 1$, and in this case the period is 2π. For all other solutions, except the steady-state solution at $(0,0)$, the radius $\|\mathbf{x}(t)\|$ is either strictly increasing or decreasing, and so we cannot have $\mathbf{x}(t+T) = \mathbf{x}(t)$ for any $T > 0$ for any t, let alone for all t. However, because every solution except the steady-state solution at $(0,0)$ "spirals in" to the unit circle exponentially fast, the motion described by this equation is* approximately periodic *for large t. We shall encounter this phenomenon of "limit cycles" again in other examples. While we have found this limit cycle by explicit computation, we shall develop method for determining when such solutions exist without relying on an explicit calculation of all solutions.*

The following plot shows some solution curves in the x, y plane, spiraling in to the unit circle from outside, and spiraling out to the unit circle from inside.



Another phenomenon of interest that shows up clearly in this plot is that while $(0,0)$ *is an equilibrium point, and the steady state solution is the only curve that ever passes through this point, it is an* unstable equilibrium *in that no matter how close* (x_0, y_0) *is to* $(0,0)$, *if it is not* exactly equal *to it, the solution will move away, and in fact will satisfy*

$$\lim_{t\to\infty} \|\mathbf{x}(t) - (0,0)\| = 1 \ .$$

That is, solutions that start *nearby this equilibrium point do not* stay *nearby this equilibrium point. The question of whether equilibrium points have this sort of* instability, *or not, is of considerable interest in applications, as we shall see.*

Many systems can be decoupled by an appropriate change of variables, and we shall develop methods for finding such changes of variables when they exist. While there is no universal method for *explicitly* computing such a change of variables there is a powerful method for exploiting *symmetry* to find them that is applicable in many interesting situations.

Example 23. *We saw in the previous example that if* $\|\mathbf{x}_0\| \leq 1$, *the the solution of* $\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t))$ *with* $\mathbf{x}(t_0) = \mathbf{x}_0$ *for*

$$\mathbf{v}(x,y) = (y,-x) + [1 - x^2 - y^2](x,y)$$

is unique and is defined for all $t \in \mathbb{R}$, *while for* $\|\mathbf{x}_0\| > 1$, *the solution is defined for all* $t > t_0$, *but not for all* $t < t_0$.

Let us consider $\mathbf{x}$ *in the unit disk; i.e.,* $\|\mathbf{x}\| \leq 1$. *We have found in (3.11) that the solution starting from* $\mathbf{x}$ *at* $t = 0$ *is given by*

$$\mathbf{x}(t) = \frac{e^t}{\sqrt{1 - \|\mathbf{x}\|^2 + \|\mathbf{x}\|^2 e^{2t}}} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \mathbf{x} \ .$$

If we define the flow transformation $\Psi_t(\mathbf{x})$ associated to the vector field on the unit disk to be position $\mathbf{x}(t)$ reached at time t by the unique solution passing through $\mathbf{x}$ at time $t = 0$, we have, by the above calculations, that

$$\Psi_t(\mathbf{x}) = \frac{e^t}{\sqrt{1 - \|\mathbf{x}\|^2 + \|\mathbf{x}\|^2 e^{2t}}} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \mathbf{x} \ .$$

You can verify directly from the formula that for all $s, t \in \mathbb{R}$ and all $\mathbf{x}$ in the unit disk, $\Psi_t \circ \Psi_s(\mathbf{x}) = \Psi_{t+s}(\mathbf{x})$. However, the easy way to see this is true is to realize that it is a direct consequence of the uniqueness and global existence that we have proved.

3.1.3 Recursively coupled systems

It is not necessary to decouple a system to solve it completely using one dimensional methods. This can also be done for *recursively coupled systems*, of which we now give an example: Consider

$$\begin{aligned} x'(t) &= 2x(t) - 3y(t) \\ y'(t) &= 2y(t) \ . \end{aligned}$$

The rate of change of x depends on both x and y, but the rate of change of y depends on y alone. Thus, for any t_0 and y_0 we can solve $y'(t) = 2y(t)$ with $y(t_0) = y_0$ by our one dimensional methods: It is a linear first order equation for the single variable y. The unique solution is

$$y(t) = y_0 e^{2(t-t_0)} \ .$$

If we now insert this into the first equation in (3.12), we obtain

$$x'(t) = 2x(t) - 3y_0 e^{2(t-t_0)} \ .$$

This too is a linear first order equation n the single variable x, except now with a time-dependent coefficient. Rearranging terms and multiplying through by e^{-2t} we find

$$\left(e^{-2t}x(t)\right)' = -3y_0 e^{2t_0} \ .$$

Integrating from t_0 to t, and taking $x(t_0) = x_0$, we find

$$e^{-2t}x(t) - e^{-2t_0}x_0 = -3y_0 e^{2t_0}(t - t_0) \ .$$

Therefore,

$$x(t) = e^{-2(t-t_0)}x_0 - 3y_0 e^{-2(t-t_0)}(t - t_0) \ .$$

Definition 10 (Recursively coupled and uncoupled first order systems)**.** *The vector field*

$$\mathbf{v}(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_n(\mathbf{x}))$$

on $\mathbb{R}^n$ is recursively coupled *in case for each $i = 1, \dots, n$, f_i depends only on x_j for $j \geq n$. That is, if*

$$\frac{\partial f_i}{\partial x_j}(\mathbf{x}) = \mathbf{0} \qquad \text{for all} \quad \mathbf{j} < \mathbf{i} \ .$$

It is uncoupled *in case for each $i = 1, \dots, n$, f_i depends only on x_i.*

Note that every decoupled vector field is recursively coupled (a more accurate terminology would be *at most recursively coupled*). Thus, the notion of a recursively coupled vector field is a generalization of the notion of a an uncoupled vector field. However, we can always apply one dimensional methods to solve any first order system associated to a recursively coupled vector field.

This is because the system can be written as

$$\begin{array}{rcl} x_1'(t) & = & f_1(x_1, \ldots, x_n) \\ x_2'(t) & = & f_2(x_2, \ldots, x_n) \\ \vdots & \vdots & \vdots \\ x_{n-1}' & = & f_{n-1}(x_{n-1}, x_n) \\ x_n' & = & f_n(x_n) \end{array}$$

The last equation is a single variable first order equation. If it can be solved to find an explicit function $x_n(t)$, then this can be substituted into the penultimate equation to obtain

$$x_{n-1}' = f_{n-1}(x_{n-1}, x_n(t))$$

which is now a non-autonomous first order equation in the single variable x_{n-1}. If this can be solved explicitly, one can substitute both $x_{n-1}(t)$ and $x_n(t)$ into the next equation up the list, to obtain another first order non-autonomous equation in a single variable, and so one, working through all of the variables one by one.

Example 24. *Let $\mathbf{v}(x,y) = (-y, y)$. This is recursively coupled since writing $\mathbf{v}(x,y) = \big(f_1(x,y), f_2(x,y)\big)$, f_2 only depends on y.*

Thus, we may solve $y' = y$ with $y(t_0) = y_0$ to find $y(t) = y_0 e^{t-t_0}$. Substituting this into $x' = -y$ we find $x'' = y_0 e^{t-t_0}$, and so

$$x(t) = \int_{t_0}^{t} y_0 e^{s-t_0} \mathrm{d}s = y_0(e^{t-t_0} - 1) + x_0 \ .$$

Finally, we have

$$\mathbf{x}(t) = \big(y_0(e^{t-t_0} - 1) + x_0, y_0 e^{t-t_0}\big) \ .$$

In other words, the flow transformation $\Psi_t(\mathbf{x})$ associated to this vector field is

$$\Psi_t(x,y) = \big(y(e^t - 1) + x, ye^t\big) \ .$$

So far in this chapter, we have introduced two "nice" types of vector fields – uncoupled and recursively coupled – for which we can use one dimensional methods to solve the corresponding system. We have also seen that systems that are not given in these nice forms may be put into such a form by a judicious change of variables. But where do these changes of variables come from?

3.1.4 Flows of time independent vector fields

In the examples of this section we have computed the flow transformations of several time independent vector fields on $\mathbb{R}^2$. The next theorem gives the composition property of the flow transformation whenever the solutions of the corresponding first order equation are unique and defined for all t.

Theorem 7. *Let* $\mathbf{v}$ *be a vector field defined on an open set* $U \subset \mathbb{R}^n$*, Suppose that for each* $\mathbf{x} \in U$ *there is a unique solution* $\mathbf{x}(t)$ *to*

$$\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t)) \ , \mathbf{x}(0) = \mathbf{x}$$

which is defined for all $t \in \mathbb{R}$ *and such that* $\mathbf{x}(t) \in U$ *for all* t*. Then the flow transformation generated by* $\mathbf{v}$*, which is defined to be the function*

$$\Psi_t(\mathbf{x}) = \mathbf{x}(t) \ ,$$

satisfies the composition law

$$\Psi_{t_1+t_2}(\mathbf{x}) = \Psi_{t_1} \circ \Psi_{t_2}(\mathbf{x})$$

for all $\mathbf{x} \in U$ *and all* $s, t \in \mathbb{R}$*.*

Proof. Let $\mathbf{x}(t)$ be the solution curve passing through $\mathbf{x}$ for $t = 0$. For Define $\mathbf{y}(u)$ by

$$\mathbf{y}(t) = \mathbf{x}(t + t_1) \ .$$

Then

$$\mathbf{y}'(t) = \mathbf{x}'(t + t_1) = \mathbf{v}(\mathbf{x}(t + t_1)) = \mathbf{v}(\mathbf{y}(t)) \ ,$$

and clearly $\mathbf{y}(0) = \mathbf{x}(t_1) = \Psi_{t_1}(\mathbf{x})$.

By definition, since $\mathbf{y}(t)$ is the unique solution of our equation passing through $\mathbf{x}(t_1) = \Psi_{t_1}(\mathbf{x})$ at $t = 0$,

$$\Psi_{t_1+t_2}(\Psi_{t_1}(\mathbf{x})) = \mathbf{y}(t_2) = \mathbf{x}(t_2 + t_1) = \Psi_{t_2+t_1}(\mathbf{x}) \ .$$

□

To apply Theorem 7, we need to know when the solutions of $\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t))$ with $\mathbf{x}(0) = \mathbf{x}$ are unique. A useful answer, as in the one dimensional case, is given in terms of Lipschitz continuity.

Definition 11 (Lipschitz vector field)**.** *A vector field* $\mathbf{v}$ *defined on an open set* $U \subset \mathbb{R}^n$ *is* Lipschitz with Lipshictz constant L *in case* $L < \infty$ *and for all* $\mathbf{x}, \mathbf{y} \in U$*,*

$$\|\mathbf{v}(\mathbf{y}) - \mathbf{v}(\mathbf{x})\| \le L\|\mathbf{y} - \mathbf{x}\| \ .$$

Our next theorem is a multidimensional generalization of Theorem 6. It includes the case of time dependent vector fields, and thus goes beyond our multidimensional examples so far. However, the proof is essentially the same for time dependent and time independent vector fields, so we may as well cover the general case now. In fact, the proof is very similar to the proof of Theorem 6 concerning the one-dimensional case.

Theorem 8 (Uniqueness for Lipschitz vector fields on $\mathbb{R}^n$)**.** *Let* U *be an open set in* $\mathbb{R}^n$*, and let* (c, d) *be an open interval in* $\mathbb{R}$*. Let* $\mathbf{v}(t, x)$ *be a continuous time dependent vector field defined on* $(c, d) \times U$*. Suppose that for some* $L < \infty$*,* $\mathbf{v}$ *is Lipschitz with Lipschitz constant* L *uniformly in* $t \in (c, d)$*. That is,*

$$\|\mathbf{v}(t, \mathbf{y}) - \mathbf{v}(t, \mathbf{x})\| \le L\|\mathbf{y} - \mathbf{x}\|$$

for all $t \in (c,d)$ *and all* $\mathbf{x}, \mathbf{y} \in U$.

Let $t_0 \in (c,d)$. *Suppose that* $\mathbf{x}(t)$ *solves* $\mathbf{x}'(t) = \mathbf{v}(t, \mathbf{x}(t))$ *with* $\mathbf{x}(t_0) = \mathbf{x}$ *for all* $t \in (c,d)$ *and that* $\mathbf{y}(t)$ *solves* $\mathbf{y}'(t) = \mathbf{v}(t, \mathbf{y}(t))$ *with* $\mathbf{y}(t_0) = \mathbf{y}$ *for all* $t \in (c,d)$, *and that* $\mathbf{x}(t), \mathbf{y}(t) \in U$ *for all* $t \in (c,d)$. *Then for all* $t \in (c,d)$,

$$e^{-L|t-t_0|}\|\mathbf{y} - \mathbf{x}\| \le \|\mathbf{y}(t) - \mathbf{x}(t)\| \le e^{L|t-t_0|}\|\mathbf{y} - \mathbf{x}\| \ . \tag{3.12}$$

In particular, there is at most one solution passing through $\mathbf{x}$ *at* $t = t_0$.

Proof. Define $z(t) = \|\mathbf{y}(t) - \mathbf{x}(t)\|^2$. Then

$$z'(t) = 2(\mathbf{y}(t) - \mathbf{x}(t)) \cdot (\mathbf{y}'(t) - \mathbf{x}'(t)) = 2(\mathbf{y}(t) - \mathbf{x}(t)) \cdot (\mathbf{v}(t, \mathbf{y}(t)) - \mathbf{v}(t, \mathbf{x}(t)) \ .$$

Therefore, by the Cauchy-Schwarz inequality,

$$|z'(t)| \le 2\|\mathbf{y}(t) - \mathbf{x}(t)\|\|\mathbf{v}(t, \mathbf{y}(t)) - \mathbf{v}(t, \mathbf{x}(t))\| \le 2L\|\mathbf{y}(t) - \mathbf{x}(t)\|^2 = 2Lz(t) \ .$$

That is,

$$-2Lz(t) \le z'(t) \le 2Lz(t) \ . \tag{3.13}$$

The second inequality in (3.13) can be written as $z'(t) - 2Lz(t) \le 0$. Multiplying both sides by e^{-t2L}, we have

$$\left(z(t)e^{-t2L}\right)' \le 0 \ .$$

Hence $z(t)e^{-t2L}$ is a non-increasing function of t. It follows that:

$$z(t)e^{-t2L} \le z(t_0)e^{-t_0 2L} \quad \text{for } t > t_0 \quad \text{and} \quad z(t)e^{-t2L} \ge z(t_0)e^{-t_0 2L} \quad \text{for } t < t_0 \ .$$

Since $z(t_0) = \|\mathbf{y} - \mathbf{x}\|^2$, this is the same as

$$\|\mathbf{y}(t) - \mathbf{x}(t)\| \le \|\mathbf{y} - \mathbf{x}\|e^{(t-t_0)L} \quad \text{for } t > t_0 \quad \text{and} \quad \|\mathbf{y}(t) - \mathbf{x}(t)\| \ge \|\mathbf{y} - \mathbf{x}\|e^{(t-t_0)L} \quad \text{for } t < t_0 \ .$$

The first inequality in (3.13) can be written as $z'(t) + 2Lz(t) \ge 0$. Multiplying both sides by e^{t2L}, we have

$$\left(z(t)e^{t2L}\right)' \ge 0 \ .$$

Hence $z(t)e^{t2L}$ is a non-decreasing function of t. It follows that:

$$z(t)e^{t2L} \ge z(t_0)e^{t_0 2L} \quad \text{for } t > t_0 \quad \text{and} \quad z(t)e^{t2L} \le z(t_0)e^{t_0 2L} \quad \text{for } t < t_0 \ .$$

Since $z(t_0) = \|\mathbf{y} - \mathbf{x}\|^2$, this is the same as

$$\|\mathbf{y}(t) - \mathbf{x}(t)\| \ge \|\mathbf{y} - \mathbf{x}\|e^{-(t-t_0)L} \quad \text{for } t > t_0 \quad \text{and} \quad \|\mathbf{y}(t) - \mathbf{x}(t)\| \le \|\mathbf{y} - \mathbf{x}\|e^{-(t-t_0)L} \quad \text{for } t < t_0 \ .$$

The two lower bounds on $\|\mathbf{y}(t) - \mathbf{x}(t)\|$ can be summarized as $\|\mathbf{y}(t) - \mathbf{x}(t)\| \ge \|\mathbf{y} - \mathbf{x}\|e^{-|t-t_0|L}$ for all t and the two upper bounds on $\|\mathbf{y}(t) - \mathbf{x}(t)\|$ can be summarized as $\|\mathbf{y}(t) - \mathbf{x}(t)\| \le \|\mathbf{y} - \mathbf{x}\|e^{|t-t_0|L}$ for all t. In particular $\mathbf{y}(0) = \mathbf{x}(0)$ implies $\mathbf{y}(t) = \mathbf{x}(t)$ for all t. □

In the next section we discus an important class of vector fields to which this theorem applies.

3.2 Linear Vector Fields

3.2.1 Linear transformations and linear vector fields

There are a number of cases in which there is a strategy for finding such a change of variables. But there is only one case in which there is (essentially) an algorithm for this purpose: The case of linear vector fields.

Definition 12 (Linear Vector Field)**.** *A function* $\mathbf{v}$ *from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$ *is a (time independent)* linear vector field *in case* $\mathbf{v}(\mathbf{x})$ *is a linear function of* $\mathbf{x}$*. Since every linear function from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$ *has a matrix representation, this means that* $\mathbf{v}$ *is linear if and only if there is an* $n \times n$ *matrix* A *such that*

$$\mathbf{v}(\mathbf{x}) = A\mathbf{x}$$

for all $\mathbf{x}$*.*

Thus, there is a one-to-one correspondence between linear vector fields on $\mathbb{R}^n$ and $n \times n$ matrices. In case $\mathbf{v}$ is a linear vector field, we may write the first order system $\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t))$ as

$$\mathbf{x}'(t) = A\mathbf{x}(t) \ . \tag{3.14}$$

While this may look like a special case, and it is, it is also enormously important to the general theory.

Our next theorem says that every linear vector field on $\mathbb{R}^n$ is Lipschitz on all of $\mathbb{R}^n$. A valid Lipschitz constant, not necessarily the smallest, is given by a measure of the "size" of an $n \times n$ matrix A that we define next.

Definition 13 (Frobenius norm)**.** *Let* A *be an* $n \times n$ *matrix whose* i,j*th entry is* $A_{i,j}$*. The* Frobenius norm *of* A *is the quantity* $\|A\|_{\mathrm{F}}$ *defined by*

$$\|A\|_{\mathrm{F}} = \sqrt{\sum_{i,j=1}^{n} |A_{i,j}|^2} \ . \tag{3.15}$$

Lemma 2. *Let* A *be an* $n \times n$ *matrix. Ley* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$*. Then*

$$\|A\mathbf{y} - A\mathbf{x}\| \leq \|A\|_{\mathrm{F}} \|\mathbf{y} - \mathbf{x}\| \ . \tag{3.16}$$

Proof. By the linearity of matrix multiplication, $A\mathbf{y} - A\mathbf{x} = A(\mathbf{y} - \mathbf{x})$. Let $\mathbf{z} = \mathbf{y} - \mathbf{x}$. Then (3.16) is the same as $\|A\mathbf{z}\| \leq \|A\|_{\mathrm{F}}\|\mathbf{z}\|$. Let $\mathbf{r}_i$ be the ith row of A. Then $(A\mathbf{z})_i = \mathbf{r}_i \cdot \mathbf{z}$, and so by the Cauchy-Schwarz inequality,

$$|(A\mathbf{z})_i| \leq \|\mathbf{r}_i\| \|\mathbf{z}\| \ .$$

Therefore,

$$\|A\mathbf{z}\| = \sqrt{\sum_{i=1}^{n} |(A\mathbf{z})_i|^2} \leq \left(\sqrt{\sum_{i=1}^{n} \|\mathbf{r}_i\|^2} \right) \|\mathbf{z}\| = \|A\|_{\mathrm{F}} \|\mathbf{z}\| \ .$$

□

Therefore, every linear vector field $\mathbf{v}(\mathbf{x})$ is Lipschitz on all of $\mathbb{R}^n$ with Lipschitz constant at most $\|A\|_{\mathrm{F}}$, and we have the following Corollary of Theorem 8 and Lemma 3.16:

Corollary 1. *Let A be an $n \times n$ matrix. for any $\mathbf{x}_0 \in \mathbb{R}^n$, there is at most one solution to*

$$\mathbf{x}'(t) = A\mathbf{x}(t) \ , \qquad \mathbf{x}(0) = \mathbf{x}_0 \ .$$

Later in the chapter, we shall show that the solution always exists, and moreover, exists for all times $t \in \mathbb{R}$.

3.2.2 The role of eigenvalues and eigenvectors

The key to solving $\mathbf{x}'(t) = A\mathbf{x}(t)$ is to find all of the eigenvalues of A. Once this is done, the system can be solved explicitly, following an algorithm, as we now explain.

Recall that a complex number μ is an *eigenvalue of* A in case there is a non-zero vector $\mathbf{v} \in \mathbb{C}^n$ such that

$$A\mathbf{v} = \mu\mathbf{v} \ . \tag{3.17}$$

In this case, $\mathbf{v}$ is called an *eigenvector* of A.

The matrices A that we consider here will always have real entries, but still it is necessary to consider complex eigenvalues and complex eigenvectors. In any case, here is the significance of (3.17) to the problem of solving $\mathbf{x}'(t) = A\mathbf{x}(t)$. If $A\mathbf{v} = \mu\mathbf{v}$, and we define

$$\mathbf{x}(t) = e^{t\mu}\mathbf{v} \ , \tag{3.18}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}e^{t\mu}\right)\mathbf{v} = \mu e^{t\mu}\mathbf{v} = \mu\mathbf{x}(t) \tag{3.19}$$

and

$$A\mathbf{x}(t) = e^{t\mu}A\mathbf{v} = \mu e^{t\mu}\mathbf{v} = \mu\mathbf{x}(t) \tag{3.20}$$

Combining (3.19) and (3.20), we see that whenever $\mathbf{v}$ is an eigenvector of A with eigenvalue μ, the formula (3.18) defines a solution $\mathbf{x}(t)$ of the $\mathbf{x}'(t) = A\mathbf{x}(t)$.

Example 25. *Consider the vector field*

$$\mathbf{v}(x,y) = (-x + 2y, 3x - 2y) = \begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix} (x,y) \ .$$

Let us write

$$A = \begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix} ,$$

and compute the eigenvectors and eigenvalues of A. The characteristic polynomial is $\det(A - tI) = t^2 + 3t + -4 = (t+4)(t-1)$*. Hence the eigenvalues are $\mu_1 = -4$ and $\mu_2 = 1$.*

Now let us find the eigenvectors. We compute

$$A - \mu_1 I = \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} \ .$$

and so $(A - \mu_1 I)\mathbf{v} = 0$ *for* $\mathbf{v}_1 = (-2, 3)$*, and the eigenvectors of* A *with eigenvalue* $\mu_1 = -4$ *are exactly the non-zero multiples of* $\mathbf{v}_1$*. In the same way, we find*

$$A - \mu_2 I = \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix} ,$$

and so $(A - \mu_2 I)\mathbf{v} = 0$ *for* $\mathbf{v}_2 = (1, 1)$*, and the eigenvectors of* A *with eigenvalue* $\mu_2 = 4$ *are exactly the non-zero multiples of* $\mathbf{v}_2$*.*

We now obtain two solutions of the equation $\mathbf{x}'(t) = A\mathbf{x}(t)$*, namely*

$$\mathbf{x}_1(t) = e^{-4t}\mathbf{v}_1 = e^{-4t}(-2, 3) \qquad \text{and} \qquad \mathbf{x}_2(t) = e^{t}\mathbf{v}_2 = e^{t}(1, 1) \ .$$

That is $x_1(t) = -2e^{-4t}$ *and* $y_1(t) = 2e^{-4t}$ *is one set of solutions of our system, and* $x_2(t) = e^t$*,* $y_2(t) = e^t$ *is another.*

We have seen in the previous example that whenever we can find eigenvectors of A, we can find solutions of $\mathbf{x}'(t) = A\mathbf{x}(t)$. These, however, are rather special solutions that trace our straight half-lines emanating from the origin (when eigenvectors and eigenvalues are real). But can these special solutions provide us with the general solutions, i.e., the solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$, $\mathbf{x}(0) = \mathbf{x}_0$ for arbitrary $\mathbf{x}_0$?

Roughly speaking, the answer is yes. A bit more precisely, in some cases, we may need to bring in a simple generalization of the eigenvector concept and consider solutions corresponding to *generalized eigenvectors*, as we shall explain shortly. However, whenever we can find a set of n linearly independent eigevectors of an $n \times n$ matrix A, we can find an explicit formula for the solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$, $\mathbf{x}(0) = \mathbf{x}_0$ for arbitrary $\mathbf{x}_0$. There are two ways of doing this, and both are useful. We introduce them in the next two subsections.

3.2.3 The superposition principle

The main theorem of this subsection says that any linear combination (superposition) of solutions of $\mathbf{x}' = A\mathbf{x}$ is again a solution of the same equation.

Theorem 9 (Superposition principle)**.** *Let* A *be an* $n \times n$ *matrix, and suppose that* $\mathbf{x}_1(t)$ *and* $\mathbf{x}_2(t)$ *are two continuously differentiable curves in* $\mathbb{R}^n$ *such that*

$$\mathbf{x}_1'(t) = A\mathbf{x}_1(t) \qquad \text{and} \qquad \mathbf{x}_2'(t) = A\mathbf{x}_2(t)$$

for all t*. For any* $a_1, a_2 \in \mathbb{R}$*, define* $\mathbf{z}(t)$ *to be the linear combination*

$$\mathbf{z}(t) = a_1\mathbf{x}_1(t) + a_2\mathbf{x}_2(t) \ .$$

Then

$$\mathbf{z},(t) = A\mathbf{z}(t) \ .$$

Proof. We compute

$$\mathbf{z}'(t) = a_1\mathbf{x}_1'(t) + a_2\mathbf{x}_2'(t) = a_1A\mathbf{x}_1(t) + a_2A\mathbf{x}_2(t) = A(a_1\mathbf{x}_1(t) + a_2\mathbf{x}_2(t)) = A\mathbf{z}'(t)$$

since matrix multiplication is linear. □

To explain the main consequence of this theorem in the simplest possible terms, suppose that $n = 2$, and that the 2×2 matrix A has 2 distinct real eigenvalues μ_1 and μ_2. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be corresponding eigenvectors. Let $\mathbf{x}_0$ be any vector in $\mathbb{R}^n$. Suppose that we can find numbers a_1 and a_2 so that

$$\mathbf{x}_0 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 \ .$$

That is, suppose that we can express any $\mathbf{x}_0$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Now define

$$\mathbf{x}(t) = a_1 e^{\mu_1 t}\mathbf{v}_1 + a_2 e^{\mu_2 t}\mathbf{v}_2 \ . \tag{3.21}$$

We claim that $\mathbf{x}(t)$ solves

$$\mathbf{x}'(t) = A\mathbf{x}(t) \ , \mathbf{x}(0) = \mathbf{x}_0 \ . \tag{3.22}$$

To verify the claim, we first compute

$$\begin{aligned} \mathbf{x}'(t) &= (a_1 e^{\mu_1 t}\mathbf{v}_1 + a_2 e^{\mu_2 t}\mathbf{v}_2)' \\ &= a_1 e^{\mu_1 t}\mu_1\mathbf{v}_1 + a_2 e^{\mu_2 t}\mu_2\mathbf{v}_2 \\ &= a_1 e^{\mu_1 t}A\mathbf{v}_1 + a_2 e^{\mu_2 t}A\mathbf{v}_2 \\ &= A(a_1 e^{\mu_1 t}\mathbf{v}_1 + a_2 e^{\mu_2 t}\mathbf{v}_2) \\ &= A\mathbf{x}(t) \ . \end{aligned} \tag{3.23}$$

Next we compute

$$\mathbf{x}(0) = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = \mathbf{x}_0 \ .$$

This verifies that (3.21) defines a solution to (3.22), valid for all $t \in \mathbb{R}$, and then by Theorem 8, it is the unique such solution.

Example 26. *Consider the equation $\mathbf{x}'(t) = A\mathbf{x}(t)$ in $\mathbb{R}^2$ where A is the 2×2 matrix*

$$A = \begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix} ,$$

that we considered in Example 25. There we found that with

$$\mathbf{v}_1 = (-2, 3) \qquad \text{and} \qquad \mathbf{v}_2 = (1, 1) \ ,$$

$A\mathbf{v}_1 = -4\mathbf{v}_1$ *and* $A\mathbf{v}_2 = \mathbf{v}_2$.

To solve the initial value problem $\mathbf{x}(t) = A\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$, it suffices to find a_1 and a_2 such that

$$\mathbf{x}_0 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 \ . \tag{3.24}$$

If we introduce the vector $\mathbf{a} = (a_1, a_2)$ and the 2×2 matrix $V = [\mathbf{v}_1, \mathbf{v}_2]$ whose first column is $\mathbf{v}_1$ and whose second column is $\mathbf{v}_2$, (3.24) is the same as, by the rules of matrix multiplication,

$$\mathbf{x}_0 = V\mathbf{a} \ .$$

In this example, the matrix V is

$$V = \begin{bmatrix} -2 & 1 \\ 3 & 1 \end{bmatrix} .$$

This is invertible, with

$$V^{-1} = \frac{1}{5}\begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix} .$$

Therefore, with $\mathbf{x}_0 = (x_0, y_0)$, we have

$$(a_1, a_2) = \frac{1}{5}\begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix}(x_0, y_0) = \frac{1}{5}(-x_0 + y_0, 3x_0 + 2y_0) .$$

Therefore, the solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$ is

$$\begin{aligned} \mathbf{x}(t) &= a_1 e^{t\mu_1}\mathbf{v}_1 + a_2 e^{t\mu_2}\mathbf{v}_2 \\ &= \frac{1}{5}(-x_0 + y_0)e^{-4t}(-2, 3) + \frac{1}{5}(3x_0 + 2y_0)e^{t}(1, 1) \\ &= \frac{1}{5}(-2(-x_0 + y_0)e^{-4t} + (3x_0 + 2y_0)e^{t}, 3(-x_0 + y_0)e^{-4t} + (3x_0 + 2y_0)e^{t}) \\ &= \frac{1}{5}((2e^{-4t} + 3e^{t})x_0 + (2e^{t} - 2e^{-4t})y_0\ ,\ (3e^{t} - 3e^{-4t})x_0 + (3e^{-4t} + 2e^{t})y_0) . \end{aligned} \tag{3.25}$$

This can be expressed in a clear form using matrix notation as

$$\mathbf{x}(t) = \frac{1}{5}\begin{bmatrix} 2e^{-4t} + 3e^{t} & 2e^{t} - 2e^{-4t} \\ 3e^{t} - 3e^{-4t} & 3e^{-4t} + 2e^{t} \end{bmatrix}\mathbf{x}_0 .$$

Therefore, the flow transformation associated to this vector if;ed is

$$\Psi_t(\mathbf{x}) = \frac{1}{5}\begin{bmatrix} 2e^{-4t} + 3e^{t} & 2e^{t} - 2e^{-4t} \\ 3e^{t} - 3e^{-4t} & 3e^{-4t} + 2e^{t} \end{bmatrix}\mathbf{x} .$$

Form this, we can easily read off the solution to any specific initial value problem, say $\mathbf{x}'(t) = A\mathbf{x}(t)$ and $\mathbf{x}(0) = (2. - 1)$. The solution is

$$\mathbf{x}(t) = \Psi_t(2, -1) = \frac{1}{5}(4e^{t} + 6e^{-4t}, -9e^{-4t} + 4e^{t}) .$$

In the previous example we were able to completely solve a 2×2 linear system $\mathbf{x}' = A\mathbf{x}$ by finding eigenvectors of the matrix A. The method we illustrated in this example is broadly applicable, as we now explain.

Let A be an $n \times n$ matrix, and suppose that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a set of n linearly independent eigenvectors of A. Let $V = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ be the $n \times n$ matrix whose jth column is $\mathbf{v}_j$. By the rules of matrix multiplication, for $\mathbf{a} = (a_1, \ldots, a_n)$,

$$V\mathbf{a} = \sum_{j=1}^{n} a_j\mathbf{v}_j .$$

By the definition of linear independence, $V\mathbf{a} = \mathbf{0}$ if and only if $\mathbf{a} = \mathbf{0}$. Since this is true for any $\mathbf{a} \in \mathbb{R}^n$, if $A\mathbf{y} = A\mathbf{y}$, $A(\mathbf{y} - \mathbf{x}) = \mathbf{0}$ and so $\mathbf{y} - \mathbf{x} = \mathbf{0}$. That is, since the columns of V are linearly independent,

$$V\mathbf{x} = V\mathbf{y} \quad \Rightarrow \quad \mathbf{x} = \mathbf{y}$$

and hence the transformation $\mathbf{a} \mapsto V\mathbf{a}$ is one to one.

The Fundamental Theorem of Linear Algebra, which says that whenever a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ is either one to one or onto, it is both, and hence invertible. Therefore, since V is one to one, V is invertible. It follows that for each $\mathbf{x}_0 \in \mathbb{R}^n$, there is a unique $\mathbf{a} \in \mathbb{R}^n$ such that $V\mathbf{a} = \mathbf{x}_0$, and hence

$$\mathbf{x}_0 = \sum_{j=1}^n a_j \mathbf{v}_j \ .$$

Therefore, if we define

$$\mathbf{x}(t) = \sum_{j=1}^n a_j e^{t\mu_j} \mathbf{v}_j \ , \tag{3.26}$$

then $\mathbf{x}(0) = \mathbf{x}_0$ and

$$\mathbf{x}'(t) = \sum_{j=1}^n a_j e^{t\mu_j} \mu_j \mathbf{v}_j = \sum_{j=1}^n a_j e^{t\mu_j} A\mathbf{v}_j = A\left(\sum_{j=1}^n a_j e^{t\mu_j} \mathbf{v}_j\right) = A\mathbf{x}(t) \ .$$

Therefore, $\mathbf{x}(t)$ satisfies $\mathbf{x}'(t) = A\mathbf{x}(t)$ and $\mathbf{x}(0) = \mathbf{x}_0$, and by Theorem 8, it is the unique solution of these equations.

Here is a useful way to express this: Define the time-dependent $n \times n$ matrix $M(t)$ by

$$M(t) = [e^{t\mu_1}\mathbf{v}_1, \ldots, e^{t\mu_n}\mathbf{v}_n] \ . \tag{3.27}$$

Then by the rules of matrix multiplication. (3.26) is equivalent to

$$\mathbf{x}(t) = M(t)\mathbf{a} \qquad \text{where} \qquad \mathbf{a} = V^{-1}\mathbf{x}_0 \ ,$$

and hence

$$\mathbf{x}(t) = M(t)V^{-1}\mathbf{x}_0. \tag{3.28}$$

We summarize our conclusion in the following theorem:

Theorem 10. *Let A be an $n \times n$ matrix, and suppose that there exists a linearly independent set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of n eigenvectors of A. Let μ_j be the eigenvalue corresponding to $\mathbf{v}_j$. Define the matrix $M(t)$ by (3.27). Then the unique solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$ is*

$$\mathbf{x}(t) = M(t)V^{-1}\mathbf{x}_0 \ .$$

In particular, the flow transformation associated to the equation $\mathbf{x}'(t) = A\mathbf{x}(t)$ is

$$\Psi_t(\mathbf{x}) = M(t)V^{-1}\mathbf{x} \ . \tag{3.29}$$

3.2.4 Decoupling linear systems by a linear change of variables

There is another way to look at the result we have obtained in Theorem 10. Let A be an $n \times n$ matrix, and suppose that there exists a linearly independent set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of n eigenvectors of A. Then we may decouple the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ with a linear change of variables, as we now explain.

Theorem 11 (Linear changes of variables for linear vector fields)**.** *Let V be any invertible $n \times n$ matrix. Let A be any $n \times n$ matrix. Let $\mathbf{x}(t)$ be a continuously differentiable curve in $\mathbb{R}^n$, and define*

$$\mathbf{y}(t) = V^{-1}\mathbf{x}(t) \ . \tag{3.30}$$

Then $\mathbf{x}(t)$ satisfies

$$\mathbf{x}'(t) = A\mathbf{x}(t) \tag{3.31}$$

for all t if and only if

$$\mathbf{y}'(t) = (V^{-1}AV)\mathbf{y}(t) \tag{3.32}$$

for all t.

Proof. Suppose that $\mathbf{x}'(t) = A\mathbf{x}(t)$. Then we compute

$$\mathbf{y}'(t) = V{-}1\mathbf{x}'(t) = V{-}1A\mathbf{x}(t) = (V^{-1}AV)V\mathbf{x}(t) = (V^{-1}AV)\mathbf{y}(y) \ . \tag{3.33}$$

Conversely, suppose that $\mathbf{y}'(t) = (V^{-1}AV)\mathbf{y}(t)$. Then

$$\mathbf{x}'(y) = V\mathbf{y}'(t) = AV\mathbf{y}(t) = A\mathbf{x}(t) \ .$$

□

The point of this theorem is that $V^{-1}AV$ may be a much simpler matrix than A, at least if we make a judicious choice of V. For instance, suppose we can find an invertible $n \times n$ matrix V such that $(V^{-1}AV)$ is a diagonal matrix; i.e.,

$$(V^{-1}AV) = \begin{bmatrix} \mu_1 & 0 & 0 & \dots & 0 \\ 0 & \mu_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & \dots & \mu_{n-1} & 0 \\ 0 & 0 & 0 & \dots & \mu_n \end{bmatrix} . \tag{3.34}$$

Then (3.33) is equivalent the decoupled system

$$y_j'(t) = \mu_j y_j(t)$$

for $j = 1, \dots n$. For each j, this has the unique solution

$$y_j(t) = e^{t\mu_j} y_j(0) \ .$$

Then we have that the unique solution of $\mathbf{y}'(t) = (V^{-1}AV)\mathbf{y}(t)$, $\mathbf{y}(0) = \mathbf{y}_0$ is

$$\mathbf{y}(t) = (e^{t\mu_1}(\mathbf{y}_0)_1, \dots, e^{t\mu_n}(\mathbf{y}_0)_n) = \begin{bmatrix} e^{t\mu_1} & 0 & 0 & \dots & 0 \\ 0 & e^{t\mu_2} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & \dots & e^{t\mu_{n-1}} & 0 \\ 0 & 0 & 0 & \dots & e^{t\mu_n} \end{bmatrix} \mathbf{y}_0 \ . \tag{3.35}$$

By Theorem 11, it follows that

$$\mathbf{x}(t) = V\mathbf{y}(t)$$

is the unique solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$ with $\mathbf{x}(0) = V\mathbf{y}_0$. By Choosing $\mathbf{y}_0 = V\mathbf{x}_0$, this gives us the solution with $\mathbf{x}(0) = \mathbf{x}_0$.

That is, whenever we can find an invertible matrix V that "diagonalizes" A, meaning that (3.34) is satisfied, we see that the the unique solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$ is given by

$$\mathbf{x}(t) = V \begin{bmatrix} e^{t\mu_1} & 0 & 0 & \dots & 0 \\ 0 & e^{t\mu_2} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & \dots & e^{t\mu_{n-1}} & 0 \\ 0 & 0 & 0 & \dots & e^{t\mu_n} \end{bmatrix} V^{-1}\mathbf{x}_0 \ . \tag{3.36}$$

In other words, the flow transformation associated to the linear vector field $A\mathbf{x}$ is

$$\Psi_t = V \begin{bmatrix} e^{t\mu_1} & 0 & 0 & \dots & 0 \\ 0 & e^{t\mu_2} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & \dots & e^{t\mu_{n-1}} & 0 \\ 0 & 0 & 0 & \dots & e^{t\mu_n} \end{bmatrix} V^{-1} \ . \tag{3.37}$$

This is a nice formula, but its utility depends on being able to find a matrix V satisfying (3.34). The next theorem tells us when and how we can do this:

Theorem 12 (Diagonalizability)**.** *Let A and V be $n \times n$ matrices. Let $\mathbf{v}_j$ denote the jth column of V so that $V = [\mathbf{v}_1, \dots \mathbf{v}_n]$. Then (3.34) is satisfied if and only if for each j,*

$$A\mathbf{v}_j = \mu_j \mathbf{v}_j$$

and the set $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly independent.

Proof. Let D denote the diagonal matrix on the right side of (3.34). Suppose that for each j, $A\mathbf{v}_j = \mu_j \mathbf{v}_j$. Then by the rules of matrix multiplication

$$AV = A[\mathbf{v}_1, \dots \mathbf{v}_n] = [A\mathbf{v}_1, \dots A\mathbf{v}_n] = [\mu_1\mathbf{v}_1, \dots \mu_n\mathbf{v}_n] = [\mathbf{v}_1, \dots \mathbf{v}_n]D = VD \ .$$

Suppose further that the columns of V are linearly independent, Then V is invertible, and multiplying $AV = VD$ on the right by V^{-1}, we obtain $A = VDV^{-1}$.

On the other hand, suppose that $A = VDV^{-1}$. Since V is invertible, its columns are linearly independent, and then it remains to show that they are also eigenvectors of A. Multiplying on the right by V, we obtain $AV = VD$. As we have shown above, this means that

$$[A\mathbf{v}_1, \dots A\mathbf{v}_n] = [\mu_1\mathbf{v}_1, \dots \mu_n\mathbf{v}_n] \ ,$$

and so $A\mathbf{v}_j = \mu_j \mathbf{v}_j$ for each j. □

Therefore, we can diagonalize an $n \times n$ matrix A by a linear change of variables if and only if we can find a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of n linearly independent eigenvectors of A. This change of variables decouples the system $\mathbf{x}(t) = A\mathbf{x}(t)$, and we then have the explicit formula (3.36) for the general solution.

Example 27. *As in Example 26, consider the equation* $\mathbf{x}'(t) = A\mathbf{x}(t)$ *in* $\mathbb{R}^2$ *where* A *is the* 2×2 *matrix*

$$A = \begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix} .$$

We have seen that that with

$$\mathbf{v}_1 = (\,-2, 3) \qquad \text{and} \qquad \mathbf{v}_2 = (1, 1) \ ,$$

$A\mathbf{v}_1 = -4\mathbf{v}_1$ *and* $A\mathbf{v}_2 = \mathbf{v}_2$. *Clearly* $\{\mathbf{v}_1, \mathbf{v}_2\}$ *is linearly independent, so we can diagonalize* A. *Let*

$$V = \begin{bmatrix} -2 & 1 \\ 3 & 1 \end{bmatrix} .$$

This is invertible, with

$$V^{-1} = \frac{1}{5}\begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix} .$$

Finally, let

$$D = \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix} .$$

Then by Theorem 12,

$$A = VDV^{-1}$$

which can be checked by direct matrix multiplication.

The flow transformation is

$$\Psi_t = V \begin{bmatrix} e^{-4t} & 0 \\ 0 & e^t \end{bmatrix} V^{-1} .$$

Multiplying this out, we find

$$\Psi_t(\mathbf{x}) = \frac{1}{5}\begin{bmatrix} 2e^{-4t} + 3e^t & 2e^t - 2e^{-4t} \\ 3e^t - 3e^{-4t} & 3e^{-4t} + 2e^t \end{bmatrix} \mathbf{x} \ ,$$

as we found before.

Many linear systems can be solved in this way. However, not every such system can. There are $n \times n$ matrices for which there does not exist any set of n linearly independent eigenvectors in $\mathbb{R}^n$.

Example 28. *Let* A *be the* 2×2 *matrix*

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} .$$

Then

$$\det(A - tI) = t^2 + 1 \ .$$

The roots of the characteristic polynomial are i and $-i$.

Clearly, no vector $\mathbf{v}$ in $\mathbb{R}^2$ can satisfy $A\mathbf{v} = i\mathbf{v}$ since the left hand side is real, and the right hand side is imaginary. But we can find a complex vector $\mathbf{z} = (z_1, z_2)$ such that $A\mathbf{z} = i\mathbf{z}$, which is the same as

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} (z_1, z_2) = (0, 0) \ .$$

Making the same computations we would with real numbers, we see that

$$\mathbf{z} = (1, i)$$

is an eigenvector of A with eigenvalue i. Similarly, we find $A(1, -i) = -i(1, -i)$.

It may seem that complex eigenvectors are useless when we seek real solutions, but this is not the case: From every complex eigenvector we will get *two* real solutions as we shall explain below.

First, let us present another example of how things might get more complicated even if the eigenvalues are real.

Example 29. *Let A be the 2×2 matrix*

$$A = \begin{bmatrix} 7 & 9 \\ -1 & 1 \end{bmatrix} \ .$$

Then

$$\det(A - tI) = t^2 - 8t + 16 = (t - 4)^2 \ .$$

The only eigenvalue is $\mu = 4$, which is a double root of the characteristic polynomial.

$$A - 4I = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \ ,$$

and so the only eigenvectors are the non-zero multiples of $\mathbf{v} = (-3, 1)$. In particular, these does not exist a set of two linearly independent eigenvectors. This matrix cannot be diagaonlized. In other words, the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ cannot be decoupled by a linear change of variables.

However, we can do something almost as good. Let $V = [\mathbf{v}, \mathbf{v}^\perp]$ so that

$$V = \begin{bmatrix} -3 & -1 \\ 1 & -3 \end{bmatrix} \ .$$

Since, by construction $V\mathbf{e}_1 - \mathbf{v}$, $V^{-1}\mathbf{v} = \mathbf{e}_1$. Then since

$$AV = A[\mathbf{v}, \mathbf{v}^\perp] = [A\mathbf{v}, A\mathbf{v}^\perp] = [4\mathbf{v}, A\mathbf{v}^\perp] \ ,$$

$$V^{-1}AV = [4V^{-1}\mathbf{v}, V^{-1}A\mathbf{v}^\perp] = [4\mathbf{e}_1, V^{-1}A\mathbf{v}^\perp] \ .$$

Since the first column of $V^{-1}AV$ is a multiple of $\mathbf{e}_1$, this matrix is upper triangular*. That is, it has the from.*

$$V^{-1}AV = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

for some numbers a, b and c, and note that we arrived at this conclusion using only the information that the first column of V was an eigenvector of A and that V was invertible, which is guaranteed by taking the second column to be $\mathbf{c}^\perp$*, because* $\{\mathbf{v}, \mathbf{v}^\perp\}$ *is linearly independent for any non-zero* $\mathbf{v}$*.*

Now carrying out the actual computations we find

$$V^{-1}AV = \begin{bmatrix} 4 & 10 \\ 0 & 4 \end{bmatrix}$$

The system

$$\mathbf{y}'(t) = (V^{-1}AV)\mathbf{y}(t)$$

is recursively coupled, and so it may be solver recursively. Writing $\mathbf{y}(t) = (u(t), v(t))$*, we have* $v'(t) = 4v(t)$*, so that* $v(t) = e^{4t}v_0$*. Then we have*

$$u'(t) = 4u(t) + 10v(t) = 4u(t) + e^{4t}10v_0 \ .$$

This is the same as

$$(u(t)e^{-4t})' = 10v_0 \ ,$$

and so $u(t) = e^{4t}(10v_0t + u_0)$*. Altogether, we have*

$$\mathbf{y}(y) = e^{4t}(10v_0t + u_0, v_0) = e^{4t}\begin{bmatrix} 1 & 10t \\ 0 & 1 \end{bmatrix}\mathbf{y}_0 \ .$$

Finally, putting $\mathbf{y}_0 = \mathbf{x}_0$ *and and* $\mathbf{x}(t) = V^{-1}\mathbf{y}(t)$*, we have that the solution to* $\mathbf{x}'(t) = A\mathbf{x}(t)$ *with* $\mathbf{x}(0) = \mathbf{x}_0$ *is*

$$\mathbf{x}(t) = V^{-1}e^{4t}\begin{bmatrix} 1 & 10t \\ 0 & 1 \end{bmatrix}V\mathbf{x}_0 \ .$$

Do the matrix multiplications explicitly, we find

$$\mathbf{x}(t) = e^{4t}\begin{bmatrix} (1+3t) & 9t \\ -t & (1-3t) \end{bmatrix}\mathbf{x}_0 \ .$$

The corresponding flow transformation is

$$\Psi_t(\mathbf{x}) = e^{4t}\begin{bmatrix} (1+3t) & 9t \\ -t & (1-3t) \end{bmatrix}\mathbf{x} \ .$$

What we have seen in the last example turns out to a particular case of a general fact: For any $n \times n$ matrix A, there is *always* a linear change of variables matrix V such that $V^{-1}AV$ is diagonal. However, in general, the matrix will be complex. As we shall see, this provides a way to solve any linear first order system – provided one can compute all of the eigenvalues of A. One can give an existence theorem for the the solutions of $\mathbf{x}'(t) = A\mathbf{x}(t)$ by developing these ideas, but there is another approach that has a number of advantages, as we explain next.

3.3 Matrix exponentials and global existence

3.3.1 The definition of the matrix exponential

Let A be an $n \times n$ matrix. In this section we define and study the matrix exponential function. Recall that for complex variables w, the exponential function e^w is defined by its power series

$$e^w = \sum_{k=1}^{\infty} \frac{1}{k!} w^k ,$$

or equivalently, by the limit

$$e^w = \lim_{n \to \infty} \left(1 + \frac{1}{n} w\right)^n .$$

Both formulas can be applied if we replace w by an $n \times n$ matrix, and the constant 1 (which is the first term in the power series) by the $n \times n$ identity matrix. To see that the series converges, we need some simple facts about the Forbenius norm that we have introduced above.

Lemma 3. *Let A and B be any $n \times n$ matrices. Then*

$$\|A + B\|_{\mathrm{F}} \le \|A\|_{\mathrm{F}} + \|B\|_{\mathrm{F}} \tag{3.38}$$

and

$$\|AB\|_{\mathrm{F}} \le \|A\|_{\mathrm{F}} \|B\|_{\mathrm{F}} . \tag{3.39}$$

Proof. Associate to and $n \times n$ matrix A the vector $\mathbf{v}_A$ in $\mathbb{R}^{n^2}$ obtained by writing out the rows of A one after the other, as a vector with n^2 entries. That is, if A has the rows $\mathbf{r}_1, \ldots, \mathbf{r}_n$,

$$\mathbf{v}_A = (\mathbf{r}_1, \ldots, \mathbf{r}_n) .$$

It is easy to check that $\|A\|_{\mathrm{F}} = \|\mathbf{v}_A\|$ where the quantity on the right is the length of $\mathbf{v}_A$ in $\mathbb{R}^{n^2}$. It is also easy to check that $\mathbf{v}_{A+B} = \mathbf{v}_A + \mathbf{v}_B$. Therefore,

$$\|A + B\|_{\mathrm{F}} = \|\mathbf{v}_A + \mathbf{v}_B\| \le \|\mathbf{v}_A\| + \|\mathbf{v}_B\| = \|A\|_{\mathrm{F}} + \|B\|_{\mathrm{F}}$$

where the inequality in the middle is the triangle inequality.

Next, for any i, j, by the Cauchy-Scwarz inequality,

$$|AB_{i,j}| = \left|\sum_{k=1}^{n} A_{i,k} B_{k,j}\right| \le \left(\sum_{k=1}^{n} (A_{i,k})^2\right)^{1/2} \left(\sum_{k=1}^{n} (B_{k,j})^2\right)^{1/2} .$$

Squaring both sides and summing over i and j we get

$$\|AB\|_{\mathrm{F}}^2 \le \sum_{i,j} \left(\sum_{k=1}^{n} (A_{i,k})^2\right) \left(\sum_{k=1}^{n} (B_{k,j})^2\right) = \left(\sum_{i,k=1}^{n} (A_{i,k})^2\right) \left(\sum_{j,k=1}^{n} (B_{k,j})^2\right) = \|A\|_{\mathrm{F}}^2 \|B\|_{\mathrm{F}}^2 .$$

□

A simple induction starting from (3.39) shows that for all integers $k \geq 1$,

$$\|A^k\|_{\rm F} \leq \|A\|_{\rm F}^k \ .$$

This is obviously true for $k = 0$ since we define $A^0 = I$.

Then since for any i, j, $|A_{i,j}| \leq \|A\|_{\rm F}$, we have that for any $t \in \mathbb{R}$,

$$\left|\frac{t^k}{k!}A^k_{i,j}\right| \leq \frac{|t|^k}{k!}\|A^k\|_{\rm F} \leq \frac{|t|^k}{k!}\|A\|_{\rm F}^k \tag{3.40}$$

Since

$$\sum_{k=0}^{\infty}\frac{|t|^k}{k!}\|A\|_{\rm F}^k < \infty$$

for all $t \in \mathbb{R}$, for each i, j, the *numerical* power series

$$\sum_{k=1}^{\infty}\frac{t^k}{k!}A^k_{i,j} \ ,$$

is absolutely convergent. It then follows that this power series has an infinite radius of convergence, and thus it defines a function of t that is continuously differentiable, and the derivative may be computed by differentiating term by term, obtaining a new power series that also has an infinite radius of convergence.

Note that we are using the theory of *numerical* power series here. For each fixed i and j, the coefficients in our power series are generated by matrix multiplication. But all we need to know at this point is that we have the upper bound (3.40) that guarantees convergence.

Therefore the power series $\sum_{k=1}^{\infty}\frac{t^k}{k!}A^k$ converges absolutely, entry by entry.

Definition 14 (Matrix exponential)**.** *Let A be an $n \times n$ matrix. The* matrix exponential *of A is the matrix*

$$e^A = \sum_{k=1}^{\infty}\frac{1}{k!}A^k \ ,$$

where $A^0 = I$, by definition.

By what we have explained above, this series converges absolutely entry by entry. Moreover, replacing A by tA, the entries of e^{tA} are all given by power series with infinite radius of convergence. Hence each entry may be differentiated, term by term in the power series. Since we differentiate t-dependent matrices term by term anyhow, this gives us

$$\frac{\rm d}{{\rm d}t}e^{tA} = \sum_{k=1}^{\infty}\frac{\rm d}{{\rm d}t}\frac{t^k}{k!}A^k = \sum_{k=1}^{\infty}\frac{t^{k-1}}{(k-1)!}A^k = A\left(\sum_{k=1}^{\infty}\frac{t^{k-1}}{(k-1)!}A^{k-1}\right) = Ae^{tA} = e^{tA}A \ .$$

Therefore, for any $\mathbf{x}_0 \in \mathbb{R}^n$, if we define $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$, we have

$$\mathbf{x}'(t) = \frac{\rm d}{{\rm d}t}\left(e^{tA}\mathbf{x}_0\right) = \left(\frac{\rm d}{{\rm d}t}e^{tA}\right)\mathbf{x}_0 = Ae^{tA}\mathbf{x}_0 = A\mathbf{x}(t) \ .$$

Moreover, it is clear that $e^{0A} = I$, and so $\mathbf{x}(t_0) = \mathbf{x}_0$.

This proves that $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$ is a solution of the equation $\mathbf{x}'(t) = A\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{c}_0$, and it is well-defined for all t. This proves that a global solution of this equation exists for every A and every x_0. Moreover, since for any A, $A\mathbf{x}$ is a Lipschitz vector field on all of $\mathbb{R}^n$, we know that the solutions are unique. Thus, $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$ is the unique solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{c}_0$, and so the flow transformation Ψ_t of the vector field $A\mathbf{x}$ is given by $\Psi_t(\mathbf{x}) = e^{tA}\mathbf{x}$.

We summarize our conclusions in a theorem:

Theorem 13. *For every $n \times n$ vector matrix A, and every $\mathbf{x}_0 \in \mathbb{R}^n$, there is a unique solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{c}_0$. This solutions is defined for all $t \in \mathbb{R}$, and is given by*

$$\mathbf{x}(t) = e^{tA}\mathbf{x}_0 \ .$$

It follows that the flow transformation Ψ_t associated to the linear vector field $A\mathbf{x}$ is $\Psi_t(\mathbf{x}) = e^{tA}\mathbf{x}$, and is therefore the linear transformation given by the matrix e^{tA}.

From the composition property of flow transformations, $\Psi_{t+s} = \Psi_t \circ \Psi_s$, proved in Theorem 7, it follows that

$$e^{(t+s)A} = e^{tA}e^{sA} \ .$$

In particular, for all t, e^{tA} is invertible and $(e^{tA})^{-1} = e^{-tA}$.

Example 30. *In some cases, one can easily work out a closed from expression for A^k for all k, and use this to explicitly compute e^{tA} directly from the definition. Here is one such case: Let*

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \ .$$

Then one easily computes

$$A^2 = -I \ , \qquad A^3 = -A \quad \text{and} \quad A^4 = I \ .$$

The $A^5 = A$, and the cycle starts again. Therefore, we have for all non-negative integers j,

$$A^{2j} = \begin{cases} I & j \text{ is even} \\ -I & j \text{ is odd} \end{cases}$$

and

$$A^{2j+1} = \begin{cases} A & j \text{ is even} \\ -A & j \text{ is odd} \end{cases}$$

Therefore, splitting the sum over k into the even and odd contributions,

$$\begin{aligned} e^{tA} &= \left(\sum_{j=0}^{\infty} \frac{t^{2j}}{(2j)!}(-1)j\right) I + \left(\sum_{j=0}^{\infty} \frac{t^{2j+1}}{(2j+1)!}(-1)\right) A \\ &= \cos tI + \sin tA \\ &= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \ . \end{aligned} \tag{3.41}$$

In Example 30 we were able to compute e^{tA} explicitly in closed from because we were able to compute a simple expression for all of the powers of A. As we now explain, we can do this whenever we can find all of the eigenvalues of A. Once we know these, the power series for e^{tA} can be computed exactly, in closed form.

The simplest case is that in which A can be diagonalized. That is, suppose that there is an invertible matrix V so that $V^{-1}AV = D$, or, what is the same, $A = VDV^{-1}$. Then as in Example 30, we can find a simple closed form expression for all of the powers of A.

It is easy to see by induction that if D is the diagonal matrix with μ_j being its jth diagonal entry, then for every positive integer k, D^k the the diagonal matrix whose jth diagonal entry is μ_k^j. That is,

$$\begin{bmatrix} \mu_1 & 0 & 0 & \dots & 0 \\ 0 & \mu_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & \dots & \mu_{n-1} & 0 \\ 0 & 0 & 0 & \dots & \mu_n \end{bmatrix}^k = \begin{bmatrix} \mu_1^k & 0 & 0 & \dots & 0 \\ 0 & \mu_2^k & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & \dots & \mu_{n-1}^k & 0 \\ 0 & 0 & 0 & \dots & \mu_n^k \end{bmatrix} .$$

The fact the we can compute arbitrary powers of D directly means we can do the same for A. Note that since $A = VDV^{-1}$,

$$A^2 = VAV^{-1}VDV^{-1} = VD^2V^{-1} .$$

Then

$$A^3 = VAV^{-1}VD^2V^{-1} = VD^3V^{-1} .$$

A simple induction shows that for all positive integers k,

$$A^k = VD^kV^{-1} . \tag{3.42}$$

Therefore, for any positive integer n,

$$\sum_{k=0}^{n} \frac{t^k}{k!} A^k = V \left(\sum_{k=0}^{n} \frac{t^k}{k!} D^k \right) V^{-1} .$$

However, by what we have noted above,

$$\sum_{k=0}^{n} \frac{t^k}{k!} D^k = \begin{bmatrix} \sum_{k=0}^{n} \frac{t^k}{k!}\mu_1^k & 0 & 0 & \dots & 0 \\ 0 & \sum_{k=0}^{n} \frac{t^k}{k!}\mu_2^k & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & \dots & \sum_{k=0}^{n} \frac{t^k}{k!}\mu_{n-1}^k & 0 \\ 0 & 0 & 0 & \dots & \sum_{k=0}^{n} \frac{t^k}{k!}\mu_n^k \end{bmatrix} ,$$

Since for each j, $\lim_{n\to\infty} \sum_{k=0}^{n} \frac{t^k}{k!}\mu_j^k = e^{t\mu_j}$, it follows that

$$e^{tA} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{t^k}{k!} A^k = V \begin{bmatrix} e^{t\mu_1} & 0 & 0 & \dots & 0 \\ 0 & e^{t\mu_2} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & \dots & e^{t\mu_{n-1}} & 0 \\ 0 & 0 & 0 & \dots & e^{t\mu_n} \end{bmatrix} V^{-1} .$$

3.3.2 Properties of the matrix exponential

To compute matrix exponentials in closed form for matrices that cannot be diagonalized, we need some other properties of the matrix exponential.

Theorem 14. *Let A and B be $n \times n$ matrices, and suppose that*

$$AB = BA \ .$$

Then

$$Ae^B = e^B A \tag{3.43}$$

and

$$e^{A+B} = e^A e^B \ . \tag{3.44}$$

Proof. Since AB = BA, $AB^2 = BAB = B^2A$, and so $AB^2 = B^2A$. Next, $AB^3 = BAB^2 = B^3A$, and se have $AB^3 = B^3A$. A simple induction shows that $AB^k = B^kA$ for all positive integers k. But then for all positive integers n,

$$A\left(\sum_{k=1}^{n} \frac{t^k}{k!}B^k\right) = \sum_{k=1}^{n} \frac{t^k}{k!}AB^k = \sum_{k=1}^{n} \frac{t^k}{k!}B^kA = \left(\sum_{k=1}^{n} \frac{t^k}{k!}B^k\right)A \ .$$

Taking the limit $n \to \infty$ gives us (3.43).

To prove (3.44) we use the uniqueness theorem for linear equations as follows: Define

$$\mathbf{x}(t) = e^{-tA}e^{t(A+B})\mathbf{x}_0 \ .$$

Then

$$\mathbf{x}'(t) = e^{-tA}(-A)e^{t(A+B}\mathbf{x}_0 + e^{-tA}(A+B)e^{t(A+B}\mathbf{x}_0 = e^{-tA}Be^{t(A+B}\mathbf{x}_0 \ .$$

Then by the fact that $AB = BA$ and (3.43),

$$e^{-tA}Be^{t(A+B}\mathbf{x}_0 = Be^{-tA}e^{t(A+B}\mathbf{x}_0 = B\mathbf{x}_0 \ .$$

Combining results, we see that

$$\mathbf{x}'(t) = B\mathbf{x}(t) \qquad \text{and} \qquad \mathbf{x}(0) = \mathbf{x}_0 \ .$$

But we know the unique solution of this equation is $e^{tB}\mathbf{x}_0$. Therefore

$$e^{-tA}e^{t(A+B}\mathbf{x}_0 = etB\mathbf{x}_0$$

for all t and all $\mathbf{x}_0$. Taking $t = 1$, and multiplying on the left by e^A, we obtain (3.44). □

Applying the theorem to tA and tB, we see that whenever $AB = BA$, we have $e^{t(A+B)} = e^{tA}e^{tB}$, and this is what we actually proved. An important case is that in which B is a multiple of the identity; i.e., $B = \mu I$ for some
mu. For any μ, and any $n \times n$ matrix A we can write

$$A = (A - \mu I) + \mu I \ .$$

Since $(A-\mu I)(\mu I) = (\mu I)(A-\mu I)$, we have

$$e^{tA} = e^{t(A-\mu I)+t\mu I} = e^{t(A-\mu I)}e^{t\mu I} \ .$$

Since for any vector $\mathbf{v}$, $e^{t\mu I}\mathbf{v} = e^{t\mu}\mathbf{v}$, this is the same as

$$e^{tA} = e^{t\mu}e^{t(A-\mu I)} \ . \tag{3.45}$$

3.3.3 Generalized eigenvectors

Recall that a non-zero vector $\mathbf{v}$ is an eigenvector of A with eigenvalue μ if and only if $(A-\mu I)\mathbf{v} = 0$, In this case, we have that

$$e^{tA}\mathbf{v} = e^{t\mu}e^{t(A-\mu I)}\mathbf{v} = e^{t\mu}\sum_{k=0}^{\infty}(A-\mu I)^k\mathbf{v} = e^{t\mu}\mathbf{v}$$

since $(A-\mu I)^k\mathbf{v} = 0$ for all $k \geq 1$. While we already knew this, the computation we have just made leads to the following important definition.

Definition 15 (Generalized eigenvalues)**.** *A non-zero vector* $\mathbf{v}$ *is a* generalized eigenvector *of* A *with eigenvalue* μ *in case for some positive integer* k*,*

$$(A-\mu I)^k\mathbf{v} = 0 \ .$$

Notice that when $(A-\mu I)^k\mathbf{v} = 0$ for some positive integer k, there is a least such integer. Let j be the least such integer. Then $\mathbf{w} = (A-\mu I)^{j-1} \neq \mathbf{0}$, but $(A-\mu I)\mathbf{w} = 0$. Therefore, $\mathbf{w}$ is an eigenvector of A with eigenvalue μ. In particular, μ is an ordinary eigenvalue. There are no "generalized eigenvalues", only generalized eigenvectors.

The point of the definition is that we get an explicit closed form solution of the equation $\mathbf{x}'(t) = A\mathbf{x}(t)$ for every generalized eigenvector. To see this suppose that

$$(A-\mu I)^n\mathbf{v} = \mathbf{0} \qquad \text{but} \qquad (A-\mu I)^{n-1}\mathbf{v} = \mathbf{0} \ .$$

Clearly for all $\ell > 0$,

$$(A-\mu I)^{n+\ell}\mathbf{v} = (A-\mu I)^{\ell}(A-\mu I)^n\mathbf{v} = 0 \ ,$$

so

$$e^{tA}\mathbf{v} = e^{t\mu}e^{t(A-\mu I)}\mathbf{v} = e^{t\mu}\left(\sum_{k=0}^{n-1}\frac{t^k}{k!}(A-\mu I)^k\mathbf{v}\right) \ . \tag{3.46}$$

The sum on the right is now a finite sun, and so it can be computed in closed form.

In some cases in which there is only one eigenvalue μ, it even turns out that *every* vector $\mathbf{v}$ is a generalized eigenvector with $(A-\mu I)^n\mathbf{v} = \mathbf{0}$ for some n. In this case (3.46) is true for every $\mathbf{v}$, which means that

$$e^{tA} = e^{t\mu}\left(\sum_{k=0}^{n-1}\frac{t^k}{k!}(A-\mu I)^k\right) \ .$$

Example 31. *Consider again the matrix* $A = \begin{bmatrix} 7 & 9 \\ -1 & 1 \end{bmatrix}$ *that we introduced in Example 29. This has only one eigenvalue,* $\mu = 4$*, and the only eigenvectors are non-zero multiples of* $(-3, 1)$*. Hence this matrix cannot be diagonalized; we have a shortage of eigenvectors. But we have no shortage of generalized eigenvectors. In fact,*

$$A - 4I = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} ,$$

and so

$$(A - 4I)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} .$$

Thus every *non-zero vector in* $\mathbb{R}^2$ *is a generalized eigenvector with eigenvalue* 4*. Moreover,*

$$e^{tA} = e^{4t}\left(\sum_{k=0}^{1} \frac{t^k}{k!}(A - \mu I)^k\right) = e^{4t}\left(I + t\begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix}\right) = e^{4t}\begin{bmatrix} (1+3t) & 9t \\ -t & (1-3t) \end{bmatrix} ,$$

which agree with what we found in Example 29.

The following theorem is an important result of linear algebra. Using it, we shall show that one can always explicitly compute e^{tA} once one knows all of the eigenvalues of A.

Theorem 15 (Generalized eigenvector basis)**.** *Let* A *be an* $n \times n$ *matrix. Suppose that* A *has* m *distinct eigenvalues* $\mu_1, \ldots, \mu_m$ *and that the characteristic polynomial of* A *factors as*

$$\prod_{j=1}^{m} (\mu_j - t)^{d_j} .$$

so that necessarily $\sum_{j=1}^{m} d_j = n$*.*

Then for each j*, whenever* $\mathbf{v}$ *is a generalized eigenvector of* A *with eigenvalue* μ_j*,* $(A - \mu_j)^{d_j}\mathbf{v} = \mathbf{0}$*, and there exist a set of* d_j *solutions of this equation that is linearly independent. Taking the union of any* m *such sets, one for each* μ_j*, gives us a set of* n *linearly independent vectors, each of which is a generalized eigenvector.*

Here is how we may apply this. Let A be any $n \times n$ matrix, and let $V = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ be an invertible matrix each of whose columns is a generalized eigenvector of A. For each $j = 1, \ldots, n$, we know how to find a solution $\mathbf{x}_j(t)$ to $\mathbf{x}'(t) = A\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{v}_j$:

$$\mathbf{x}_j(t) = e^{t\mu_j} \sum_{k=0}^{d_j - 1} \frac{t^k}{k!}(A - \mu_j I)^k \mathbf{v}_j .$$

Then for any $\mathbf{a} = (a_1, \ldots, a_n)$,

$$\mathbf{x}(t) = \sum_{j=1}^{n} a_j \mathbf{x}_j(t)$$

is a solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$ with $\mathbf{x}(0) = \sum_{j=1}^{n} a_j \mathbf{v}_j = V\mathbf{a}$. Therefore, if we choose $\mathbf{a} = V^{-1}\mathbf{x}_0$, $\mathbf{x}(t)$ solves $\mathbf{x}'(t) = A\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$. But this means that $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$. Therefore, for all $\mathbf{x}_0$,

$$e^{tA}\mathbf{x}_0 = \sum_{j=1}^{n} a_j \mathbf{x}_j(t) = [\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)]V^{-1}\mathbf{x}_0 .$$

It follows that

$$e^{tA} = [\mathbf{x}_1(t), \dots, \mathbf{x}_n(t)]V^{-1} \ .$$

Of course, to carry this out, one has to factor the characteristic polynomial of A. The Fundamental Theorem of Algebra says that this can always be done, except that even if the coefficients of the polynomial are real, as they will be when A is a real $n \times n$ matrix, the roots may be complex.

However, since the identity $e^{t(A+G)} = e^{tA}e^{tB}$ is an identity between power series with an infinite radius of convergence, it holds also in the complex case. Hence the identity $e^{tA} = e^{t(A-\mu I)}e^{t\mu I}$ is valid even when μ is complex. In the next section we investgate the complex case.

3.4 Complex solutions of linear first order systems

3.4.1 Complex vectors and complex matrices

Let $\mathbb{C}^n$ denote the set of all n dimensional vectors with complex entries. That is, $\mathbf{z} \in \mathbb{C}^n$ is an ordered n-tuple $(z_1, \dots, z_n)$ of complex numbers. For all $w \in \mathbb{C}$ and all $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$, we define

$$w\mathbf{z} = (wz_1, \dots, wz_n) \ ,$$

which is exactly like scalar multiplication in the real case, except now the numbers are complex.

Likewise, for $\mathbf{z} = (z_1, \dots, z_n)$ and $\mathbf{w} = (w_1, \dots, w_n)$ we define the vector sum $\mathbf{z} + \mathbf{w}$ by

$$\mathbf{z} + \mathbf{w} = (z_1 + w_1, \dots, z_n + w_n) \ .$$

This is exactly as in the real case, except now the numbers are complex. For any complex number $z = x + iy$ with x and y real, we call x the *real part* of z, and denote it by $\Re(z)$. We call y the *imaginary part* of z, and denote it by $\Im(z)$. The *complex conjugate* of $z = x + iy$ is the complex number $\overline{z}$ given by $\overline{z} = x - iy$. Notice that

$$\Re(z) = \frac{1}{2}(z + \overline{z}) \qquad \text{and} \qquad \Im(z) = \frac{1}{2i}(z - \overline{z}) \ .$$

Consider any $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$. For each $j = 1, \dots, n$, let $x_j = \Re(z_j)$ and let $y_j = \Im(z_j)$. Then if we define $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ by $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$, we have that

$$\mathbf{z} = \mathbf{x} + i\mathbf{y} \ .$$

We refer to $\mathbf{x}$ as the *real part of* $\mathbf{z}$ and to $\mathbf{y}$ as *the imaginary part* of $\mathbf{z}$, and we write $\mathbf{x} = \Re(\mathbf{z})$ and $\mathbf{y} = \Im(\mathbf{z})$.

Given a vector $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$, we define its *complex conjugate* to be the vector $\overline{\mathbf{z}}$ given by

$$\overline{\mathbf{z}} = (\overline{z}_1, \dots, \overline{z}_n) \ .$$

It is easy to see that if $\mathbf{x} = \Re(\mathbf{z})$ and $\mathbf{y} = \Im(\mathbf{z})$, then

$$\overline{\mathbf{z}} = \mathbf{x} - i\mathbf{y} \ .$$

In particular,

$$\Re(\mathbf{z}) = \frac{1}{2}(\mathbf{z} + \overline{\mathbf{z}}) \qquad \text{and} \qquad \Im(\mathbf{z}) = \frac{1}{2i}(\mathbf{z} - \overline{\mathbf{z}}) \ .$$

Let Z be a complex $n \times n$ matrix; i.e., an $n \times n$ matrix with complex entries. Then each column of Z is a vector $\mathbf{c}_j$ in $\mathbb{C}^n$, and we denote this by writing $Z = [\mathbf{c}_1, \ldots, \mathbf{c}_n]$ exactly as in the real case. We define the matrix-vector product of Z with $\mathbf{z} \in \mathbb{C}^n$ by

$$Z\mathbf{z} = \sum_{j=1}^{n} z_j \mathbf{c}_j$$

where $\mathbf{c}_j$ is the jth column of $\mathbf{z}$ and where z_j is the jth entry of $\mathbf{z}$. Notice that when all of the vectors happen to be real, i.e., their imaginary parts are zero, this reduces to the definition for real matrices and vectors.

Likewise, if Z and W are two $n \times n$ complex matrices with $W = [\mathbf{w}_1, \ldots, \mathbf{w}_n]$, we define their matrix product ZW by

$$ZW = [Z\mathbf{w}_1, \ldots, Z\mathbf{w}_n] \ .$$

Notice that if all of the entries of the $n \times n$ matrix of A are real, so that $A = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ with $\mathbf{v}_1, \ldots, bv_n \in \mathbb{R}^n \subset \mathbb{C}^n$, and $\mathbf{z} = \mathbf{x} + \mathbf{y}$ with $\mathbf{x}, by \in \mathbb{R}^n$,

$$A\mathbf{z} = \sum_{j=1}^{n} (x + iy)\mathbf{v}_j = \sum_{j=1}^{n} x_j \mathbf{v}_j + i \sum_{j=1}^{n} y_j \mathbf{v}_j = A\mathbf{x} + i\mathbf{y} \ .$$

3.4.2 Geometry in $\mathbb{C}^n$

We may identify each complex number $z = x + iy$, $x, y \in \mathbb{R}$, with the vector $(x, y) \in \mathbb{R}^2$. Under this identification, addition of complex number corresponds to addition of vectors in $\mathbb{R}^2$. This identification is the origin of the term "complex plane" that is often used to refer to $\mathbb{C}$.

We define the *magnitude* of the complex number $z = x_i y$ to be the length of the corresponding vector (x, y) in $\mathbb{R}^2$. We denote the magnitude of z by $|z|$. Then, with $z = x + iy$,

$$|z| = \sqrt{x^2 + y^2} = \sqrt{\overline{z}z} \ .$$

In the same way, let $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$, and let $\mathbf{z} = \mathbf{x} + i\mathbf{y}$ be the decomposition of $\mathbf{z}$ into its real and imaginary parts. We may identify $\mathbf{z}$ with the vector $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2n}$, whose first n entries are from $\mathbf{x}$, and whose second n entries are from $\mathbf{y}$. Let $\mathbf{w} = \mathbf{u} + i\mathbf{v}$ be another vector in $\mathbb{C}^n$ given together with it decomposition into its real and imaginary parts. We then define the *inner product* of $\mathbf{w}$ and $\mathbf{z}$, $\langle \mathbf{w}, \mathbf{z} \rangle$ by

$$\langle \mathbf{w}, \mathbf{z} \rangle = \overline{\mathbf{w}} \cdot \mathbf{z} \ ,$$

where the dot product of two vectors $\mathbf{w}, bz$ in $\mathbb{C}^n$ is given by

$$\mathbf{w} \cdot \mathbf{z} = \sum_{j=1}^{n} w_j z_j \ .$$

The difference between the dot product and the inner product is the complex conjugate applied to the vector on the left in the inner product. Notice that for any complex number c,

$$\langle \mathbf{w}, c\mathbf{z} \rangle = c\langle \mathbf{w}, \mathbf{z} \rangle \qquad \text{and} \qquad \langle c\mathbf{w}, \mathbf{z} \rangle = \overline{c}\langle \mathbf{w}, \mathbf{z} \rangle \ . \tag{3.47}$$

Since with $\overline{z}z = |z|^2$, we have that

$$\langle \mathbf{z}, \mathbf{z} \rangle = \sum_{j=1}^{n} |z_j|^2 = \sum_{j=1}^{n} (x_j^2 + y_j^2) = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2.$$

Notice that this is the same as the length of the vector $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2n}$.

Therefore, we define the *magnitude* $\|\mathbf{z}\|$ of a vector $\mathbf{z} \in \mathbb{C}^n$ by

$$\|\mathbf{z}\| = \sqrt{\langle \mathbf{z}, \mathbf{z} \rangle}\ .$$

Notice that for all complex c,

$$\|c\mathbf{z}\| = |c|\|\mathbf{z}\|\ .$$

Then, as a direct consequence of the identity $\|\mathbf{z}\| = \|(\mathbf{x}, \mathbf{y})\|$ and the Minkowski inequality in R^{2n}, we deduce the Minkowski inequality in $\mathbb{C}^n$:

$$\|\mathbf{z} + \mathbf{w}\| \leq \|\mathbf{z}\| + \|\mathbf{w}\|\ .$$

Furthermore, one easily checks from the definition that if $\mathbf{u} = \Re(\mathbf{w})$, $\mathbf{v} = \Im(\mathbf{w})$, $\mathbf{x} = \Re(\mathbf{z})$ and $\mathbf{y} = \Im(\mathbf{y})$,

$$\Re\left(\langle \mathbf{w}, \mathbf{z} \rangle\right) = \mathbf{u} \cdot \mathbf{x} + \mathbf{v} \cdot \mathbf{y} = (\mathbf{u}, \mathbf{v}) \cdot (\mathbf{x}, \mathbf{y})\ .$$

Then by the Cauchy-Schwarz inequality in $\mathbb{R}^{2n}$,

$$|\Re\left(\langle \mathbf{w}, \mathbf{z} \rangle\right)| \leq \|(\mathbf{u}, \mathbf{v})\|\|(\mathbf{x}, \mathbf{y})\| = \|\mathbf{w}\|\|\mathbf{z}\|\ . \tag{3.48}$$

Next, notice that there is some $\theta \in [0, 2\pi)$ such that

$$|\langle \mathbf{w}, \mathbf{z} \rangle| = e^{i\theta} \langle \mathbf{w}, \mathbf{z} \rangle\ .$$

Then by (3.47), $e^{i\theta}\langle \mathbf{w}, \mathbf{z} \rangle = \langle \mathbf{w}, e^{i\theta}\mathbf{z} \rangle$, and altogether,

$$|\langle \mathbf{w}, \mathbf{z} \rangle| = \langle \mathbf{w}, e^{i\theta}\mathbf{z} \rangle = |\Re(\langle \mathbf{w}, e^{i\theta}\mathbf{z} \rangle)|\ .$$

Combining this with (3.48) we have

$$|\langle \mathbf{w}, \mathbf{z} \rangle| \leq \|\mathbf{w}\|\|e^{i\theta}\mathbf{z}\| = |e^{i\theta}|\|\mathbf{w}\|\|\mathbf{z}\| = \|\mathbf{w}\|\|\mathbf{z}\|\ .$$

The treatment of angles in $\mathbb{C}^n$ takes a little care. Recall that for two non-zero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the angle between them is

$$\theta(\mathbf{x}, \mathbf{y}) = \arccos\left(\frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}\right)\ ,$$

and this has the property that for all $a > 0$, the angle between $a\mathbf{x}$ and $\mathbf{y}$ is the same as the angle between $\mathbf{x}$ and $\mathbf{y}$. That is, angles depend only on direction, and not length. On the other hand, for all $a < 0$, the angle between $a\mathbf{x}$ and $\mathbf{y}$ is the same as the angle between $-\mathbf{x}$ and $\mathbf{y}$.

The angle between two complex vectors $\mathbf{w}$ and $\mathbf{z}$ will be defined so that for any two non-zero vectors $\mathbf{w}$ and $\mathbf{z}$, the angle between $\mathbf{w}$ and $c\mathbf{z}$ is the same as the angle between $\mathbf{w}$ and $\mathbf{z}$ for all complex $c \neq 0$. The difference between the real case and the complex case is that the complex plane is not ordered, so it makes no sense to attempt to distinguish between positive an negative multiples. Therefore, in the complex case, we define the angle between $\mathbf{w}$ and $\mathbf{z}$ in $\mathbb{C}^n$ to be

$$\theta(\mathbf{w},\mathbf{z}) = \arccos\left(\frac{|\langle \mathbf{w},\mathbf{z}\rangle|}{\|\mathbf{w}\|\|\mathbf{z}\|}\right) .$$

Notice that by the Cauchy-Scwarz inequality in $\mathbb{C}^n$, $0 \le |\langle \mathbf{w},\mathbf{z}\rangle| \le \|\mathbf{w}\|\|\mathbf{z}\|$ so that the argument of the arccos function is within its domain of definition, so that the angle is well-defined.

Since $|\langle \mathbf{w}, c\mathbf{z}\rangle| = |\langle c\mathbf{w},\mathbf{z}\rangle| = |c||\langle \mathbf{w},\mathbf{z}\rangle|$ and $\|c\mathbf{w}\| = c\|\mathbf{w}\|$ and likewise $\|c\mathbf{z}\| = |c|\|\mathbf{z}\|$ for all complex c, it follows that for all complex $c \neq 0$,

$$\theta(c\mathbf{w},\mathbf{z}) = \theta(\mathbf{w},\mathbf{z}) = \theta(\mathbf{w},\mathbf{z}) ,$$

and

$$0 \le \theta(\mathbf{w},\mathbf{z}) \le 2\pi .$$

That is, in the complex case, angles are always acute since the angle between $\mathbf{w}$ and $\mathbf{z}$ is the same as the angle between $\mathbf{w}$ and $-\mathbf{z}$. We are in a sense measuring the angle between the lines through these vectors, which is what we would have gotten has we taken the absolute value of the dot product in the real case.

3.4.3 Continuously differentiable curves in $\mathbb{C}^n$

Let $\mathbf{z}(t)$ be a function from the open interval $(a,b) \subset \mathbb{R}$ to $\mathbb{C}^n$. We say that $\mathbf{z}(t)$ is continuous at $t = t_0 \in (a,b)$ in case

$$\lim_{t\to t_0} \|\mathbf{z}(t) - \mathbf{z}(t_0)\| = 0 .$$

We say that $\mathbf{z}(t)$ is continous in case it is continuous at each $t_0 \in (a,b)$, its domeain of definition.

We say that $\mathbf{z}(t)$ is differentiable at t_0 with derivative $\mathbf{w}$ in case

$$\lim_{t\to t_0} \frac{\|\mathbf{z}(t) - [\mathbf{z}(t_0) + (t-t_0)\mathbf{w}]\|}{|t-t_0|} = 0 . \tag{3.49}$$

Let $\mathbf{x}(t) = \Re(\mathbf{z}(t)$ and let $\mathbf{y}(t) = \Im(\mathbf{z}(t)$. Let $\mathbf{u} = \Re(\mathbf{w})$ and let $\mathbf{v} = \Im(\mathbf{w})$. Then

$$\|\mathbf{z}(t) - [\mathbf{z}(t_0) + (t-t_0)\mathbf{w}]\| = \|(\mathbf{x}(t) - [\mathbf{x}(t_0) + (t-t_0)\mathbf{u}]) + i(\mathbf{y}(t) - [\mathbf{v}(t_0) + (t-t_0)\mathbf{v}])\| ,$$

and so

$$\|\mathbf{z}(t) - [\mathbf{z}(t_0) + (t-t_0)\mathbf{w}]\|^2 = \|\mathbf{x}(t) - [\mathbf{x}(t_0) + (t-t_0)\mathbf{u}]\|^2 + \|\mathbf{y}(t) - [\mathbf{v}(t_0) + (t-t_0)\mathbf{v}])\|^2 . \tag{3.50}$$

It follows that whenver (3.49) is satisfied, then

$$\lim_{t\to t_0} \frac{\|\mathbf{x}(t) - [\mathbf{x}(t_0) + (t-t_0)\mathbf{u}]\|}{|t-t_0|} = 0 ,$$

and so $\mathbf{x}(t)$ is differentiable at t_0 with derivative $\mathbf{x}'(t_0) = \mathbf{u}$. By the same reasoning, $\mathbf{y}(t)$ is differentiable at $t = t_0$ with $\mathbf{y}'(t) = \mathbf{v}$. Conversely, suppose that $\mathbf{x}(t)$ is differentiable at t_0 with derivative $\mathbf{x}'(t_0) = \mathbf{u}$, and that $\mathbf{y}(t)$ is differentiable at $t = t_0$ with $\mathbf{y}'(t) = \mathbf{v}$. Then

$$\lim_{t\to t_0} \frac{1}{|t-t_0|}\left(\|\mathbf{x}(t) - [\mathbf{x}(t_0) + (t-t_0)\mathbf{u}]\|^2 + \|\mathbf{y}(t) - [\mathbf{v}(t_0) + (t-t_0)\mathbf{v}])\|^2\right)^{1/2} = 0 ,$$

and then by (3.50), $\mathbf{z}(t)$ is differentiable at $t = t_0$ with derivative $\mathbf{z}'(t) = \mathbf{w}$.

In summary, $\mathbf{z}(t)$ is differentiable at $t = t_0$ if and only if $\mathbf{x}(t) = \Re(\mathbf{z}(t))$ and $\mathbf{y}(t) = \Im(\mathbf{z}(t)$ are both differentiable at $t = t_0$, and in this case

$$\mathbf{z}'(t_0) = \mathbf{x}'(t_0) + i\mathbf{y}'(t_0) \ .$$

We say that a curve $\mathbf{z}(t)$ defined on $(a, b) \subset \mathbb{R}$ is continuously differentiable on (a, b) in case $\mathbf{z}(t)$ is differentiable at each $t \in (a, b)$ an $\mathbf{z}'(t)$ is a continuous function of t in (a, b).

3.4.4 Complex solutions and real solutions

We now know what it means for a curve $\mathbf{z}(t)$ in $\mathbb{C}^n$ to be continuously differentiable, and we know how to define the matrix vector product $A\mathbf{z}(t)$ where A is any $n \times n$ matrix. Hence we know what it means for $\mathbf{z}(t)$ to sole the forst order linear system

$$\mathbf{z}'(t) = A\mathbf{z}(t) \ .$$

Let $\mathbf{x}(t) = \Re(\mathbf{z}(t)$ and let $\mathbf{y}(t) = \Im(\mathbf{z}(t))$. Then we have seen that $\mathbf{z}'(t) = \mathbf{x}'(t) + i\mathbf{y}(t)$.

Now suppose that A is real; i.e., each entry of A is real. Then we have seen that

$$A\mathbf{z}(t) = A\mathbf{x}(t) + iA\mathbf{y}(y) \ .$$

Therefore, when A is real, $\mathbf{z}'(t) = A\mathbf{z}(t)$ if and only if

$$\mathbf{x}'(t) + i\mathbf{y}'(t) = A\mathbf{x}(t) + iA\mathbf{y}(t) \ .$$

But since two vectors in $\mathbb{C}^n$ are equal if and only if their real and imaginary parts are both equal, $\mathbf{z}'(t) = A\mathbf{z}(t)$ if and only if

$$\mathbf{x}'(t) = A\mathbf{x}(t) \qquad \text{and} \qquad \mathbf{y}'(t) = A\mathbf{y}(t) \ .$$

Thus when A is real, every single complex solution of $\mathbf{z}'(t) = A\mathbf{z}(t)$, provides *two real solutions* of $\mathbf{x}'(t) = A\mathbf{x}(t)$, namely $\mathbf{x}(t) = \Re(\mathbf{z}(t))$ and $\mathbf{y}(t) = \Im(\mathbf{z}(t))$.

Example 32. *Let A be the 2×2 matrix*

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \ .$$

Then, as we saw in Example 28, the eigenvalues of A are i and $-i$, and the corresponding eigenvactors are

$$\mathbf{z}_1 = (1, i) \qquad \text{and} \qquad \mathbf{z}_2 = (1, -i) \ .$$

Let us define

$$\mathbf{z}(t) = e^{it}\mathbf{z}_1 \ .$$

Then

$$\mathbf{z}'(t) = e^{it}i\mathbf{z}_1 = e^{it}A\mathbf{z}_1 = A(e^{it}\mathbf{z}_1) = A\mathbf{z}(t) \ .$$

Thus, $\mathbf{z}(t)$ *is a complex solution. To find its real and imaginary parts, we use Euler's formula* $e^{it} = \cos t + i\sin t$ *to write*

$$\mathbf{z}(t) = (\cos t + i\sin t)(1, i) = \big(\cos t + i\sin t, i\cos t - \sin t\big) = \big(\cos t, -\sin t\big) + i\big(\sin t, cost\big) \ .$$

That is,

$$\mathbf{x}(t) = \Re(\mathbf{z}(t)) = \big(\cos t, -\sin t\big) \qquad \text{and} \qquad \mathbf{y}(t) = \Im(\mathbf{z}(t)) = \big(\sin t, cost\big)$$

are two real solutions of the equation. In other words, $\mathbf{x}'(t) = A\mathbf{x}(t)$ *and* $\mathbf{y}'(t) = A\mathbf{y}(t)$ *as you can easily check.*

Whenever A is an $n \times n$ real matrix, the complex eigenvalues and eigenvectors, if any, come in pairs. Suppose that $\Im(\mu) \neq 0$, and $\mathbf{z} \in \mathbb{C}^n$ is not the zero vector. Suppose that $A\mathbf{z} = \mu\mathbf{z}$. Then since all of the entries of A are real, it follows that

$$\overline{A\mathbf{z}} = A\overline{\mathbf{z}} \ .$$

Also, since for any two complex number w, z, $\overline{(wz)} = (\overline{w})(\overline{z})$, it follows that

$$\overline{(\mu\mathbf{z})} = \overline{\mu}(\overline{\mathbf{z}}) \ .$$

Therefore,

$$A\mathbf{z} = \mu\mathbf{z} \quad \Rightarrow \quad A\overline{\mathbf{z}} = \overline{\mu}\,\overline{\mathbf{z}} \ .$$

In other words, $\overline{\mathbf{z}}$ is an eigenvector with eigenvalue $\overline{\mu}$.

The real and imaginary parts of $\mathbf{z}(t)$ and $\overline{\mathbf{z}(t)}$ are the same up to a sign, and so while each provides us with two real solutions when A is real, up to a sign they provide us with the *same* two solutions.

3.4.5 The general structure of the matrix exponential

Let A be an $n \times n$ matrix. Let $\mu_1, \ldots, \mu_n$ be the roots of the characteristic polynomial, repeated according to the number of times they show up in the factorization of the characteristic polynomial of A. That is,

$$\det(A - tI) = \prod_{j=1}^{n} (\mu_j - t) \ .$$

Let d_j be the number of time the number μ_j is repeated as a root, so that $d_j = 1$ for all j if and only if A has n distinct eigenvlaues.

These may be complex, so we write $\lambda_j = \Re(\mu_j)$ and $\kappa_j = \Im(\mu_j)$, so that

$$\mu_j = \lambda_j + i\kappa_j \ .$$

We know that there exists a linearly independent set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ os generalized eigenvectors of A so that for each j,

$$(A - \mu_j)^{d_j}\mathbf{v}_j = \mathbf{0} \ ,$$

although this might also be true for some lower power of the matrix $(A - \mu_j)$. In any case, no higher power is needed.

Then if we define

$$\mathbf{z}_j(t) = e^{t\mu_j}\left(\sum_{k=0}^{d_j}(A-\mu_j I)\mathbf{v}_j\right)\mathbf{v}_j \ , \tag{3.51}$$

we have that for each j.

$$\mathbf{z}_j'(t) = A\mathbf{z}_j(t) \ , \quad \mathbf{z}_j(0) = \mathbf{v}_j \ .$$

Finally, define an $n \times n$ matrix $M(t)$ by

$$M(t) = [\mathbf{z}_1(t), \ldots, \mathbf{z}_n(t)] \ .$$

That is, the jth column of $M(t)$ is the solution $\mathbf{z}_j(t)$. Differentiating entry by enry, or, what is the same, column by column,

$$M'(t) = [\mathbf{z}_1'(t), \ldots, \mathbf{z}]_n(t)] = [A\mathbf{z}_1(t), \ldots, A\mathbf{z}_n(t)] = A[\mathbf{z}_1(t), \ldots, \mathbf{z}_n(t)] = AM(t) \ .$$

Also,

$$M(0) = [\mathbf{z}_1(0), \ldots, \mathbf{z}_n(0)] = [\mathbf{v}_1, \ldots, \mathbf{v}_n] = V \ .$$

Therefore,multiplying on the right be the constant matrix V^{-1} we obtain the time dependent matrix $M(t)V^{-1}$ which satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}(M(t)V^{-1}) = A(M(t)V^{-1}) \qquad \text{and} \qquad (M(0)V^{-1}) = I \ .$$

But the unique matrix with these properties is e^{tA}. Therefore,

$$e^{tA} = M(t)V^{-1} = [\mathbf{z}_1(t), \ldots, \mathbf{z}_n(t)][\mathbf{v}_1, \ldots, \mathbf{v}_n]^{-1} \ . \tag{3.52}$$

If the matrix A is real, e^{tA} will be real, as is obvious from its power series definition. It is nor obvious from the formulas that the right hand side will be real, but we have just shown that it must be.

Example 33. *Once more, let A be the 2×2 matrix*

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \ .$$

Then, as we saw in Example 28, the eigenvalues of A are i and $-i$, and the corresponding eigenvactors are

$$\mathbf{v}_1 = (1, i) \qquad \text{and} \qquad \mathbf{v}_2 = (1, -i) \ .$$

The two complex solutions corresponding to these eigenvecotrs are

$$\mathbf{z}_1(t) = e^{it}(1, i) \qquad \text{and} \qquad \mathbf{z}_2(t) = e^{-it}(1, -i) \ .$$

The matrix $M(t)$ is then

$$M(t) = \begin{bmatrix} e^{it} & e^{-it} \\ ie^{it} & -ie^{-it} \end{bmatrix}$$

and

$$V = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \qquad \text{so that} \qquad V^{-1} = \frac{1}{2i}\begin{bmatrix} i & 1 \\ i & -1 \end{bmatrix} .$$

We now compute that

$$e^{tA} = M(t)V^{-1} = \frac{1}{2i}\begin{bmatrix} e^{it} & e^{-it} \\ ie^{it} & -ie^{-it} \end{bmatrix}\begin{bmatrix} i & 1 \\ i & -1 \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} ,$$

as we found before. Notice that we obtain the real matrix e^{tA} *as the product of two complex matrices.*

Now we come to an important theorem that follows easily from what we know about the general structure of the matrix exponential.

Theorem 16 (Stability Theorem)**.** *Let* A *be an* $n \times n$ *matrix. Let* $\{\mu_1, \dots, \mu_n\}$ *be its eigenvalues. Suppose that each of these has a strictly negative real part, and define*

$$\lambda = \max\{\Re(\mu_1), \dots, \Re(\mu_n)\} < 0 .$$

Then for any κ *with*

$$0 < \kappa < -\lambda ,$$

there is a constant C_κ *such that*

$$\|e^{tA}\|_{\mathrm{F}} \leq C_\kappa e^{-t\kappa}$$

for all $t > 0$*. In particular, for all* $\mathbf{x}_0 \in \mathbb{R}^n$*, the solution of* $\mathbf{x}'(t)$ *with* $\mathbf{x}(0) = \mathbf{x}_0$ *satisfies* $\lim_{t\to 0} e^{tk}\|\mathbf{x}(t)\| = 0$ *for all* $0 < \kappa < -\lambda$*.*

However, if even on eigenvalue has a strictly positive real part, then there is an $\mathbf{x}_0 \in \mathbb{R}^n$ *so that he solution of* $\mathbf{x}'(t)$ *with* $\mathbf{x}(0) = \mathbf{x}_0$ *satisfies* $\lim_{t\to\infty} e^{-ta}\|\mathbf{x}(t)\| = \infty$ *for some* $a > 0$*, meaning that the solution "blows up" exponentially fast.*

Proof. We have seen that there is a set $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of n linearly independent generalized eigenvectors of A. Let $\mathbf{z}_j(t)$ be the solution of $\mathbf{z}'(t) = A\mathbf{z}(t)$ with $\mathbf{z}_j(0) = \mathbf{v}_j$. This solution is given by (3.51), and we see from tis that each $\mathbf{z}_j(t)$ has the form

$$\mathbf{z}_j(t) = e^{t\mu_j}\mathbf{p}_j(t)$$

where each entry of $\mathbf{p}_j(t)$ is a polynomial in t of degree at most d_j. Then since

$$|e^{t\mu_j}| = |e^{t\lambda_j}e^{it\kappa_j}| = e^{t\lambda_j}|e^{it\kappa_j}| = e^{t\lambda_j} ,$$

$$\|\mathbf{z}_j(t)\| \leq e^{t\lambda_j}\|\mathbf{p}_j(t)\| .$$

Since the exponential function grows faster than any power, we see that if $\lambda_j < 0$, then

$$\lim_{t\to\infty} \|\mathbf{z}_j(t)\| = 0 .$$

In fact, we can say more: Since for any $\epsilon > 0$,

$$\lim_{t\to\infty} e^{-\epsilon t}\|\mathbf{p}_j(t)\| = 0 ,$$

There is a $C_{j,\epsilon}$ so that

$$e^{-\epsilon t}\|\mathbf{p}_j(t)\| \le C_{j,\epsilon} \qquad \text{for all} \quad t \ge 0 \ .$$

We then have

$$\|\mathbf{z}_j(t)\| \le C_{j,\epsilon}e^{t(\lambda_j+\epsilon)} \ . \tag{3.53}$$

Thus we see that if $\lambda_j < 0$, then $\|\mathbf{z}_j(t)\|$ converges to zero exponentially fast, almost at rate $e^{t\lambda}$.

Now suppose that all of the eigenvalues of A have a strictly negative real part. That is for some λ,

$$\lambda_j \le \lambda < 0 \ .$$

Then for any $0 < \kappa < -\lambda$, there is a constant C so that

$$\|\mathbf{z}_j(t)\| \le Ce^{-tb} \ .$$

From this it follows that the Frobenius norm of the matrix $M(t)$ satisfies

$$\|M(t)\|_{\mathrm{F}} \le \sqrt{n}Ce^{-tb} \ .$$

Finally, from this we learn that

$$\|e^{tA}\|_{\mathrm{F}} \le (\|V^{-1}\|_{\mathrm{F}}\sqrt{n}C)e^{-bt} \ .$$

On the other hand, A has any eigenvalue μ with $\lambda = \Re(\mu) > 0$, then there is an eigenvector $\mathbf{v}$ with this eigenvalue and $\mathbf{z}(t) = e^{t\mu}\mathbf{v} = e^{tA}\mathbf{v}$. Calculating as above,

$$\|e^{t\mu}\mathbf{v}\| = e^{t\lambda}\|\mathbf{v}\| \le \|e^{tA}\|_{\mathrm{F}}\|\mathbf{v}\| \ .$$

In this case there are solutions of $\mathbf{x}'(t) = A\mathbf{x}(t)$ with $\|\mathbf{x}(t)\|$ gowing exponenitally fast, and also $\|e^{tA}\|_{\mathrm{F}} \ge e^{t\lambda}$, so that $\|e^{tA}\|_{\mathrm{F}}$ exponentially fast. □

Except in case all of the eigenvalues of A are purely imaginary, we have either $\lim_{t\to\infty}\|e^{tA}\|_{\mathrm{F}} = 0$ or else $\lim_{t\to\infty}\|e^{tA}\|_{\mathrm{F}} = \infty$. We shall study the important case in whihc all eigenvalues are purely imaginary in the next chapter.

3.5 Exercises

1. Let $\mathbf{v}(x,y)$ be the vector field defined on the right half-plane $U = \{(x,y) \ : \ x > 0\}$ by

$$\mathbf{v}(x,y) = \left(x \ , \ -\frac{1}{x^2} - 2y + x^2y^2\right) \ .$$

The system corresponding to this vector field is recursively coupled since the rate of change of x depends on x alone. This can be used to solve the system, but the system can also be completely decoupled by change of variables. There is a method for finding such a change of variables, but at this point in the course our goal is only to become familiar with how systems of differential equations transform under changes of variables. So we will start with the change of variables as a given.

(a) Define

$$u(x,y) = -\ln x \qquad \text{and} \qquad v(x,y) = x^2 y\ .$$

The transformation $(x,y) \to (u,v)$ invertible transforms the right half-plane onto all of $\mathbb{R}^2$. Compute the inverse transformation.

(b) Suppose that $\mathbf{x}(t)$ solves $\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t)$. Define $\mathbf{u}(t) = (u(\mathbf{x}(t)), v(\mathbf{x}(t))$. Using the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}t} u(\mathbf{x}(t)) = \frac{\partial}{\partial x} u(\mathbf{x}(t)) x'(t) + \frac{\partial}{\partial y} u(\mathbf{x}(t)) y'(t)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} v(\mathbf{x}(t)) = \frac{\partial}{\partial x} v(\mathbf{x}(t)) x'(t) + \frac{\partial}{\partial y} v(\mathbf{x}(t)) y'(t)\ ,$$

find the vector field $\mathbf{w}(u,v)$ on the u,v plane such that

$$\mathbf{u}'(t) = \mathbf{w}(\mathbf{u}(t))\ .$$

You should find that this vector field describes a completely decoupled system.

(c) Solve the system $\mathbf{u}'(t) = \mathbf{w}(\mathbf{u}(t))$ by separately solving the decoupled one dimensional equations. Show that the solution of this equation with $\mathbf{u}(0) = (u_o, v_0)$ exists for all t and is unique if and only if $|v_0| \le 1$.

(d) Use the inverse transformation you found in part **(a)** to express the solution of $\mathbf{u}'(t) = \mathbf{w}(\mathbf{u}(t))$ with $\mathbf{u}(0) = \mathbf{u}_0 = (u(x_0,y_0), v(x_0,y_0))$ in terms of x and y. Show that the resulting curve $\mathbf{x}(t)$ satisfies $\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t))$ with $\mathbf{x}(0) = \mathbf{x}_0$.

(e) Show that the solution of $\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t))$ with $\mathbf{x}(0) = \mathbf{x}_0$ exists for all time and is unique if and only if $|x_0^2 y_0| \le 1$, and give the solution for all such (x_0, y_0).

(f) Now go back to the original equation and use the fact that $x' = x$ is solved by $x(t) = x_0 e^t$ to convert the equation for y into a Ricatti equation, and solve this. Compare your two solutions.

2. Consider the vector field $\mathbf{v}(\mathbf{x})$ defined on $U = \{(x,y)\ :\ x > |y|\}$ defined by

$$\mathbf{v}(x,y) = (x - y\sqrt{x^2 - y^2}\ ,\ y - x\sqrt{x^2 - y^2})\ .$$

Notice that on the boundary of U, the vecotr field is tangent to the boundary, so that the vector field does not ever carry the solution out of U.

Define the change of variables

$$u(x,y) = \frac{1}{2}\left(\frac{x+y}{x-y}\right) \qquad \text{and} \qquad v(x,y) = \sqrt{x^2 - y^2}\ .$$

(a) Let $\mathbf{x}(t)$ be any continuously differentiable curve in $\mathbb{R}^2$ with values in U. Define a curve

$$\mathbf{u}(t) = (u(\mathbf{x}(t)), v(\mathbf{x}(t)))\ .$$

Find a vector field $\mathbf{w}$ defined on an open set $V \subset \mathbb{R}^2$ so that

$$\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t)) \qquad \text{if and only if} \qquad \mathbf{u}'(t) = \mathbf{w}(\mathbf{u}(t))\ .$$

(b) Find the general solution of $\mathbf{u}'(t) = \mathbf{w}(\mathbf{u}(t))$ with $\mathbf{u}(0) \in V$.

(c) Find the general solution of $\mathbf{x}'(t) = \mathbf{w}(\mathbf{u}(t))$ with $\mathbf{x}(0) \in U$, and find the corresponding flow transformation.

3. Consider the differential equation $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 4 \\ 0 & 0 & -2 \end{bmatrix} .$$

Find the general solution $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$ in closed form. That is, compute e^{tA}. (Note that this system is recursively coupled.)

4. Consider the differential equation $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} -4 & 2 \\ 5 & -1 \end{bmatrix} .$$

(a) Find the general solution $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$ in closed form. That is, compute e^{tA}.

(b) Find all $\mathbf{x}_0$ such that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$.

5. Consider the differential equation $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix} .$$

(a) Find the general solution $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$ in closed form. That is, compute e^{tA}.

(b) Find all $\mathbf{x}_0$ such that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$.

6. Consider the differential equation $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} -4 & 2 \\ 9 & -1 \end{bmatrix} .$$

(a) Find the general solution $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$ in closed form. That is, compute e^{tA}.

(b) Find all $\mathbf{x}_0$ such that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$.

7. Consider the differential equation $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} 5 & -1 \\ 8 & 1 \end{bmatrix} .$$

(a) Find the general solution $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$ in closed form. That is, compute e^{tA}.

(b) Find all $\mathbf{x}_0$ such that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$.

8. Compute e^{tA} for

$$A = \begin{bmatrix} -1 & 2 & 2 \\ -2 & -1 & 1 \\ -2 & -1 & -1 \end{bmatrix} .$$

9 Compute e^{tA} for

$$A = \begin{bmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{bmatrix}.$$

10. Compute e^{tA} for

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}.$$

11. Compute e^{tA} for

$$A = \begin{bmatrix} 4 & -1 & 2 \\ 1 & 2 & 2 \\ -2 & 2 & 1 \end{bmatrix}.$$

Chapter 4

NON-LINEAR AND NON-AUTONOMOUS SYSTEMS

4.1 Time-dependent vector fields

4.1.1 Driven linear systems and Duhamel's formula

Let A be an $n \times n$ matrix, and let $\mathbf{g}(t)$ a continuous curve in $\mathbb{R}^n$. The equation

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{g}(t) \tag{4.1}$$

describes a *driven linear system*. We can think of $\mathbf{g}(t)$ a representing the effects of a "driving forces" and we shall soon see examples in which this interpretation is natural.

If we define the vector field $\mathbf{v}(t, \mathbf{x})$ by

$$\mathbf{v}(t, \mathbf{x}) = A\mathbf{x} + \mathbf{g}(t) ,$$

we see that our equation has the form $\mathbf{x}'(t) = \mathbf{v}(t, \mathbf{x}(t))$. Notice that

$$\|\mathbf{v}(t, \mathbf{y})\| - \mathbf{v}(t, \mathbf{x}) = \|A(\mathbf{y} - \mathbf{x})\| \leq \|A\|_{\mathrm{F}} \|\mathbf{y} - \mathbf{x}\| ,$$

and so regardless of how $\mathbf{g}(t)$ is defined, this vector field satisfies the Lipschitz condition that we have shown implies uniqueness. Hence, whenever solutions of (4.1) exist, there will be exactly one solution passing through $\mathbf{x}_0$ at $t = t_0$, no matter how $\mathbf{x}_0$ and t_0 are chosen.

Now let us turn to solving the equation. The one dimensional version of (4.1) is

$$x'(t) = ax(t) + g(t) .$$

To solve this, we would bring all of the terms involving $x(t)$ to the left, and multiply through by e^{-at}. to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}[e^{-at}x(t)] = e^{-at}g(t) .$$

Supposing the $x(t_0) = x_0$, and integrating both sides from t_0 to t, we find

$$e^{-ta}x(t) = e^{-t_0}x_) = \int_{t_0}^{t} e^{-as}g(s)\mathrm{d}s \ ,$$

and finally we obtain

$$x(t) = e^{(t-t_0)a}x_) + \int_{t_0}^{t} e^{(t-s)a}g(s)\mathrm{d}s \ .$$

Using what we have learned about the matrix exponential functions, we can solve (4.1) in essentially the same way. Let us seek the solution of (4.1) that satisfies $\mathbf{x}(t_0) = \mathbf{x}_0$. We may re-write (4.1) as

$$\mathbf{x}'(t) - A\mathbf{x}(t) = \mathbf{g}(t) \ .$$

Multiplying both sides by e^{-tA} we obtain

$$e^{-tA}(\mathbf{x}'(t) - A\mathbf{x}(t)) = e^{-tA}\mathbf{g}(t) \ .$$

But the left hand side is the t derivative of $e^{-tA}\mathbf{x}(t)$, so we have

$$\frac{\mathrm{d}}{\mathrm{d}t}[e^{-tA}\mathbf{x}(t)] = e^{-tA}\mathbf{g}(t) \ .$$

Integrating from t_0 to t, we obtain

$$e^{-tA}\mathbf{x}(t) - e^{-t_0A}\mathbf{x}_0 = \int_{t_0}^{t} e^{-sA}\mathbf{g}(s)\mathrm{d}s \ .$$

Multiplying through by e^{tA} and regrouping terms we find, using properties of the matrix exponential such as $e^{tA}e^{-sA} = e^{(t-s)A}$,

$$\mathbf{x}(t) = e^{(t-t_0)A}\mathbf{x}_0 + \int_{t_0}^{t} e^{(t-s)A}\mathbf{g}(s)\mathrm{d}s \ . \tag{4.2}$$

Since we have assumes $\mathbf{g}(t)$ to be continuous, the integral on the right exists. Moreover, by the Fundamental Theorem of Calculus,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^{t} e^{(t-s)A}\mathbf{g}(s)\mathrm{d}s &= e^{(t-s)A}\mathbf{g}(s)\Big|_{s=t} + \int_{t_0}^{t} Ae^{(t-s)A}\mathbf{g}(s)\mathrm{d}s \\ &= \mathbf{g}(t) + A\left(\int_{t_0}^{t} e^{(t-s)A}\mathbf{g}(s)\mathrm{d}s\right) \ . \end{aligned}$$

Combining this with $(e^{(t-t_0)A}\mathbf{x}_0)' = Ae^{(t-t_0)A}\mathbf{x}_0$, we see that if we define $\mathbf{x}(t)$ by (4.2), then $\mathbf{x}(t)$ satisfies (4.1). Moreover, it is clear that $\mathbf{x}(t_0) = bx_0$. Finally, be what we have noted above concerning uniqueness, (4.2) is the only solution of (4.1) that passes through $\mathbf{x}_0$ at $t = t_0$. The formula (4.2) for this solution is known as *Duhamel's formula.*

Theorem 17. *Let A be an $n \times n$ matrix and $\mathbf{g}(t)$ a continuous curve in $\mathbb{R}^n$. Let $t_0 \in \mathbb{R}$ and $\mathbf{x}_0 \in \mathbb{R}^n$. Then there is a unique solution of (4.1) such that $\mathbf{x}(t_0) = \mathbf{x}_0$, and this solution exists for all $t \in \mathbb{R}$ and is given by Duhamel's formula (4.2).*

Example 34. *As in Example 26, consider the equation* $\mathbf{x}'(t) = A\mathbf{x}(t)$ *in* $\mathbb{R}^2$ *where* A *is the* 2×2 *matrix*

$$A = \begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix} .$$

Let

$$\mathbf{g}(t) = (1, t) \ .$$

We shall now use Duhamel's formula to produce the general solution of

$$\mathbf{x}'(t) = A\mathbf{x}(t) = \mathbf{g}(t) \qquad \text{with} \qquad \mathbf{x}(0) = \mathbf{x}_0 \ .$$

We have seen in Example 26 that

$$e^{tA} = \frac{1}{5}\begin{bmatrix} 2e^{-4t} + 3e^t & 2e^t - 2e^{-4t} \\ 3e^t - 3e^{-4t} & 3e^{-4t} + 2e^t \end{bmatrix} \mathbf{x} \ .$$

It often helps to make a change of variable in computing the integral $\int_0^t e^{(t-s)A}\mathbf{g}(s)\mathrm{d}s$*. Let* $u = t - s$ *so that* $\mathrm{d}u = -\mathrm{d}s$*, and so* $u = t$ *when* $s = 0$ *and* $u = 0$ *when* $s = t$*. Therefore*

$$\int_0^t e^{(t-s)A}\mathbf{g}(s)\mathrm{d}s = \int_0^t e^{uA}\mathbf{g}(t-u)\mathrm{d}u \ .$$

We now compute compute

$$\begin{aligned}
\int_0^t e^{uA}\mathbf{g}(t-u)\mathrm{d}u &= \frac{1}{5}\int_0^t (2e^{-4u} + 3e^u \ , \ 3e^u - 3e^{-4u})\mathrm{d}u \\
&+ \frac{1}{5}\int_0^t (\ 2e^u - 2e^{-4u} \ , \ 3e^{-4u} + 2e^u)(t-u)\mathrm{d}u \\
&= \frac{1}{20}(12e^t - 2e^{-4t} - 10 \ , \ 12e^t + 3e^{-4t} - 15) \\
&+ \frac{1}{80}(32e^t - 2e^{-4t} - 40t - 30 \ , \ 32e^t + 3e^{-4t} - 20t - 35) \\
&= \frac{1}{16}(16e^t - 2e^{-4t} - 8t - 14 \ , \ 16e^t + 3e^{-4t} - 4t - 19) \ .
\end{aligned}$$

Then, with $\mathbf{x}_0 = (x_0, y_0)$*,*

$$e^{tA}\mathbf{x}_0 = \frac{1}{5}([3e^t + 2e^{-4t}]x_0 + [2e^t - 2e^{-4t}]y_0 \ , \ [3e^t - 3e^{-4t}]x_0 + [2e^t + 3e^{-4t}]y_0) \ .$$

Putting it all together, the solution is

$$\begin{aligned}
\mathbf{x}(t) &= \frac{1}{5}([3e^t + 2e^{-4t}]x_0 + [2e^t - 2e^{-4t}]y_0 \ , \ [3e^t - 3e^{-4t}]x_0 + [2e^t + 3e^{-4t}]y_0) \\
&= \frac{1}{16}(16e^t - 2e^{-4t} - 8t - 14 \ , \ 16e^t + 3e^{-4t} - 4t - 19) \ .
\end{aligned}$$

Our next example is extremely important. It concerns a second order equation, since it comes from an important case of Newton's equation. However, our usual reduction of order method brings this equation we consider into a form to which Duhamel's formula may be applied.

Example 35. *Consider the equation*

$$mx''(t) = -kx(t) + f(t) \tag{4.3}$$

which describes the motion of an object of mass m at the end of a spring subject to Hooke's law with a spring constant k, and also an externally applied force $f(t)$ Since $k/m > 0$ we may define $\kappa > 0$ by $\kappa = k/m$. The quantity $\sqrt{\kappa}$ is the natural frequency *of the spring. The period of oscillation is $T = 2\pi/\omega$.*

If we define $y(t) = x'(t)$, and $\mathbf{x}(t) = (x(t), y(t))$ and $\mathbf{g}(t) = (0, m^{-1}f(t))$, our second order equations is equivalent to the first order driven system

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{g}(t)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -\kappa & 0 \end{bmatrix} .$$

Computing $\det(A - tI)$*, we find $t^2 + \kappa$, so the eigenvalues of A are $\pm i\sqrt{\kappa}$.*

$$A - i\sqrt{\kappa}I = \begin{bmatrix} -i\sqrt{\kappa} & 1 \\ -\kappa & -i\sqrt{\kappa} \end{bmatrix} .$$

Hence $(1, i\sqrt{\kappa})$ is an eigenvector of A with eigenvalue $i\sqrt{\kappa}$. The corresponding complex solution of $\mathbf{z}'(t) = A\mathbf{z}(t)$ is

$$\begin{aligned} \mathbf{z}(t) &= e^{i\sqrt{\kappa}t}(1, i\sqrt{\kappa}) \\ &= (\cos(\sqrt{\kappa}t) + i\sin(\sqrt{\kappa}t))(1, i\sqrt{\kappa}) \\ &= \big(\cos(\sqrt{\kappa}t) - \sqrt{\kappa}\sin(\sqrt{\kappa}t)\big) + i\big(\sin(\sqrt{\kappa}t), \sqrt{\kappa}\cos(\sqrt{\kappa}t)\big) . \end{aligned} \tag{4.4}$$

Using these two real solutions, we build the matrix exponential:

$$e^{tA} = \begin{bmatrix} \cos(\sqrt{\kappa}t) & \sin(\sqrt{\kappa}t) \\ -\sqrt{\kappa}\sin(\sqrt{\kappa}t) & \sqrt{\kappa}\cos(\sqrt{\kappa}t) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\kappa} \end{bmatrix}^{-1} = \begin{bmatrix} \cos(\sqrt{\kappa}t) & \sqrt{\kappa}^{-1}\sin(\sqrt{\kappa}t) \\ -\sqrt{\kappa}\sin(\sqrt{\kappa}t) & \cos(\sqrt{\kappa}t) \end{bmatrix} .$$

Now Duhamel's formula gives us that

$$\begin{aligned} \mathbf{x}(t) &= \begin{bmatrix} \cos(\sqrt{\kappa}(t-t_0)) & \sqrt{\kappa}^{-1}\sin(\sqrt{\kappa}(t-t_0)) \\ -\sqrt{\kappa}\sin(\sqrt{\kappa}(t-t_0)) & \cos(\sqrt{\kappa}(t-t_0)) \end{bmatrix} \mathbf{x}_0 \\ &+ \int_{t_0}^{t} \begin{bmatrix} \cos(\sqrt{\kappa}(t-s)) & \sqrt{\kappa}^{-1}\sin(\sqrt{\kappa}(t-s)) \\ -\sqrt{\kappa}\sin(\sqrt{\kappa}(t-s)) & \cos(\sqrt{\kappa}(t-s)) \end{bmatrix} (0, m^{-1}f(t))\mathrm{d}s . \end{aligned}$$

In particular,

$$x(t) = x_0\cos(\sqrt{\kappa}(t-t_0)) + \frac{y_0}{\sqrt{\kappa}}\sin(\sqrt{\kappa}(t-t_0)) + \frac{1}{m\sqrt{\kappa}}\int_{t_0}^{t}\sin(\sqrt{\kappa}(t-s))f(s)\mathrm{d}s \tag{4.5}$$

is the unique solution of (4.3) with $x(t_0) = x_0$ and $x'(t_0) = y_0$.

To go further, we need an explicit formula for the driving force f. An interesting case arises when this too is periodic with some frequency η. (Thus, the period is $2\pi/\eta$.) Now suppose that

$$f(t) = \alpha\cos(\omega t + \phi_0) \tag{4.6}$$

where α is the amplitude, *ω is the* frequency *and ϕ_0 is the* phase shift. *We may assume without loos of generality that $\alpha > 0$ and $\omega > 0$.*

Recall that $\sin(A+B) = \sin(A)\cos(B) + \sin(B)\cos(A)$. *From this we obtain*

$$\sin(A)\cos(B) = \frac{1}{2}[\sin(A+B) + \sin(A-B)] \ .$$

Thus,

$$\sin(\sqrt{\kappa}(t-s))\cos(\omega s + \phi_0) = \frac{1}{2}[\sin(\sqrt{\kappa}t + \phi_0 - (\sqrt{\kappa} - \omega)s) + \sin(\sqrt{\kappa}t - \phi_0 - (\sqrt{\kappa} + \omega)s]$$

We can now easily do the integration from $t = 0$ to t. (Taking $t_0 = 0$ simplifies the following calculations, and we can change the time later once we have these computations are done.)

$$\begin{aligned}\int_0^t \sin(\sqrt{\kappa}(t-s))\cos(\omega s + \phi_0)\mathrm{d}s &= \frac{1}{2}\frac{1}{\sqrt{\kappa}-\omega}[\cos(\omega t + \phi_0) - \cos(\sqrt{\kappa}t + \phi_0)] \\ &+ \frac{1}{2}\frac{1}{\sqrt{\kappa}+\omega}[\cos(\omega t + \phi_0) - \cos(-\sqrt{\kappa}t + \phi_0)] \ ,\end{aligned}$$

where we assume for the time being that $\omega \neq \pm\sqrt{\kappa}$. (We shall remove this condition below.) To simplify the result, define

$$\eta := \frac{\sqrt{\kappa}+\omega}{2} \qquad \text{and} \qquad \xi := \frac{\sqrt{\kappa}-\omega}{2} \ .$$

Then

$$\begin{aligned}\cos(\omega t + \phi_0) - \cos(\sqrt{\kappa}t + \phi_0) &= \cos([\eta t + \phi_0] - \xi t) - \cos([\eta t + \phi_0] + \xi t) \\ &= 2\sin(\eta t + \phi_0)\sin(\xi t)\end{aligned}$$

where we have used

$$\cos(A-B) - \cos(A+B) = 2\sin(A)\sin(B) \ .$$

Therefore,

$$\frac{1}{2}\frac{1}{\sqrt{\kappa}-\omega}[\cos(\omega t + \phi_0) - \cos(\sqrt{\kappa}t + \phi_0)] = 2\sin(\eta t + \phi_0)\frac{\sin(\xi t)}{\xi} \ .$$

Likewise,

$$\begin{aligned}\cos(\omega t + \phi_0) - \cos(-\sqrt{\kappa}t + \phi_0) &= \cos([\xi t + \phi_0] - \eta t) - \cos([\xi t + \phi_0] + \eta t) \\ &= 2\sin(\phi_0 - \xi t)\sin(\eta t)\end{aligned}$$

Therefore,

$$\frac{1}{2}\frac{1}{\sqrt{\kappa}+\omega}[\cos(\omega t + \phi_0) - \cos(-\sqrt{\kappa}t + \phi_0)] = 2\sin(\phi_0 - \xi t)\frac{\sin(\eta t)}{\eta} \ .$$

Altogether, (4.5) in the case $t_) = 0$*, becomes*

$$\begin{aligned} x(t) &= x_0\cos(\sqrt{\kappa}t) + \frac{y_0}{\sqrt{\kappa}}\sin(\sqrt{\kappa}t) \\ &+ \frac{2\alpha}{m\sqrt{\kappa}}\left[\sin(\phi_0 - \xi t)\frac{\sin(\eta t)}{\eta} + \sin(\eta t + \phi_0)\frac{\sin(\xi t)}{\xi}\right] \ . \end{aligned} \tag{4.7}$$

If we define $\tilde{t} = t - t_0$*, so that* $t = t_0$ *corresponds to* $\tilde{t} = 0$*, the driving force becomes*

$$f(t) = \alpha\cos(\omega(\tilde{t} + t_0) + \phi_0) = \alpha\cos(\omega\tilde{t} + (\omega t_0 + \phi_0)) = \alpha\cos(\omega\tilde{t} + \widetilde{\phi}_0) \ ,$$

where $\widetilde{\phi}_0 = \omega t_0 + \phi_0$*. Thus, (4.7) is valid with* t *replaced by* $\tilde{t} = t - t_0$ *and* ϕ_0 *replaced by* $\widetilde{\phi}_0 = \omega t_0 + \phi_0$*. Making these substitutions, we obtain the formula for a general starting time.*

The previous example solves a very important equation. We shall soon learn how to decouple a general class of linear systems into parts that obey the equation we have just solved. This will provide much useful information on the behavior of vibrating mechanical systems.

As long as $\omega \neq \sqrt{\kappa}$, so that $\eta \neq 0$, the function in (4.7) is bounded. Consider the case in which ω is very close to $\sqrt{\kappa}$, but not equal to it. Then η is closes to zero, and the dominant term in the formula for $x(t)$ is the one with η in the denominator. We have

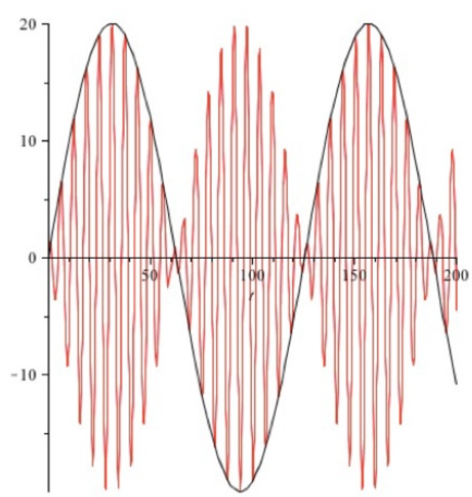
$$x(t) \approx \frac{2\alpha}{m\sqrt{\kappa}}\sin(\phi_0 - \xi t)\frac{\sin(\eta t)}{\eta} \tag{4.8}$$

If we plot this we see a high-frequency – ξ – sinusoidal oscillation with large amplitude (since η is in the denominator) that is *modulated* by a low frequency oscillation $\sin(\eta t)$. This produces the phenomenon of *beats*. If you strike two tuning forks with close frequencies, you will hear a very low frequency modulation of the ringing.

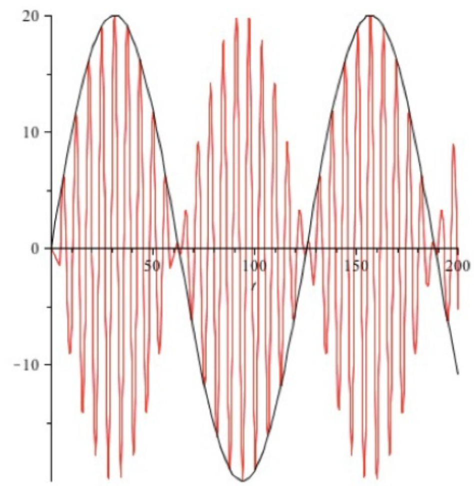
To get a visual understanding of this, let us look at some plots of $x(t)$ where we take $x_0 = y_0 = 0$, $\phi_0 = \pi/2m$, and $\alpha = \sqrt{\kappa}/2$ in (4.7) so that

$$x(t) = \cos(\xi t)\frac{\sin(\eta t)}{\eta} + \cos(\eta t)\frac{\sin(\xi t)}{\xi} \ .$$

The next plot shows $w(t)$ for $\xi = 1/20$ and $\eta = 1$, for $0 \leq t \leq 200$:

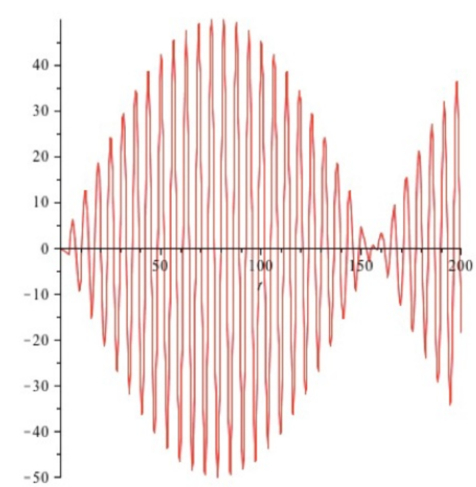


Also shown in the plot is the "envelope curve" $\frac{1}{20}\sin\left(\frac{1}{20}t\right)$. The second term in $w(t)$ dominates the first one, and if we leave the latter out, the plot is essentially unchanged:

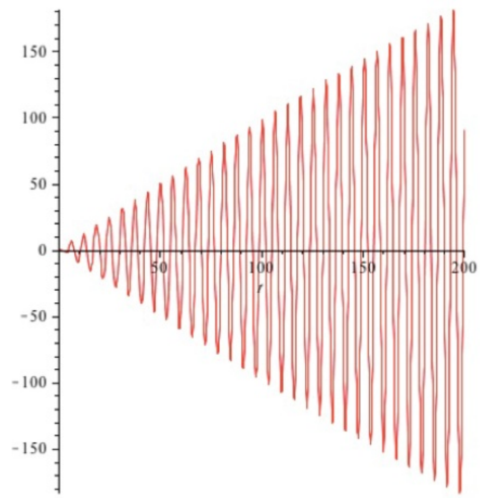


This is an illustrates the phenomenon of *beats* described above: You see here high-frequency (rapid) oscillation, modulated by a low frequency amplitude oscillation. If you listened to this pattern of vibrations played with the fast oscillations played in the audible range, you would hear this recency at a sinusoidally varying volume, with the volume modulations at the longer period. The volume modulations are called beats.

If we lower ξ further, both the amplitude and period of the beats increases. Here is the plot for $\eta = 1$ and $\xi = 1/50$:



Here is the plot for $\eta = 1$ and $\xi = 1/300$:



What we see in the last plot is very close to *resonance*. Resonance occurs when the driving frequency ω (or $-\omega$) is exactly equal to $\sqrt{\kappa}$. As ω approaches $\sqrt{\kappa}$, ξ approaches zero and η approaches $\sqrt{\kappa}$. Thus, by l'Hospital's rule,

$$\lim_{\omega\to\sqrt{\kappa}}[\cos(\xi t)\frac{\sin(\eta t)}{\eta}+\cos(\eta t)\frac{\sin(\xi t)}{\xi}]=\frac{\sin(\sqrt{\kappa}t)}{\sqrt{\kappa}}+t\cos(\sqrt{\kappa}t)\ .$$

The amplitude of the oscillations now grows linearly, without bound.

In an actual mechanical system, once the oscillation become large enough, some sot of breakdown will occur. For example, if you take a wine glass, and tap it with a fork, say, you will hear a "ping". The frequency of this ping is the *natural frequency* $\sqrt{\kappa}$ of the wine glass.

If you now turned on an oscillator producing sound at exactly (or nearly exactly) this frequency, the glass would begin to vibrate with larger and larger oscillations, and eventually shatter.

Suspension bridge failures have been caused by resonance. The first example on record occurred with the Broughton Suspension Bridge that was built in 1826 across the River Irwell near Manchester, England. This was among the first suspension bridges built anywhere in Europe. On April 12, 1831, troops of the $60th$ rifle Corps were marching across the bridge four abreast and marching in step – and unfortunately, the frequency of their march was one of the natural frequencies of the bridge. Resonance occurred, the oscillations built up – which the soldiers found amusing – and then, spoiling the amusement, the bridge collapsed, dumping 40 men into the river. Since this event, soldiers everywhere break cadence when crossing bridges.

A similar collapse occurred with the Tacoma Narrows Bridge November 7, 1940. In this case, wind caused some of the cables to vibrate at a natural frequency of certain twisting motions of the bridge. The resonance produced larger and larger twists, eventually destroying the bridge.

Of course to understand bridge collapse, we must look at *systems of coupled oscillators*, which we shall do later. The key method for analyzing these is to make a change of variables, which always exists, that decouples the system. Each decoupled component my be solved by the formula (4.7) we have derived here. Thus, we have already accomplished a significant part of building a methodology for analyzing vibrating mechanical systems.

4.1.2 Stability of equilibrium points

Let $\mathbf{v}$ be a *time independent* vector field defined on some open subset U of $\mathbb{R}^n$. In this subsection, we shall suppose that $\mathbf{v}$ is continuously differentiable. Suppose that $\mathbf{x}_\star$ is an equilibrium point of $\mathbf{v}$. That is, $\mathbf{v}(\mathbf{x}_\star) = \mathbf{0}$. Associated to this equilibrium point is the steady state solution of $\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t))$ with $\mathbf{x}(t) = \mathbf{x}_\star$ for all t. Since $\mathbf{v}(\mathbf{x})$ is Lipschitz on an open set around $\mathbf{x}_\star$ (since $\mathbf{v}$ is continuously differentiable), it follows that for any t_0, the equilibrium solution is the *only* solution to $\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t))$ with $\mathbf{x}(t_0) = \mathbf{x}_\star$.

We shall now analyze the *stability* of this equilibrium point. Roughly speaking, the equilibrium point $\mathbf{x}_\star$ is stable in case for all $\mathbf{x}_0$ sufficiently close to $\mathbf{x}_\star$, the solution of $\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t))$ with $\mathbf{x}(t_0) = \mathbf{x}_0$ is such that $\mathbf{x}(t)$ remains close to $\mathbf{x}_\star$ for all $t > t_0$, or, better yet, not only stays close, but satisfies $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}_\star$. We now make this precise.

Definition 16 (Stability)**.** *An equilibrium point $\mathbf{x}_\star$ of a vector field $\mathbf{v}$ is* Lyapunov stable *in case for every $\epsilon > 0$, there is $\delta > 0$ so that whenever $\mathbf{x}(t)$ solves $\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t))$ with $\|\mathbf{x}(t_0) - \mathbf{x}_\star\| < \delta$, then the solution exists for all $t > t_0$ and*

$$\|\mathbf{x}(t) - \mathbf{x}_\star\| < \epsilon$$

for all $t > t_0$.

The equilibrium point $\mathbf{x}_\star$ is asymptotically stable *if it is Lyapunov stable and moreover if there is a $\delta > 0$ such that whenever $\mathbf{x}(t)$ solves $\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t))$ with $\|\mathbf{x}(t_0) - \mathbf{x}_\star\| < \delta$,*

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}_\star \ .$$

Our first examples concern the case in which $\mathbf{v}(\mathbf{x})$ is linear, so that for some $n \times n$ matrix A, $\mathbf{v}(\mathbf{x}) = A\mathbf{x}$. For any $n \times n$ matrix, $A\mathbf{0} = \mathbf{0}$, so that $\mathbf{x}_\star = \mathbf{0}$ is always an equilibrium point for any linear vector field. Whenever A is invertible, the only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$ so that in this case $\mathbf{0}$ is the only equilibrium point. Let us now analyze the stability of $\mathbf{x}_\star = \mathbf{0}$ for a linear vector field $A\mathbf{x}$.

Example 36. *Once more, consider $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. As we have seen, in this case*

$$e^{tA} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \ .$$

Therefore, the solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$ is

$$\mathbf{x}(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \mathbf{x}_0 \ .$$

Since the transformation generated by e^{tA} is a rotation, $\|e^{tA}\mathbf{x}_0\| = \|\mathbf{x}_0\|$ for all t, and hence for all solutions, $\|\mathbf{x}(t)\| = \|\mathbf{x}_0\|$. That is,

$$\|\mathbf{x}(t) - \mathbf{0}\| = \|\mathbf{x}_0 - \mathbf{0}\| \ . \tag{4.9}$$

Now choose any $\epsilon > 0$, and take $\delta = \epsilon$. Then we see that whenever $\|\mathbf{x}_0 - \mathbf{0}\| < \delta$, $\|\mathbf{x}(t) - \mathbf{0}\| < \epsilon$ for all $t > 0$ (and in this case, for all t). Therefore, the equilibrium point $\mathbf{0}$ is Lyapunov stable in *this case. However, it is not asymptotically stable: We see from (4.9) that the only case in which* $\lim_{t\to\infty} \|\mathbf{x}(t) - \mathbf{0}\| = 0$ *is that in which* $\mathbf{x}_0 = \mathbf{0}$.

Example 37. *Consider* $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. *In this case*

$$e^{tA} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{bmatrix} .$$

Therefore, the solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$ *with* $\mathbf{x}(0) = \mathbf{x}_0$ *is* $\mathbf{x}(t) = e^{-t}\mathbf{x}_0$. *It follows that*

$$\|\mathbf{x}(t) - \mathbf{0}\| = e^{-t}\|\mathbf{x}_0 - \mathbf{0}\| . \tag{4.10}$$

In particular, $\|\mathbf{x}(t) - \mathbf{0}\|$ *is monotone decreasing so that the equilibrium point* $\mathbf{0}$ *is Lyapunov stable. Moreover,* $\lim_{t\to\infty} \|\mathbf{x}(t) - \mathbf{0}\| = 0$, *so that it is also asymptotically stable.*

There is a much more general version of this. We have seen that if A *is any* $n \times n$ *matrix all of whose eigenvalues have strictly negative real parts that for some* $a > 0$, *and* $C < \infty$,

$$\|e^{tA}\|_{\mathrm{F}} \le C^{-ta} .$$

It follows that in this case, for all $\mathbf{x}_0 \in \mathbb{R}^n$, *the solution of* $\mathbf{x}'(t) = A\mathbf{x}(t)$ *with* $\mathbf{x}(0) = \mathbf{x}_0$ *satisfies*

$$\|\mathbf{x}(t)\| \le Ce^{-ta}\|\mathbf{x}_0\| .$$

Let $\epsilon > 0$, *and take* $\delta = \epsilon/C$. *Then if* $\|\mathbf{x}_0\| < \delta$,

$$\|\mathbf{x}(t) - \mathbf{0}\| \le Ce^{-ta}\frac{\epsilon}{C} \le \epsilon$$

for all $t > 0$. *Thus,* $\mathbf{0}$ *is Lyapunov stable. Moreover,*

$$\lim_{t\to\infty} \|\mathbf{x}(t) - \mathbf{0}\| = 0 ,$$

and so $\mathbf{0}$ *is also asymptotically stable. On the other hand, if there is any eigenvalue that does not have a strictly negative real part, there is a solution that does not tend to zero as* t *tends to infinity. Thus, for linear systems,* $\mathbf{0}$ *is asymptotically stable if and only if all of the eigenvalues of* A *have a strictly negative real part.*

Example 38. *Consider* $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. *In this case*

$$e^{tA} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{bmatrix} .$$

Therefore, the solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$ *with* $\mathbf{x}(0) = \mathbf{x}_0$ *is* $\mathbf{x}(t) = (e^{-t}x_0, e^{t}y_0)$. *It follows that of* $\mathbf{y}_0 \neq 0$,

$$\lim_{t\to\infty} \|\mathbf{x}(t) - \mathbf{0}\| = \infty , \tag{4.11}$$

so that the equilibrium point $\mathbf{0}$ *is not even Lyapunov stable, let alone asymptotically stable.*

We now turn to the analysis of non-linear systems. Our main goal is to give conditions for asymptotic stability by relating solutions of non-linear systems in the vicinity of an equilibrium point to driven linear systems. To do this we *linearize*.

For $\mathbf{x}$ close to $\mathbf{x}_\star$, we have

$$\mathbf{v}(\mathbf{x}) \approx \mathbf{v}(v_0) + [D_{\mathbf{v}}(\mathbf{x}_\star)](\mathbf{x} - \mathbf{x}_\star) = [D_{\mathbf{v}}(\mathbf{x}_\star)](\mathbf{x} - \mathbf{x}_\star)$$

since $\mathbf{v}(\mathbf{x}_\star) = \mathbf{0}$.

The Jacobian matrix $[D_{\mathbf{v}}(\mathbf{x}_\star)]$ plays the key role in the analysis of the *stability* of the equilibrium point $\mathbf{x}_\star$, as we now explain. To simplify our notation, let us define

$$A = [D_{\mathbf{v}}(\mathbf{x}_\star)] \ .$$

Then for all $\mathbf{x}$ close to $\mathbf{x}_\star$, $\mathbf{v}(\mathbf{x}) \approx A(\mathbf{x} - \mathbf{x}_\star)$.

Let us suppose that $\mathbf{x}(t)$ is any solution of $\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t))$. We introduce $\mathbf{y}(t) = \mathbf{x}(t) - \mathbf{x}_\star$. Since $\mathbf{x}_\star$ is constant,

$$\mathbf{y}'(t) = \mathbf{x}'(t) = \mathbf{v}(\mathbf{x}) \approx A(\mathbf{x} - \mathbf{x}_\star) = A\mathbf{y}(t) \ .$$

That is, for $\mathbf{x}(t)$ close to the equilibrium point $\mathbf{x}_\star$, the relative position $\mathbf{y}(t) = \mathbf{x}(t) - \mathbf{x}_\star$ is an approximate solution of the differential equation

$$\mathbf{z}'(t) = A\mathbf{z}(t) \ . \tag{4.12}$$

This linear system is the *linearization* of the non-linear system $\mathbf{x}'(t) = \mathbf{x}(\mathbf{x}(t))$ around the equilibrium point $\mathbf{x} = \mathbf{x}_\star$. We have seen in Example 37 that $\mathbf{0}$ is asymptotically stable for (4.12) if and only if all of the eigenvalues of A have a strictly negative real part.

One might hope that the non-linear system would be asymptotically stable if and only if its linearization is asymptotically stable. This turns out to be the case.

Theorem 18. *Let $\mathbf{v}$ be a continuously differentiable vector field defined on an open set U in $\mathbb{R}^n$. Suppose that $\mathbf{x}_\star \in U$ is an equilibrium point for $\mathbf{v}$. Let $A = [D_{\mathbf{v}}(\mathbf{x}_\star)]$. Then $\mathbf{x}_\star$ is asymptotically stable if and only if all of the eigenvalues of A have strictly negative real parts.*

Proof. Suppose that all of the eigenvalues of A have strictly negative real parts. Let $\mathbf{x}(t)$ solve $\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t))$ with $\mathbf{x}(0) = \mathbf{x}_0$. Let $\mathbf{y}(t) = \mathbf{x}(t) - \mathbf{x}_\star$. Then

$$\mathbf{y}'(t) = \mathbf{x}'(t) = \mathbf{v}(\mathbf{x}_\star + \mathbf{y}(t)) \qquad \text{with} \qquad \mathbf{y}(0) = \mathbf{x}_0 - \mathbf{x}_\star \ . \tag{4.13}$$

Define $\mathbf{r}(t)$ by

$$\mathbf{v}(\mathbf{x}_\star + \mathbf{y}(t)) = A\mathbf{y}(t) + \mathbf{r}(t) \ . \tag{4.14}$$

Altogether, $\mathbf{y}(t)$ satisfies the driven linear system

$$\mathbf{y}'(t) = A\mathbf{y}(t)) + \mathbf{r}(t) \qquad \text{with} \qquad \mathbf{y}(0) = \mathbf{y}_0 = \mathbf{x}_0 - \mathbf{x}_\star \ , \tag{4.15}$$

and therefore by Duhamel's Formula,

$$\mathbf{y}(t) = e^{tA}\mathbf{y}_0 + \int_0^t e^{(t-s)A}\mathbf{r}(s)\mathrm{d}s \ . \tag{4.16}$$

We now show that as long as $\mathbf{y}(t)$ stays small, the driving term $\mathbf{r}(t)$ stays really small. By the definition of differentiability of vector valued functions, for all $\epsilon > 0$, there is a $\delta > 0$ so that

$$\|\mathbf{x} - \mathbf{x}_\star\| \leq \delta \quad \Rightarrow \qquad \|\mathbf{v}(\mathbf{x}) - A(\mathbf{x} - \mathbf{x}_\star)\| \leq \epsilon\|\mathbf{x} - \mathbf{x}_\star\| \ . \tag{4.17}$$

Since (4.17) can be written as

$$\|\mathbf{y}\| \leq \delta \quad \Rightarrow \qquad \|\mathbf{v}(\mathbf{x}_\star + \mathbf{y}) - A\mathbf{y}\| \leq \epsilon\|\mathbf{y}\| \ . \tag{4.18}$$

we have that

$$\|\mathbf{r}(t)\| \leq \epsilon\|\mathbf{y}(t)\| \tag{4.19}$$

as long as $\|\mathbf{y}(t)\| \leq \delta$.

Since all of the eigenvalues of A have strictly negative real parts, there is an $a > 0$ and a $C < \infty$ so that $\|e^{tA}\|_{\mathrm{F}} \leq Ce^{-ta}$. We will apply the bounds we have derived with the choice

$$\epsilon = \frac{a}{3C} \ . \tag{4.20}$$

Let $1 \geq \delta > 0$ be chosen so that (4.17) is valid for this choice of ϵ.

Consider initial data with

$$\|\mathbf{y}_0\| \leq \frac{\delta}{4} \ . \tag{4.21}$$

We claim that the solution satisfies $\|\mathbf{y}(t)\| < 3\delta/4$ for all $t > 0$. If this is not the case, there is some time $t > 0$ for which $\mathbf{y}(t) = 3\delta/4$. Then, by continuity, there is a first such time. Thus, under our hypothesis, we may suppose that $\|\mathbf{y}(t)\| = 3\delta/4$, but $\|\mathbf{y}(s)\| \leq 3\delta/4$ for all $0 \leq s \leq t$. We shall show this leads to contradiction.

Applying our bounds to (4.16) we obtain, using Minkowski's inequality in the first line,

$$\begin{aligned} \|\mathbf{y}(t)\| &\leq Ce^{-ta}\|\mathbf{y}_0\| + C\int_0^t e^{-(t-s)a}\|\mathbf{r}(s)\|\mathrm{d}s \\ &\leq Ce^{-ta}\|\mathbf{y}_0\| + \epsilon Ce^{-ta}\int_0^t e^{-(t-s)a}\|\mathbf{y}(s)\|\mathrm{d}s \end{aligned} \tag{4.22}$$

where we have used (4.19) in the last line.

Since by the choice of t, $\|\mathbf{y}(s)\| \leq 3\delta/4$ for all $0 \leq s \leq t$,

$$\int_0^t e^{-(t-s)a}\|\mathbf{y}(s)\|\mathrm{d}s \leq \frac{3\delta}{4}\int_0^t e^{-a(t-s)}\mathrm{d}s \leq \frac{3\delta}{4}\int_0^\infty e^{-as}\mathrm{d}s = \frac{3\delta}{4a} \ .$$

Altogether,

$$\|\mathbf{y}(t)\| \leq C\|\mathbf{y}_0\| + \epsilon C\frac{3\delta}{4a} \ .$$

Combining with (4.20) and (4.21), we obtain

$$\|\mathbf{y}(t)\| \leq \frac{\delta}{2} \ .$$

This contradicts $\|\mathbf{y}(t)\| = 3\delta/4$. Hence $\|\mathbf{y}(t)\| < 3\delta/4$ for all $t > 0$. In particular, (4.19) and (4.22) are valid for all $t > 0$, so that

$$\|\mathbf{y}(t)\| \leq Ce^{-ta}\|\mathbf{y}_0\| + \frac{a}{3}\int_0^t e^{-(t-s)a}\|\mathbf{y}(s)\|\mathrm{d}s \ . \tag{4.23}$$

Now define

$$y(t) = e^{ta/2}\|\mathbf{y}(t)\|$$

so that $\|\mathbf{y}(t)\| = e^{-ta/2}y(t)$. Inserting this in (4.23) we obtain

$$y(t) \le Ce^{-ta/2}\|\mathbf{y}_0\| + \frac{a}{3}\int_0^t e^{-(t-s)a/2}y(s)\mathrm{d}s \ . \tag{4.24}$$

Now fix any $T > 0$, and suppose that the maximum of $y(t)$ on $[0,T]$ occurs at $t = t_1$. Then

$$\begin{aligned} \max_{0\le t\le T} y(t) = y(t_1) &\le Ce^{-t_1a/2}\|\mathbf{y}_0\| + \frac{a}{3}\int_0^{t_1} e^{-(t_1-s)a/2}y(s)\mathrm{d}s \\ &\le C\|\mathbf{y}_0\| + \left(\max_{0\le t\le T} y(t)\right)\frac{a}{3}\int_0^{t_1} e^{-(t_1-s)a/2}\mathrm{d}s \\ &\le C\|\mathbf{y}_0\| + \frac{2}{3}\left(\max_{0\le t\le T}\|\mathbf{y}(t)\|\right) \end{aligned}$$

We conclude that for all $T > 0$,

$$\max_{0\le t\le T} y(t) \le 3C\|\mathbf{y}_0\| \ ,$$

provided (4.21) is satisfied, and recalling the definition of $y(t)$, we see that when (4.21) is satisfied,

$$\|\mathbf{y}(t)\| \le e^{-ta/2}3C\|\mathbf{y}_0\| \ .$$

That is, whenever $\|\mathbf{x}_0 - \mathbf{x}_\star\| \le \delta/4$, then $\|\mathbf{x}(t) - \mathbf{x}_\star\| \le e^{-ta/2}3C\|\mathbf{x}_0 - \mathbf{x}_\star\|$ for all $t > 0$. This proves that $\mathbf{x}_\star$ is asymptotic stability: Any solution that starts near $\mathbf{x}_\star$ stays near $\mathbf{x}_\star$, and converges to $\mathbf{x}_\star$. In fact, we have shown that the convergence is exponentially fast.

□

Example 39. *Consider the vector field*

$$\mathbf{v}(x,y) = ((y-x)(1-x-y) \ , \ x(1/2+y)) \ .$$

The first component equals zero along the linear $y = x$ and $x + y = 1$. The second component equals zero along the lines $x = 0$ and $y = -1/2$. We have an equilibrium point exactly where either of the first two lines intersects either of the second two, since then both components will be zero.

Hence, there are four equilibrium points:

$$(0,0) \ , \quad (0,1) \ , \quad (-1/2,-1/2) \ , \quad (3/2,-1/2) \ .$$

The Jacobian matrix of $\mathbf{v}$ at $\mathbf{x}$ is

$$[D_{\mathbf{v}}(\mathbf{x})] = \begin{bmatrix} 2x-1 & 1-2y \\ \frac{1}{2}+y & x \end{bmatrix} .$$

Evaluating $[D_{\mathbf{v}}(\mathbf{x})]$ at $\mathbf{x} = (0,0)$, we find

$$[D_{\mathbf{v}}(0,0)] = \begin{bmatrix} -1 & 1 \\ \frac{1}{2} & 0 \end{bmatrix} .$$

The characteristic polynomial is

$$t^2 + t - \tfrac{1}{2} \ .$$

Therefore, the eigenvalues are

$$\mu_1 = \frac{\sqrt{3}-1}{2} \qquad \text{and} \qquad \mu_2 = -\frac{\sqrt{3}+1}{2} \ .$$

Both are real, and $\mu_1 > 0$, so this equilibrium point is unstable.

Evaluating $[D_{\mathbf{v}}(\mathbf{x})]$ at $\mathbf{x} = (0,1)$, we find

$$[D_{\mathbf{v}}(0,1)] = \begin{bmatrix} -1 & 1 \\ \frac{1}{2} & 0 \end{bmatrix} \ .$$

The characteristic polynomial is

$$t^2 + t + \tfrac{3}{2} \ .$$

Therefore, the eigenvalues are

$$\mu_1 = -\frac{1+i\sqrt{5}}{2} \qquad \text{and} \qquad \mu_2 = -\frac{1-i\sqrt{5}}{2} \ .$$

Both are complex with imaginary part $-\frac{1}{2}$, so this equilibrium point is asymptotically unstable.

Evaluating $[D_{\mathbf{v}}(\mathbf{x})]$ at $\mathbf{x} = (-1/2, -1/2)$, we find

$$[D_{\mathbf{v}}(0,0)] = \begin{bmatrix} -2 & 2 \\ 0 & -\frac{1}{2} \end{bmatrix} \ .$$

The characteristic polynomial is

$$t^2 + \frac{5}{2}t + 1 \ .$$

Therefore, the eigenvalues are

$$\mu_1 = -2 \qquad \text{and} \qquad \mu_2 = -\tfrac{1}{2} \ .$$

Both are real and negative, so this equilibrium point is asymptotically unstable.

Evaluating $[D_{\mathbf{v}}(\mathbf{x})]$ at $\mathbf{x} = (3/2, -1/2)$, we find

$$[D_{\mathbf{v}}(0,0)] = \begin{bmatrix} 2 & 2 \\ 0 & \frac{3}{2} \end{bmatrix} \ .$$

The characteristic polynomial is

$$t^2 - \frac{7}{2}t + 3 \ .$$

Therefore, the eigenvalues are

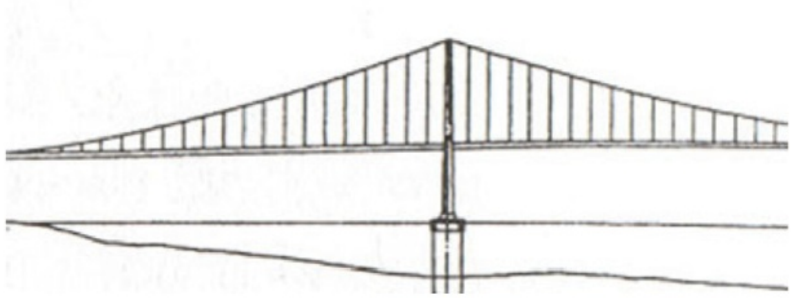
$$\mu_1 = 2 \qquad \text{and} \qquad \mu_2 = \tfrac{3}{2} \ .$$

Both are real and positive, so this equilibrium point is unstable.

4.2 Oscilations of mechanical systems

The methods we have been studying provide an effective means for analyzing mechanical systems near equilibrium, either freely oscilating, or being driven by periodic driving forces.

The systems we shall study typically require a large number of coordinates for their description. For instance, we might be considering a suspension bridge, and then we would like to keep track of the position in $\mathbb{R}^3$ of each of the points at both ends of each cable.



As you see, this will require a large number of coordinates. We can combine them all into one vector $\mathbf{x} \in \mathbb{R}^n$, for large n. The vector $\mathbf{x}$ describes the *configuration* of the system.

In this subsection we shall consider *conservative* mechanical systems. There are mecahanical systems in which the force $\mathbf{F}(\mathbf{x})$ is minus the gradient of a *potential energy function* $V(\mathbf{x})$, so that

$$\mathbf{F}(\mathbf{x}) = -\nabla V(\mathbf{x}) \ . \tag{4.25}$$

The reason for the name is that for such systems, the *total energy* is conserved, as we explain below.

Let M be a "mass matrix", which you can think of as being a diagonal $n \times n$ matrix whose diagonal entries are the masses associated to the various coordinate pointsl. More generally, for systems described by *Lagrangian dynamics*, M is a symmetric $n \times n$ matrix all of whose eigenvalues are strictly positive. We shall not go into Lagrangian dynamics now, but shall introduce it when we come to the Calculus of Variations later in the course. In the meantime, we shall take the applicability of the equation (4.26) for granted, and shall study its solutions, together with those of a related equation that includes the effects of periodic external driving forces.

Newton's Second Law, mass times accelleration equals force, together with the force specified in (4.25) give us

$$M\mathbf{x}''(t) = -\nabla V(\mathbf{x}(t) \ . \tag{4.26}$$

Define the *Hamiltonian function*, or *energy function*, $H(\mathbf{x}, \mathbf{y})$ on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$H(\mathbf{x}, \mathbf{y}) = \frac{1}{2}\mathbf{y} \cdot M\mathbf{y} + V(\mathbf{x}) \ , \tag{4.27}$$

Then for any solution $\mathbf{x}(t)$ of (4.26), using the symmetry of M, the chain rule,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} H(\mathbf{x}(t), \mathbf{x}'(t)) &= \frac{1}{2}\mathbf{x}''(t() \cdot M\mathbf{x}'(t)\frac{1}{2}\mathbf{x}'(t() \cdot M\mathbf{x}''(t) + \nabla V(\mathbf{x}(t)) \cdot \mathbf{x}'(t) \\ &= [M\mathbf{x}''(t) + \nabla V(\mathbf{x}(t))] \cdot \mathbf{x}'(t) = 0 \ . \end{aligned}$$

Therefore, for any solution of (4.26),

$$H(\mathbf{x}(t), \mathbf{x}'(t)) = H(\mathbf{x}(0), \mathbf{x}'(0))$$

for all t. This *constant of the motion* is known the *energy.* The fact that it is 'conserved' is the origin of the terminology 'conervative system'.

Now suppose that $\mathbf{x}_\star$ is a strict local minimum of V. Then every solution of (4.26) with $\mathbf{x}(0) = \mathbf{x}_\star$ and $\mathbf{x}'(0) = \mathbf{0}$ is starting at a strict local minimum of H, and therefore must remain at this strict local minimum. That is. the equulibrium solution

$$\mathbf{x}(t) = \mathbf{x}_\star \quad \text{for all } t \tag{4.28}$$

is the only solution of (4.26) with $\mathbf{x}(0) = \mathbf{x}_\star$ and $\mathbf{x}'(0) = \mathbf{0}$

However, much more is true. Supose that

$$\mathbf{x}(0) \approx \mathbf{x}_\star \qquad \text{and} \qquad \mathbf{x}'(0) \approx \mathbf{0}\ .$$

Then

$$H(\mathbf{x}(0), \mathbf{x}'(0) \approx H(\mathbf{x}_\star, \mathbf{0}) = V(\mathbf{x}_\star)\ .$$

The conservation law then implies that

$$H(\mathbf{x}(t), x'(t)) = H(\mathbf{x}(0), \mathbf{x}'(0) \approx V(\mathbf{x}_\star)$$

for all t. This in turn implies that

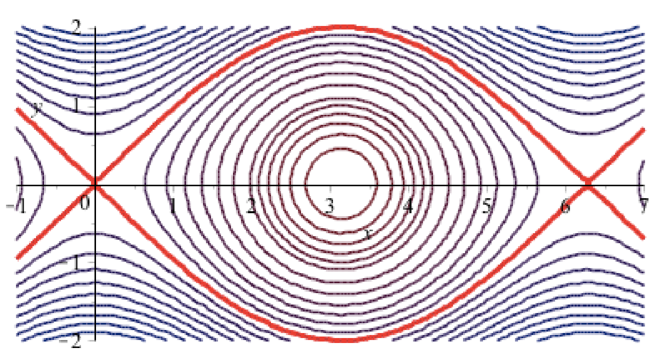
$$\mathbf{x}(t) \approx \mathbf{x}_\star \qquad \text{and} \qquad \mathbf{x}'(t) \approx \mathbf{0}$$

for *all* t. That is, the conservation law forces any solution starting near a strict local minimum of V with a sufficiently small initial velocity to stay near the staedy-state solution (4.28) for all time.

Example 40. *Let us consider a simple one dimensional case for whihc we can graphically illustrate the discussion just above. Let* $V(x) = \cos(x)$ *and consider unit mass. Then* $x_\star = \pi$ *is a strict local minimum, and*

$$H(x, y) = \frac{1}{2}y^2 + \cos x\ .$$

Here is a contour plot of the Hamiltonian:



The minimum value of H is -1 which is achieved at $x_\star = \pi$, and at every odd integer multiple of π. While H itself is unbounded above, the potential energy *has maxima when x is an even integer multiple of π.*

Since $\nabla V(\mathbf{x}) = \mathbf{0}$ at local maxima as well as local minima of V, for any integer k, $x(t) = 2k\pi$ for all t is a stationary solution of

$$x'' = \sin x \ ,$$

which is the Newton equation for this simple system. Since $x'(t) = 0$ for all t for any such solution, these stationary solutions have the energy

$$H(2k\pi, 0) = 1 \ .$$

However, theres are not the only soltions with energy equal to 1. The contour curve given by $H(x,y) = 1$ is plotted boldly in red. At each of the equauilibrium points $(2k\pi, 0)$, two of whihc are shown, four *branches of this contour curve meet. The upper branch that is fully shown here has $y > 0$, and hence the motion is to the right along this branch. It runs between the equilibrium point at $(0,0)$ and the equilibrium point at $(2\pi, 0)$. The lower branch runs in the oppsoite direction.*

For all values of E with $-1 < E < 1$, the connected components of the contour curves of $H(x,y)$ are simple closed curves. For values of E close to -1, they are nearly circles. As the value of E appoaches $+1$, the become more elongated, and evetually merge with the red curves. For $E > 1$, the curves are not closed: They oscilate up and down but run along above or below the whole length of the x-axis.

The important point to notice is that if $(x(0), x'(0))$ is close to $(\pi, 0)$, then the motion takes place on a small closed curve about $(\pi, 0)$ that is very close to circular, the more so the closer $(x(0), x'(0))$ is to $(\pi, 0)$. Thus, $(x(t), x'(t))$ stays close to $(\pi, 0)$ for all t.

What we have seen in the previous example is quite general: If $\mathbf{x}_\star$ is a strict local minimum of $V(\mathbf{x})$, then solutions of (4.26) with $(\mathbf{x}(0), \mathbf{x}'(0))$ in a sufficiently small neighborhood of $(\mathbf{x}_\star, \mathbf{0})$ will remain in a small neighborhood of $(\mathbf{x}_\star, \mathbf{0})$ for all t.

Within this small nieghboorhod, we can effectively approximate (4.26) by a linear system that an be solved explicitly. As we shall see, this yields much useful information about the exact solutions of (4.26). The approximation process is known as *linearization.*

4.2.1 Linearization of Newton's equations near a potential energy minimum

Let $\mathbf{x}_\star$ be a strict local minimum of V, and suppose that V is twice continuously differentiable. Introduce a new variable $\mathbf{z}(t) = \mathbf{x}(t) - \mathbf{x}_\star$, which measures the displacement from equilibrium. Then since $\mathbf{x}_\star$ is constant,

$$\mathbf{z}''(t) = \mathbf{x}''(t) = -\nabla V(\mathbf{x}(t)) = -\nabla V(\mathbf{z}(t) + \mathbf{x}_\star) \ .$$

By the definition of the Hessian matrix, and the fact that $\nabla V(\mathbf{x}_\star) = \mathbf{0}$,

$$\nabla V(\mathbf{z} + \mathbf{x}_\star) = [\mathrm{Hess}_V(\mathbf{x}_\star)]\mathbf{z} + o(\|\mathbf{z}\|) \ .$$

The approximation that consist of ignoring the $o(\|\mathbf{z}\|)$ remainder term is known as *linearization*: Linearizing about $\mathbf{x}_\star$, we have

$$\nabla V(\mathbf{z}(t) + \mathbf{x}_\star) \approx [\mathrm{Hess}_V(\mathbf{x}_\star)]\mathbf{z}(t) \ ,$$

where $[\mathrm{Hess}_V(\mathbf{x}_\star)]$ is the Hessian matrix of V at $\mathbf{x}_\star$.

Recall that the Hessian matrix of a twice continuously differentiable function is always symmetric. Recall that the Spectral Theorem says that for every symmetric $n \times n$ matrix, there is an orthonormal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ of $\mathbb{R}^n$ consisting of eigenvectors of that matrix.

We have already assumed that $\mathbf{x}_\star$ is a strict local minimum of V. This implies in particular that all eigenvalues of $[\mathrm{Hess}_V(\mathbf{x}_\star)]$ are non-negative. We shall generally require a more robust form of stability, namely, that all of the eigenvalues of $[\mathrm{Hess}_V(\mathbf{x}_\star)]$ are *strictly* positive. This brings us to a basic definition:

Definition 17 (Positive definite matrix)**.** *An $n \times n$ matrix A is* positive define *in case it is symmetric, and for all $\mathbf{x} \neq \mathbf{0}$,*

$$\mathbf{x} \cdot A\mathbf{x} > 0 \ .$$

The next theorem provides a means to check whether a matrix is positive definite – by computing its eigenvalues.

Theorem 19. *A symmetric $n \times n$ matrix M is positive definite if an only if each of its eigenvalues is positive.*

Proof. Suppose M is positive definite. Let $\mathbf{u}$ be an eigenvector of M so that $M\mathbf{u} = \mu\mathbf{u}$ for some $\mu \in \mathbb{R}$. We must show that $\mu > 0$. Since M is positive definite, and since $\mathbf{u} \neq \mathbf{0}$ by virtue of being an eigenvector, $\mathbf{u} \cdot M\mathbf{u} > 0$. But

$$\mathbf{u} \cdot M\mathbf{u} = \mathbf{u} \cdot \mu\mathbf{u} = \mu\|\mathbf{u}\|^2 > 0$$

and so $\mu > 0$ as was to be shown.

Next, suppose that each of the eigenvalues of M is strictly positive. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvecotrs of M; we shall write $A\mathbf{u}_j = \mu_j\mathbf{u}_j$, $j = 1, \dots, n$. Next, for any $\mathbf{x} \in \mathbb{R}^n$,

$$\mathbf{x} = \sum_{j=1}^{n} (\mathbf{u}_j \cdot \mathbf{x})\mathbf{u}_j \ .$$

Thus,

$$\mathbf{x} \cdot M\mathbf{x} = \sum_{j=1}^{n} \mu_j(\mathbf{u}_j \cdot \mathbf{x})^2 \geq 0 \ ,$$

and since each μ_j is stricly positive, the sum is zero if and only if $\mathbf{u}_j \cdot \mathbf{x} = 0$ for each j, and this means $\mathbf{x} = \mathbf{0}$. □

Of course, any diagonal matrix with striclty positive entries is a positive definite matrix. It turns out that it is useful if we do not insist that the mass matrix M be diagonal, but allow it to be a general positive definite matrix. We now come to the basic equation in the theory of machanical vibrations:

Definition 18 (The mechanical vibration equation)**.** *Let M and A be $n\times n$ symmetric matrices, each with the property that all of its eigenvalues are strictly positive. The* mechanical vibration equation *is the second order linear equation in $\mathbb{R}^n$*

$$M\mathbf{z}''(t) = -A\mathbf{z}(t)\ . \tag{4.29}$$

It is in reduced form if $M = I_{n\times n}$, the $n\times n$ identity matrix.

We have explained how such equations arise: The matrix A will be the Hessian of a potential energy function at a local minimum $\mathbf{x}_\star$, and $\mathbf{z}$ specifies the difference between the current configuration, and the local minimum $\mathbf{x}_\star$. To proceed, we need a few simple facts about symmetric matrices with strictly positive eigenvlaues.

Lemma 4. *Let A and B be positive definite $n\times n$ matrices. Then $A+B$ is positive definite.*

Proof. Since A and B are symmetric, so is $A+B$. Moreover, for any $\mathbf{x}\neq\mathbf{0}$,

$$\mathbf{x}\cdot(A+B)\mathbf{x} = \mathbf{x}\cdot A\mathbf{x} + \mathbf{x}\cdot B\mathbf{x} > 0\ .$$

□

Lemma 5. *Let A be a positive definite $n\times n$ matrix. Then A is invertible. Moreover, if C is an other invertible $n\times n$ matrix, then C^tAC is positive definite.*

Proof. Suppose $A\mathbf{x} = 0$. Then $\mathbf{x}\cdot A\mathbf{x} = 0$, and this means that $\mathbf{x} = 0$. Thus, the only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$, and then by the Fundamental Theorem of Linear Algebra, A is invertible.

Next, if A is positive definite and C is invertible, then for any $\mathbf{x}\neq\mathbf{0}$,

$$\mathbf{x}\cdot(C^tAC)\mathbf{x} = (C\mathbf{x})\cdot A(C\mathbf{x}) > 0$$

since $C\mathbf{x}\neq\mathbf{0}$. □

The next theorem tells us that positive define matrices have a positive define *square root for matrix multiplication*, and tells us how to compute this matrix square root.

Theorem 20. *Let M be a positive definite $n\times n$ matrix. Then there is a unique positive definite matrix $M^{1/2}$ such that $(M^{1/2})^2 = M$. Moreover, let $\{\mathbf{u}_1,\dots,\mathbf{u}_n\}$ be any orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of M, let $U = [\mathbf{u}_1,\dots,\mathbf{u}_n]$ and let $D^{1/2}$ be the $n\times n$ diagonal matrix whose jth diagonal entry is $\sqrt{m_j}$ where $M\mathbf{u}_j = m_j\mathbf{u}_j$. Then*

$$M^{1/2} = UD^{1/2}U^t\ . \tag{4.30}$$

Proof. Since the columns of U are orthonormal, $U^{-1} = U^t$. Therefore,

$$[UD^{1/2}U^t][UD^{1/2}U^t] = UD^2U^t$$

where the diagonal entries of D^2 are the eigenvalues of M. But then $M = UD^2U^t = UD^2U^{-1}$ is equivalend to $U^{-1}MU = D^2$ which is true whenver the columns of U are linearly independent eigenvectors of M, D^2 is diagonal, and the diagonal entries of D^2 are the corresponding eigenvlaues.

Thus we always have at least one positive definite square root which is given by (4.30). The uniqueness is left as an exercise. □

Example 41. *Let* $M = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$. *The characteristic polynomial of* M *is* $(t-25)(t-9)$ *so that the eigenvalues are* $m_1 = 25$ *and* $m_2 = 9$, *and thus,* M *is positive definite. The corresponding normalized eigenvectors are*

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}(1,1) \qquad \text{and} \qquad \mathbf{u}_1 = \frac{1}{\sqrt{2}}(1,-1)\ .$$

Therefore

$$D^{1/2} = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \qquad \text{and} \qquad U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\ .$$

Therefore,

$$M^{1/2} = UD^{1/2}U^t = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\ .$$

As you can check,

$$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}^2 = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}\ ,$$

so we have indeed computed a matrix square root. Acoording to the theorem, this is the only matrix square root that is itself positive definite.

4.2.2 Reduction to reduced form

We now explain how every equation of the form $M\mathbf{z}''(t) = -A\mathbf{z}(t)$ with M and A positive definite may be put in reduced form.

Multiply both sides of $M\mathbf{z}''(t) = -A\mathbf{z}(t)$ by $M^{-1/2}$ to obtain

$$M^{1/2}\mathbf{z}''(t) = -M^{-1/2}A\mathbf{z}(t)\ . \tag{4.31}$$

Next, since $M^{-1/2}M^{1/2}$ is the identity matrix, we may freely insert $M^{-1/2}M^{1/2}$ between A and $\mathbf{z}$ on the right. We obtain

$$M^{1/2}\mathbf{z}''(t) = -[M^{-1/2}AM^{-1/2}]M^{1/2}\mathbf{z}(t)\ . \tag{4.32}$$

Now let us define

$$\mathbf{y}(t) = M^{1/2}\mathbf{z}''(t) \qquad \text{and} \qquad K = [M^{-1/2}AM^{-1/2}]\ . \tag{4.33}$$

Then we have

$$\mathbf{y}''(t) = -K\mathbf{y}(t)\ . \tag{4.34}$$

Furthermore, all of the transformations we have made are invertible, and so (4.34) is equivalent to (4.29). Finally, by Lemma 5, K is positive definite.

We summarize:

Theorem 21. *Let* M *and* A *be positive definite* $n \times n$ *matrices. Let* $\mathbf{x}(t)$ *be a twice continuously differentiable curve, and define* $L = M^{-12/}AM^{-1/2}$ *and* $\mathbf{y}(t) = M^{1/2}\mathbf{x}(t)$. *Then*

$$M\mathbf{x}''(t) = -A\mathbf{x}(t) \quad \Longleftrightarrow \quad \mathbf{y}''(t) = -K\mathbf{y}(t)\ .$$

Example 42. *Consider the system*

$$\begin{aligned} 4x_1''(t) &= -8x_1(t) - 4x_2(t) \\ x_2''(t) &= -4x_1(t) - 5x_2(t) \end{aligned}$$

with the initial conditions

$$x_1(0) = -1\ ,\ x_2(0) = 1\ ,\ x_1'(0) = 1/2 \quad \text{and} \quad x_2'(0) = 2\ .$$

Introducing

$$M = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} 8 & 4 \\ 4 & 5 \end{bmatrix}\ ,$$

and $\mathbf{x}(t) = (x_1(t), x_2(t))$*, we can write this system as*

$$M\mathbf{x}''(t) = -A\mathbf{x}(t)\ .$$

In this case, it is easy to see that $M^{1/2} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$*, and so*

$$K = M^{-1/2}AM^{-1/2} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}\ .$$

Then with $\mathbf{y}(t) = M^{1/2}\mathbf{x}(t) = (2x_1(t), x_2(t))$*, we have*

$$\mathbf{y}''(t) = -K\mathbf{y}(t)\ ,$$

and the initial conditions transform to

$$\mathbf{y}(0) = (-2, 1) \quad \text{and} \quad \mathbf{y}'(0) = (1, 2)\ .$$

In the next subsection, we shall see how to solve systems of this type.

4.2.3 Normal modes

We are now in a position to decouple the system $\mathbf{y}'' = -K\mathbf{y}$ into n independent one-variable equations. To do this, let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of K. We denote the jth eigenvalue of K by κ_j; i.e.,

$$K\mathbf{u}_j = \kappa_j\mathbf{u}_j\ ,$$

for each $j = 1, \ldots, n$, and since K is positive definite, $\kappa_j > 0$ for each j.

Suppose that $\mathbf{y}''(t) = -K\mathbf{y}(t)$. Let us take the dot product of both sides with $\mathbf{u}_j$, On the left we get

$$\mathbf{u}_j \cdot \mathbf{y}''(t) = (\mathbf{u}_j \cdot \mathbf{y}(t))''\ .$$

On the right we get, using the transpose identity $\mathbf{x} \cdot B\mathbf{y} = B^t\mathbf{x} \cdot \mathbf{y}$ and the fact that K is symmetric,

$$-\mathbf{u}_j \cdot K\mathbf{y}(t) = -(K\mathbf{u}_j) \cdot \mathbf{y}(t) = -\kappa_j(\mathbf{u}_j \cdot \mathbf{y}(t))\ .$$

Thus, if we define

$$w_j(t) = \mathbf{u}_j \cdot \mathbf{y}(t) \ , \tag{4.35}$$

we have shown that

$$w_j''(t) = -\kappa_j w_j(t) \ , \tag{4.36}$$

Conversely, suppose that $\{w_1(t), \ldots, w_n(t)\}$ are such that (4.36) is satisfied for each $j = 1, \ldots, n$, Then, defining

$$\mathbf{y}(t) = \sum_{j=1}^{n} w_j(t)\mathbf{u}_j \ , \tag{4.37}$$

we have

$$\mathbf{y}''(t) = -K\mathbf{y}(t) \tag{4.38}$$

and

$$\mathbf{y}(0) = \sum_{j=1}^{n} w_j(0)\mathbf{u}_j \qquad \text{and} \qquad \mathbf{y}'(0) = \sum_{j=1}^{n} w_j'(0)\mathbf{u}_j \ . \tag{4.39}$$

Therefore, if we can solve the one-variable equation

$$w''(t) = -\kappa w(t) \qquad \text{with} \qquad w(0) = a \quad \text{and} \quad w(0) = b \ , \tag{4.40}$$

for $\kappa > 0$ and a and b arbitrary, we can solve

$$\mathbf{y}''(t) = -K\mathbf{y}(t) \qquad \text{with} \qquad \mathbf{y}(0) = \mathbf{a} \quad \text{and} \quad \mathbf{y}(0) = \mathbf{b} \ , \tag{4.41}$$

for K positive define and $\mathbf{a}$ and $\mathbf{b}$ arbitrary, and we obtain the solution in the form of a sum (4.37). The special solutions that are the terms of this sum are called *normal modes*, and the decomposition of the solution into a sum of these special solutions is called a *normal mode decomposition.* We refer to (4.40) as the *normal mode equation.*

Now let us solve the normal mode equation. We reduce it to a first order system, defining $v(t) = w'(t)$ and $\mathbf{w}(t) = (w(t), v(t))$, so that (4.40) is equivalent to

$$\mathbf{w}'(t) = \begin{bmatrix} 0 & 1 \\ -\kappa & 0 \end{bmatrix} \mathbf{w}(t) \qquad \text{with} \qquad \mathbf{w}(0) = \mathbf{w}_0 := (a, b) \ . \tag{4.42}$$

The characteristic polynomial of the matrix $L := \begin{bmatrix} 0 & 1 \\ -\kappa & 0 \end{bmatrix}$ is $t^2 + \kappa = 0$. This has the roots $\pm i\sqrt{\kappa}$. and so we find that $(1, i\sqrt{\kappa})$ is an eigenvector of L with eigenvalue $i\sqrt{\kappa}$. Thus,

$$e^{i\sqrt{\kappa}}(1, i\sqrt{\kappa})$$

is a complex solution of our equation, and the real and imaginary parts give us the two real solutions we need to compute e^{tL}. Carrying out the simple computations, we find

$$e^{tL} = \begin{bmatrix} \cos(\sqrt{\kappa}t) & \frac{1}{\sqrt{\kappa}}\sin(\sqrt{\kappa}t) \\ -\sqrt{\kappa}\sin(\sqrt{\kappa}t) & \cos(\sqrt{\kappa}t) \end{bmatrix} . \tag{4.43}$$

Therefore, the solution of (4.40) is

$$w(t) = a\cos(\sqrt{\kappa}t) + \frac{b}{\sqrt{\kappa}}\sin(\sqrt{\kappa}t) \ . \tag{4.44}$$

Example 43. *Let* $K = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$. *We shall use a normal modes decomposition to solve*

$$\mathbf{y}''(t) = -K\mathbf{y}(t) \qquad \text{with} \qquad \mathbf{y}(0) = (-2,1) \quad \text{and} \quad \mathbf{y}'(0) = (1,2)\ .$$

To find the normal modes, we first compute the eigenvalues. The characteristic polynomial is $t^2 - 7t + 6 = (t-6)(t-1)$ *so the eigenvalues are* $\mu_1 = 1$ *and* $\mu_2 = 6$. *we compute:*

$$K - I_{2\times 2} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\ .$$

Any vector orthogonal orthogonal to the rows of this matrix is an eigenvector with eigenvalue 1. *Let us choose the unit vector*

$$\mathbf{u}_1 = \frac{1}{\sqrt{5}}(-2,1)\ .$$

Since K *is symmetric, the orthogonal unit vector*

$$\mathbf{u}_2 = \frac{1}{\sqrt{5}}(1,2)$$

is an eigenvector with eigenvalue 6.

In the normal mode expansion

$$\mathbf{y}(t) = w_1(t)\mathbf{u}_1 + w_2(t)\mathbf{u}_2\ ,$$

$w_1(t)$ *is the solution of*

$$w_1''(t) = -w_1(t) \qquad \text{with} \qquad w_1(0) = \mathbf{u}_1 \cdot \mathbf{y}(0) = \sqrt{5} \qquad \text{and} \qquad \mathbf{u}_1 \cdot \mathbf{y}'(0) = 0\ ,$$

and $w_2(t)$ *is the solution of*

$$w_2''(t) = -6w_2(t) \qquad \text{with} \qquad w_2(0) = \mathbf{u}_2 \cdot \mathbf{y}(0) = 0 \qquad \text{and} \qquad \mathbf{u}_1 \cdot \mathbf{y}'(0) = \sqrt{5}\ ,$$

Then from (4.44) we have

$$w_1(t) = \sqrt{5}\cos(t) \qquad \text{and} \qquad w_2(t) = \sqrt{\frac{5}{6}}\sin(\sqrt{6}t)\ .$$

Finally,

$$\begin{aligned} \mathbf{y}(t) &= w_1(t)\mathbf{u}_1 + w_2(t)\mathbf{u}_2 \\ &= \cos(t)(-2,1) + \sqrt{6}\sin(\sqrt{6}t)(1,2) \\ &= (\sin(\sqrt{6}t) - 2\cos(t)\ ,\ 2\sin(\sqrt{6}t) + \cos(t)) \end{aligned}$$

Now recall that the system we have solved in this example is the reduced form of the system introduced in Example 42. To concert back to the original variables $(x_1(t), x_2(t))$, *all we need do is use*

$$\mathbf{x}(t) = M^{-1/2}\mathbf{y}(t)$$

where M *is the mass matrix* $M = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$ *from Example 42. We find*

$$(x_1(t), x_2(t)) = (\sin(\sqrt{6}t)/2 - \cos(t)\ ,\ 2\sin(\sqrt{6}t) + \cos(t))\ .$$

4.3 Driven oscillations of Mechanical Systems

4.3.1 The normal mode decomposition for a driven system

We now suppose that there is a time-dependent external force $\mathbf{f}(t)$ acting our mechanical system. The equations of motion become

$$M\mathbf{x}''(t) = -\nabla V(\mathbf{x}(t)) + \mathbf{f}(t) \ .$$

If we linearize about a local minimum $\mathbf{x}_\star$ of V, and let A denote the Hessian of V at $\mathbf{x}_\star$, we obtain that $\mathbf{z}(t0 = \mathbf{x}(t) - \mathbf{x}_\star$ satisfies

$$M\mathbf{z}''(t) = -A\mathbf{z}(t) + \mathbf{f}(t) \ .$$

Because $\mathbf{x}_\star$ is a local minimum, all of the eigenvlaues of A are non-negative, but as above, we shall assume a little more, namely that all of the eigenvalues are strictly positive.

Multiplying through by $M^{-1/2}$ as before, we obtain

$$M^{1/2}\mathbf{z}''(t) = -[M^{-1/2}AM^{-1/2}]M^{1/2}\mathbf{z}(t) = M^{1/2}\mathbf{f}(t) \ .$$

Therefore, defining

$$\mathbf{y}(t) := M^{-1/2}\mathbf{z}(t) \ , \quad K := M^{-1/2}AM^{-1/2} \quad \text{and} \quad \mathbf{g}(t) = M^{1/2}\mathbf{f}(t) \ ,$$

our equation is equivalent to

$$\mathbf{y}''(t) = -K\mathbf{y}(t) + \mathbf{g}(t) \ . \tag{4.45}$$

We shall reduce to normal modes, just as in the previous section. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be an orthonormal basis of eigenvectors of K with $K\mathbf{u}_j = \kappa_j\mathbf{u}_j$ for $j = 1, \dots, n$. Then, just as in the previous section, we can represent the solution to (4.45) in the form

$$\mathbf{y}(t) = \sum_{j=1}^{n} w_j(t)\mathbf{u}_j \tag{4.46}$$

where $w_j(t)$ solves

$$w_j''(t) = -\kappa_j w_j(t) + g_j(t) \tag{4.47}$$

where

$$g_j(t) = \mathbf{u}_j \cdot \mathbf{g}(t) \ .$$

We have already found a formula for the solution of

$$w''(t) = -\kappa w(t) + g(t) \qquad \text{with} \quad w(0) = a \quad \text{and} \quad w'(0) = b \ , \tag{4.48}$$

Indeed, this equation is, apart from a change in notation, nothing other that (4.3) and so we have from the solution of (4.5) that

$$w(t) = a\cos(\sqrt{\kappa}t) - \frac{b}{\sqrt{\kappa}}\sin(\sqrt{\kappa}t) + \frac{1}{\sqrt{\kappa}}\int_0^t \sin(\sqrt{\kappa}(t-s))g(s)\mathrm{d}s \ . \tag{4.49}$$

This formula is as far as we can go without specifying the forcing function $g(t)$.

4.3.2 Periodic forcing

Now suppose that

$$g(t) = \alpha\cos(\omega t + \phi_0) \tag{4.50}$$

where α is the *amplitude*, ω is the *frequency* and ϕ_0 is the *phase shift*.

We have already computed the the integrals in (4.49) for this choice of $g(t)$ in the first section of this chapter. Adapting the notation slightly, the result is that when the forcing is given by (4.49), the solution becomes

$$\begin{aligned} w(t) &= a\cos(\sqrt{\kappa}t) - \frac{b}{\sqrt{\kappa}}\sin(\sqrt{\kappa}t) \\ &+ \frac{2\alpha}{\sqrt{\kappa}}\left[\sin(\phi_0 - \xi t)\frac{\sin(\eta t)}{\eta} + \sin(\eta t + \phi_0)\frac{\sin(\xi t)}{\xi}\right] \end{aligned} \tag{4.51}$$

where

$$\eta := \frac{\sqrt{\kappa}+\omega}{2} \qquad \text{and} \qquad \xi := \frac{\sqrt{\kappa}-\omega}{2} \ .$$

Now let us put everything together, and solve the system

$$M\mathbf{x}''(t) = -\nabla V(\mathbf{x}(t)) + \mathbf{f}(t) \ ,$$

subject ot

$$\mathbf{x}(0) = \mathbf{x}_0 \qquad \text{and} \qquad \mathbf{x}'(0) = \mathbf{v}_0 \ . \tag{4.52}$$

We shall suppose that for some ficed vector $\mathbf{v}$,

$$\mathbf{f}(t) = \cos(\omega t + \phi_0)\mathbf{v} \ . \tag{4.53}$$

Then, reducing to normal form, we define $\mathbf{y}(t) = M^{1/2}\mathbf{x}(t)$, $\mathbf{g}(t) = M^{-1/2}\mathbf{f}(t)$ and $K = M^{-1/2}LM^{-1/2}$. The equivalent normal system is

$$\mathbf{y}''(t) = -K\mathbf{y}(t) + \mathbf{g}(t) \ ,$$

with

$$\mathbf{y}(0) = M^{1/2}\mathbf{x}_0 \qquad \text{and} \qquad \mathbf{x}'(0) = M^{-1/2}\mathbf{v}_0 \ .$$

We now make the normal mode expansion of $\mathbf{y}(t)$ to completely decopuple our system:

$$\mathbf{y}(t) = \sum_{j=1}^{n} w_j(t)\mathbf{u}_j \tag{4.54}$$

where $w_j(t)$ solves

$$w_j''(t) = -\kappa_j w_j(t) + g_j(t) \tag{4.55}$$

with

$$w_j(0) = \mathbf{u}_j \cdot M^{1/2}\mathbf{x}_0 \qquad \text{and} \qquad w_j'(0) = \mathbf{u}_j \cdot M^{1/2}\mathbf{v}_0 \ .$$

and where

$$g_j(t) = \mathbf{u}_j \cdot \mathbf{g}(t) = \mathbf{u}_j \cdot M^{-1/2}\mathbf{f}(t) = (\mathbf{u}_j \cdot M^{-1/2}\mathbf{v})\cos(\omega t + \phi_0) \ . \tag{4.56}$$

Notice that the quantity corresponding to α is $\mathbf{u}_j \cdot M^{-1/2}\mathbf{v}$. If $\mathbf{u}_j$ happens to be orthogonal to $M^{-1/2}\mathbf{v}$, the amplitude of forcing for the jth normal mode is zero, and there is no forcing on this mode. This, even if $\omega = \sqrt{\kappa_j}$, there will be no resonance for this mode if $\mathbf{u}_j \cdot M^{-1/2}\mathbf{v} = 0$. Once all of the $w_j(t)$ are computed, (4.54) can be summed to produce $\mathbf{y}(t)$. Finally, we have $\mathbf{x}(t) = M^{-1/2}\mathbf{y}(t)$.

We have just described a method for solving, by means of a general decoupling procedure, the system

$$M\mathbf{x}''(t) = -L\mathbf{x}(t) + \mathbf{f}(t)$$

in anynumber of degrees of freedom, as long as the forceing term has the form

$$\mathbf{f}(t) = \cos(\omega t + \phi_0)\mathbf{v} \ .$$

This may seem like a severe restriction, but it is not. The reason lie in the superposition principle. Suppose that for each $j = 1, \ldots, N$, $\mathbf{x}_j(t)$ solves

$$M\mathbf{x}_j''(t) = -L\mathbf{x}_j(t) + \mathbf{f}_j(t)$$

where M and L do not depend on j, but the forcing term does. Now define

$$\mathbf{x}(t) = \sum_{j=1}^{N} \mathbf{x}_j(t) \qquad \text{and} \qquad \mathbf{f}(t) = \sum_{j=1}^{N} \mathbf{f}_j(t) \ .$$

Then since differentiation and matrix multiplication are linear; i.e., since

$$\left(\sum_{j=1}^{N} \mathbf{x}_j(t) \right)'' = \sum_{j=1}^{N} \mathbf{x}_j''(t) = \mathbf{x}''(t) \qquad \text{and} \qquad L\left(\sum_{j=1}^{N} \mathbf{x}_j(t) \right) = \sum_{j=1}^{N} L\mathbf{x}_j(t) = L\mathbf{x}(t) \ ,$$

and likewise for multiplication by M, $\mathbf{x}(t)$ satisfies

$$M\mathbf{x}''(t) = -L\mathbf{x}(t) + \mathbf{f}(t) \ .$$

Therefore, we know how to solve this equation for any forcing term of the form

$$\mathbf{f}(t) = \sum_{j=1}^{N} \cos(\omega_j t + \phi_j)\mathbf{v}_j \ .$$

As we shall see in our investigation of the vibrating string problem, a fairly general class of driving forces can be written in this way.

As a final remark, it reamins to take the initial data into account. The usual way to do this is arrange that $\mathbf{f}_1 = 0$ and $\mathbf{x}_1(0) = \mathbf{x}(0)$ and $\mathbf{x}_1'(0) = \mathbf{x}'(0)$, and to then solve the remaining equatons, for $j \geq 2$, subject to $\mathbf{x}_j(0) = \mathbf{0}$ and $\mathbf{x}_j'(0) = \mathbf{0}$. That is, we solve the unforces equation with the desired initial conditions, and then add on the solution for each foring term started with zero intiial conditions.

4.4 Exercsies

1. Let $A = \begin{bmatrix} 9 & 9 \\ -1 & 1 \end{bmatrix}$. Find the solution of the ststem

$$\mathbf{x}'(t) = A\mathbf{x}(t) + e^{-4t}(1, t) \qquad \text{with} \qquad \mathbf{x}(0) = (1, 1) \ .$$

2. Let $A = \begin{bmatrix} 5 & 3 \\ 31 & 5 \end{bmatrix}$. Find the solution of the ststem

$$\mathbf{x}'(t) = A\mathbf{x}(t) + e^{-8t}(t,t) \qquad \text{with} \qquad \mathbf{x}(0) = (0,1) \ .$$

3. Let A be the matrix $A = \begin{bmatrix} 0 & 1 \\ -\kappa & 0 \end{bmatrix}$.

(a) Compute A^2, A^3 and A^4 Observe the patterns, and deduce a formula for A^k for all positive integers k. (You will probably want to consider even and odd k separately.)

(b) Use the results of part **(a)** to compute e^{tA}.

4. Find the solution of $x''(t) = -x(t) + f(t)$ with $x(0) = 0$ and $x'(0) = 0$ where

$$f(t) = \begin{cases} f_0 t & 0 \le t \le \pi \\ f_0(2\pi - t) & \pi \le t \le 2\pi \\ 0 & t \ge 2\pi \end{cases}$$

and where f_0 is a constant.

5. In simplify the results obtained from applying Duhamel's formula to the forced vibration problem, we used the fact that once can always write an expresion of the form, say,

$$\cos(\omega_1 t) + \cos(\omega_2 t)$$

as a multiple of the product of tho trigonometric functions with the frequencies $\frac{(}{\omega}_1 \pm \omega_2)$. Do this explicitly in the following cases.

(a)$\cos(5t) - \cos(3t)$

(b) $\cos(5t) + \cos(4t)$

(c) $\sin(5t) - \sin(2t)$

(d) $\sin(6t) - \sin(3t)$

6. This problem concerns oscillation that are damped by friction. We will consider a fricative force of the form $-ax'(t)$ where $a > 0$. That is the force is a negative multiple of the velocity. Combining this with the spring force, again assumed to be given by Hooke's Law, we have the Newton equation

$$mx''(t) = -kx(t) - ax'(t) \tag{4.57}$$

(a) Introduce $y(t) = x'(t)$, and $\mathbf{x}(t) = (x(t), y(y))$ and $\mathbf{g}(t) = (0, \frac{1}{m}f(t))$. Find a 2×2 matrix B so that (4.58) is equivalent to

$$\mathbf{x}'(t) = B\mathbf{x}(t) \ .$$

(b) Compute e^{tB}. There will be three cases, according to whether $(a/m)^2 > 4(k/m)$, $(a/m)^2 = 4(k/m)$ and $(a/m)^2 < 4(k/m)$.

(b) The system is said to be *critically damped* in case $(a/m)^2 = 4(k/m)$, *underdamped* in case $(a/m)^2 < 4(k/m)$, and *overdamped* in case $(a/m)^2 > 4(k/m)$ Find the general solution in the critically damped case. Show that no matter what thei nitial data are, $x(t)$ crosses the origin at most one time.

(c) Show also that in the overdamped case, no matter what thei nitial data are, $x(t)$ crosses the origin at most one time.

(d) Sow that in the underdamped case, as long as $(x(0), x'(0)) \neq (0,0)$, the solution crosses the orign infinitely many times.

7. In this problem we consider driven oscillations with friction taken into account as in the previous exercise, so that we have the Newton equation

$$mx''(t) = -kx(t) - ax'(t) + f(t) \tag{4.58}$$

where m is the mass, k is the spring constant, and $f(t)$ is the driving force.

(a) Using Duhamel's formula and the results of the prvious exercise, find integral formulas for the solution of (4.58). You will need 3 formulas, depending on whether $(a/m)^2 > 4(k/m)$, $(a/m)^2 = 4(k/m)$ or $(a/m)^2 < 4(k/m)$.

(b) Solve (4.58) with $x(0) = 0$,x'(0) =0, $f(t) = \cos(t)$, $m = 1$, $a = 1$ and $k = 5/4$.

(c) Solve (4.58) with $x(0) = 0$,x'(0) =0, $f(t) = \cos(t)$, $m = 1$, $a = 1$ and $k = 1/4$.

8. Consider the vector field

$$\mathbf{v}(x,y) = \big((x+y)(x-y-1), (x+y-2)(x-y+1)\big) \ .$$

(a) Find all equilibrium points of **v**, and determine which, if any, are asymptotically stable, and which if any are unstable.

(b) Do the same for

$$\mathbf{v}(x,y) = \big((x+y-2)(x-y+1)\ , (x+y)(x-y-1)\big) \ .$$

9. Consider the vector field

$$\mathbf{v}(x,y) = (x - xy, y + 2xy) \ .$$

(a) Find all equilibrium points of **v**, and determine which, if any, are asymptotically stable, and which if any are unstable.

(b) Do the same for

$$\mathbf{v}(x,y) = (y + 2xy\ , x - xy, y + 2xy) \ .$$

10. Consider the vector field

$$\mathbf{v}(x,y) = \big(-(2+y)(x+y)\ ,\ -y(1-x)\big) \ .$$

(a) Find all equilibrium points of **v**, and determine which, if any, are asymptotically stable, and which if any are unstable.

(b) Do the same for

$$\mathbf{v}(x,y) = \big((2+y)(x+y)\ ,\ -y(1-x)\ \big)\ .$$

11. Consider the vector field

$$\mathbf{v}(x,y) = \big(y(2+x-x^2)\ ,(2+x)(y-x)\big)\ .$$

(a) Find all equilibrium points of $\mathbf{v}$, and determine which, if any, are asymptotically stable, and which if any are unstable.

(b) Do the same for

$$\mathbf{v}(x,y) = \big((2+x)(y-x)\ ,y(2+x-x^2)\big)\ .$$

Chapter 5

PICARD'S THEOREM AND FLOW TRANSFORMATIONS FOR SYSTEMS

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5.1 Picard's Theorem

5.1.1 The equivalent integral equation

In this chapter we prove the fundamental existence theorem for systems of first order differential equations. Let $[a, b]$ be a bounded closed interval in $\mathbb{R}$. Throughout this chapter, we assume that $\mathbf{v}(\mathbf{x}, t)$ is a continuous function from $\mathbb{R}^n \times [a, b]$ to $\mathbb{R}^n$, and that moreover, for some $L < \infty$,

$$\|\mathbf{v}(\mathbf{y}, t) - \mathbf{v}(\mathbf{x}, t)\| \leq L\|\mathbf{y} - \mathbf{x}\| \tag{5.1}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and all $t \in [a, b]$.

Suppose that $t_0 \in (a, b)$ and $\mathbf{x}(t)$ is a continuous curve on $[a, b]$ that is continuously differentiable on (a, b) and satisfies

$$\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t), t) \quad \text{with} \quad \mathbf{x}(t_0) = x_0 \ . \tag{5.2}$$

Then, by the Fundamental Theorem of Calculus, for all $t \in [a, b]$,

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{v}(\mathbf{x}(s), s)\mathrm{d}s \ . \tag{5.3}$$

Conversely, Let $\mathbf{x}(t)$ be any continuous curve in $\mathbb{R}^n$ defined on $[a, b]$ such that (5.3) is true for all $t \in [a, b]$. Since the indefinite integral of a continuous function is continuously differentiable, again by the Fundamental Theorem of Calculus, the right hand side of (5.3) is continuously differentiable, and thus $\mathbf{x}(t)$ is not only continuous on $[a, b]$, it is continuously differentiable on (a, b). Moreover, taking the derivative, we find that $\mathbf{x}(t)$ satisfies (5.2).

Therefore, the problem of solving the differential equation (5.2) is the same as the problem of solving the integral equation (5.3). The latter may be viewed as a *fixed-point problem*, and this is the key to constructing solutions of it.

With $t_0 \in (a, b)$ and $\mathbf{x}_0 \in \mathbb{R}^n$ fixed, let $\mathcal{C}$ denote the set of curves $\mathbf{x}(t)$ in $\mathbb{R}^n$ that are defined and continuous on $[a, b]$ and satisfy $\mathbf{x}(t_0) = \mathbf{x}_0$. It will be helpful to make sure our notation is completely unambiguous.

As is usual, the symbol $\mathbf{x}(t)$ serves double duty in most of this text: It stand for the *function* which sends t to the value $\mathbf{x}(t)$, i.e., the curve $\mathbf{x}(t)$, and it stands for the *point* $\mathbf{x}(t) in \mathbb{R}^n$. In this chapter we shall use upper case boldface letters such as $\mathbf{X}$ to denotes elements of $\mathcal{C}$, which are functions, or curves, defined on $[a, b]$. When we refer to the value of the function $\mathbf{X}$ at time t, we shall write$\mathbf{X}|_t$ to denote the evaluation of the function $\mathbf{X}$ at the input value t. This notation would quickly get cumbersome, so wherever it is unambiguous, we write

$$\mathbf{X}|_t = \mathbf{x}(t) \ ,$$

using, $\mathbf{x}(t)$, the lower case version of the same letter. That is, in this notation $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ are curves in $\mathcal{C}$ and $\mathbf{x}(r) = \mathbf{X}|_r$, $\mathbf{y}(s) = \mathbf{Y}|_s$ and $\mathbf{z}(t) = \mathbf{Z}|_t$ are points along these curves.

Define a function Ψ from $\mathcal{C}$ to $\mathcal{C}$ as follows:

$$\Psi(X)|_t = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{v}(\mathbf{x}(s), s)\mathrm{d}s \ . \tag{5.4}$$

The right hand side defines a new curve in $\mathcal{C}$, so this formula specifies a function from $\mathcal{C}$ to $\mathcal{C}$. We can now write (5.3) in a vert compact form: The curve $\mathbf{X}$ satisfies (5.3) if and only if

$$X = \Psi(X) \ , \tag{5.5}$$

in other words, if and only if X is a *fixed point* of the transformation Ψ.

As we shall see, this fixed point equation can be solved by *iteration.* Fix *any* $\mathbf{X}_0 \in \mathcal{C}$, and define a sequences of curves $\}\mathbf{X}_n\} \in \mathcal{C}$ by

$$\mathbf{X}_n = \Psi(\mathbf{X}_{n-1})$$

for all $n \geq 1$. We shall show that if

$$b - a \leq \frac{1}{2L} \ ,$$

then for each $t \in [a, b]$, $\lim_{n\to\infty} bx_n(t)$ will exist, and moreover, if we let $\mathbf{x}(t)$ denote the limit, this defines a continuous curve in $\mathbb{R}^n$ with $\mathbf{x}(t_0) = \mathbf{x}_0$. That is, it defines an element $\mathbf{X}$ of $\mathcal{C}$. We shall show that, as you might expect from the construction, that $X = \Psi(X)$. By what we have explained above, X is the solution we seek of (5.3), and hence (5.2).

This will give us of the existence of solutions on some interval $[a, b]$ about the starting time. But then when we get close to the end of the interval, we can repeat the argument from where we have gotten and extend the solution for another $1/2L$ units of time. Continuing in this way, we construct a solution for all times t, and we already know that this solution is unique.

Before carrying out the proof, it is instructive to work through some simple examples, which we do in the next subsection.

5.1.2 Some simple examples of Picard iteration

Let us consider $n = 1$, and let $v(x,t) = kx$ for some constant a. This is a very simple case, and we know that the solutions of $x'(t) = kx(t)$ with $x(0) = x_0$ is $x(t) = e^{tk}x_0$.

Fix any bounded interval $[a,b]$ about 0. For any $X \in \mathcal{C}$, we have

$$\Psi(X)|_t = x_0 + \int_0^t v(x(s))\mathrm{d}s = x_0 + k\int_0^t x(s)\mathrm{d}s \ .$$

(Since $n = 1$, we have dropped the boldface.)

Let X_0 denote the constant function $x(t) = x_0$ for all t. This is certainly an element of $\mathcal{C}$ in this case. Then,

$$X_1|_t = \Psi(X_0)|_t = x_0 + k\int_0^t x_0\mathrm{d}s = x_0\left(1 + kt\right) \ .$$

Next,

$$X_2|_t = \Psi(X_1)|_t = x_0 + k\int_0^t x_0\left(1 + ks\right)\mathrm{d}s = x_0\left(1 + kt + \frac{k^2t^2}{2}\right) \ .$$

It is then easy to see that for all positive integers n,

$$x_n(t) = x_0\sum_{j=0}^{n}\frac{k^jt^j}{j!} \ ,$$

and so

$$\lim_{n\to\infty} x_n(t) = x_0e^{kt} \ .$$

For our next example, consider $n = 1$ and $v(x) = x^2$. This vector field is not Lipschitz on all of $\mathbb{R}$, but it is Lipschitz on any bounded interval $[a,b]$. Fix $x_0 \in R$ and let $[a.b]$ be any closed interval about $t = 0$. Then for any $X \in \mathcal{C}$, we have

$$\Psi(X)|_t = x_0 + \int_0^t x^2(s)\mathrm{d}s \ .$$

Let X_0 denote the constant function $x(t) = x_0$ for all t, as before. Then,

$$X_1|_t = \Psi(X_0)|_t = x_0 + \int_0^t x_0^2\mathrm{d}s = x_0\left(1 + x_0t\right) \ .$$

Next,

$$X_2|_t = \Psi(X_1)|_t = x_0 + \int_0^t x_0^2\left(1 + x_0s\right)^2\mathrm{d}s = x_0\left(1 + x_0t + x_0^2t^22 + \frac{1}{3}x_0^3t^3\right) \ .$$

The exact solution is

$$x(t) = \frac{x_0}{1 - x_0t} \ ,$$

and for $|x_0t| < 1$, we have the geometric series representation of the solution

$$x(t) = x_0\sum_{j=0}^{\infty}x_0^jt^j \ .$$

In this example, the Picard iteration does not produce this exact power series representation of the solution, but another, closely related sequence of polynomial approximations – notice that the first 3 terms of $x_2(t)$ are the first three terms of the geometric series, and it you computed x_3, you would find that then the first 4 terms agree.

5.1.3 The distance between two curves

To prove that our sequences of curves $\{X_n\}$ in $\mathcal{C}$ converges to a curve $X \in \mathcal{C}$, we need to give a precise meaning to the notion of the distance between two curves in $\mathcal{C}$. The following definition turns out to be very useful for our purpose.

Definition 19. *Let $\mathcal{C}$ be the set of continuous curves on $[a, b]$ with values in $\mathbb{R}^n$, For any $\mathbf{X}, \mathbf{Y}$ in$\mathcal{C}$, we define*

$$d(\mathbf{X}, \mathbf{Y}) = \max_{t \in [a,b]} \{\|bx(t) - by(t)\| \} \ . \tag{5.6}$$

Note that the function sending t to $\|bx(t) - by(t)\|$ is a continuous real-valued function, and then since $[a, b]$ is closed and bounded, there is some $t_{\max} \in [ab]$ such that

$$d(\mathbf{X}, \mathbf{Y}) = \|\mathbf{x}(t_{\max}) - \mathbf{y}(t_{\max})\| \ , \tag{5.7}$$

so that $0 \leq d(\mathbf{X}, \mathbf{Y}) < \infty$, and thus d is a well-defined function from $\mathcal{C} \times \mathcal{C}$ to $[0, \infty)$.

It is easy to see that $d(\mathbf{X}, \mathbf{Y}) = d(\mathbf{Y}, \mathbf{X})$ and that $d(\mathbf{X}, \mathbf{Y}) = 0$ if and only if $\mathbf{X} = \mathbf{Y}$; i.e., $\mathbf{x}(t) = \mathbf{y}(t)$ for all $t \in [a, b]$. We next show that the distance function d satisfies the triangle inequality, and therefore that it is a proper *metric* on the set $\mathcal{C}$.

Lemma 6. *Let $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ be curves in $\mathcal{C}$. Then*

$$d(\mathbf{X}, \mathbf{Z}) \leq d(\mathbf{X}, \mathbf{Y}) + d(\mathbf{Y}, \mathbf{Z}) \ .$$

Proof. By (5.7) there is some $t_{\max} \in [a, b]$ so that $d(\mathbf{X}, \mathbf{Z}) = \|\mathbf{x}(t_{\max}) - \mathbf{z}(t_{\max})\|$. But then by the triangle inequality in $\mathbb{R}^n$,

$$\|\mathbf{x}(t_{\max}) - \mathbf{z}(t_{\max})\| \leq \|\mathbf{x}(t_{\max}) - \mathbf{y}(t_{\max})\| + \|\mathbf{y}(t_{\max}) - \mathbf{z}(t_{\max})\| \ .$$

Therefore we have

$$d(\mathbf{X}, \mathbf{Z}) \leq \|\mathbf{x}(t_{\max}) - \mathbf{y}(t_{\max})\| + \|\mathbf{y}(t_{\max}) - \mathbf{z}(t_{\max})\| \ .$$

However, by definition,

$$\|\mathbf{x}(t_{\max}) - \mathbf{y}(t_{\max})\| \leq d(\mathbf{X}, \mathbf{Y}) \qquad \text{and} \qquad \|\mathbf{y}(t_{\max}) - \mathbf{z}(t_{\max})\| \leq d(\mathbf{Y}, \mathbf{Z}) \ .$$

□

The next lemma is the key to the success of the iteration.

Lemma 7. *Let $\mathbf{v}$ satisfy (5.1), and suppose that*

$$b - a \leq \frac{1}{2L} \ . \tag{5.8}$$

Then for any $\mathbf{X}, \mathbf{Y} \in \mathcal{C}$

$$d(\Psi(\mathbf{X}), \Psi(\mathbf{Y})) \leq \frac{1}{2} d(\mathbf{X}, \mathbf{Y}) \ . \tag{5.9}$$

Proof. For each $t \in [a,b]$,

$$\Psi(\mathbf{X})|_t - \Psi(\mathbf{Y})|_t = \int_{t_0}^{t} [\mathbf{v}(\mathbf{x}(s),s) - \mathbf{v}(\mathbf{y}(s),s)]\mathrm{d}s,$$

and so

$$\begin{aligned} \|\Psi(\mathbf{X})|_t - \Psi(\mathbf{Y})|_t\| &= \left\| \int_{t_0}^{t} [\mathbf{v}(\mathbf{x}(s),s) - \mathbf{v}(\mathbf{y}(s),s)]\mathrm{d}s \right\| \\ &\le \int_{t_0}^{t} \|\mathbf{v}(\mathbf{x}(s),s) - \mathbf{v}(\mathbf{y}(s),s)\|\mathrm{d}s \\ &\le \int_{t_0}^{t} L\|\mathbf{x}(s) - \mathbf{y}(s)\|\mathrm{d}s \\ &\le \int_{t_0}^{t} L \max_{r\in[a,b]}\{\|\mathbf{x}(r) - \mathbf{y}(r)\|\}\mathrm{d}s \\ &= L|t| \le L(b-a)d(\mathbf{X},\mathbf{Y}) \ . \end{aligned}$$

Then

$$d(\Psi(\mathbf{X}),\Psi(\mathbf{Y})) = \max_{t\in[a,b]}\{\|\Psi(\mathbf{X})|_t - \Psi(\mathbf{Y})|_t\|\} \le L(b-a)d(\mathbf{X},\mathbf{Y}) \ .$$

□

Now let $\mathbf{X}_0$ be the constant function with value $\mathbf{x}_0$ for all $t \in [a,b]$. Define

$$\mathbf{X}_1 = \Psi(\mathbf{X}_0) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{v}(\mathbf{x}_0,s)\mathrm{d}s \ .$$

Notice that

$$d(\mathbf{X}_1,\mathbf{X}_0) = \max_{t\in[a,b]} \left\| \int_{t_0}^{t} \mathbf{v}(\mathbf{x}_0,s)\mathrm{d}s \right\| \le (b-a) \max_{t\in[a,b]} \|\mathbf{v}(\mathbf{x}_0,t)\| \ ,$$

which is finite since $\|\mathbf{v}(\mathbf{x}_0,t)\|$ depends continuously on t in the bounded interval $[a,b]$.

Next, by Lemma refconlem, if we inductively define $X_n = \Psi(X_{n-1})$ for all positive integers n, we have, when $(b-1)L \le 1/2$,

$$d(\mathbf{X}_2,\mathbf{X}_1) = d(\Psi(\mathbf{X}_1,\Psi(\mathbf{X}_0)) \le \frac{1}{2}d(\mathbf{X}_1,\mathbf{X}_0)$$

and then

$$d(\mathbf{X}_3,\mathbf{X}_2) = d(\Psi(\mathbf{X}_2,\Psi(\mathbf{X}_1)) \le \frac{1}{2}d(\mathbf{X}_2,\mathbf{X}_1) \le \frac{1}{4}d(\mathbf{X}_1,\mathbf{X}_0) \ .$$

Continuing in this way, we deduce

$$d(\mathbf{X}_n,\mathbf{X}_{n-1}) \le 2^{-(n-1)}d(\mathbf{X}_1,\mathbf{X}_0) \ . \tag{5.10}$$

The next Lemma is a simple consequence of (5.10) and Lemma 6.

Lemma 8. *With $\{X_n\}$ defined as above, and $(b-1)L \le 1/2$, the sequence is a Cauchy sequence for the metric d on $\mathcal{C}$, meaning that for each $\epsilon > 0$, there is an N_ϵ such for all $m,n \ge N_\epsilon$,*

$$d(\mathbf{X}_n,\mathbf{X}_m) \le \epsilon \ . \tag{5.11}$$

More specifically, for all $n > m$,

$$d(\mathbf{X}_n,\mathbf{X}_m) \le 2^{-m}d(\mathbf{X}_1,\mathbf{X}_0) \ . \tag{5.12}$$

Proof. By the telescoping sum identity, for all $n > m$,

$$\mathbf{X}_n - \mathbf{X}_m = \sum_{j=m+1}^{n} (\mathbf{X}_j - \mathbf{X}_{j-1}) \ .$$

Then by the triangle inequality of Lemma 6.

$$\begin{aligned} d(\mathbf{X}_n - \mathbf{X}_m) &\leq \sum_{j=m+1}^{n} d(\mathbf{X}_j, \mathbf{X}_{j-1}) \\ &\leq \left(\sum_{j=m+1}^{n} \leq 2^{-(j-1)} \right) d(\mathbf{X}_1, \mathbf{X}_0) \\ &\leq \left(\sum_{j=m+1}^{\infty} \leq 2^{-(j-1)} \right) d(\mathbf{X}_1, \mathbf{X}_0) = 2^{-m} d(\mathbf{X}_1, \mathbf{X}_0) \ . \end{aligned}$$

Now simply choose N_ϵ to the the least integer m such that $2^{-m} d(\mathbf{X}_1, \mathbf{X}_0) \leq \epsilon$. □

5.1.4 Convergence of the iteration

We are now ready to prove the main result:

Theorem 22 (Picard's Theorem)**.** *Let* $\mathbf{v}$ *be a continuous vector field on* $\mathbb{R}^n \times [a,b]$ *satisfying the Lipschitz condition (5.1). Let* $\mathbf{x}_) \in \mathbb{R}^n$ *and let* $t_) \in (a,b)$*, and let* Ψ *be given by (5.4). Then whenever* $(b-a)L \leq 1/2$*, there is a unique* $X \in \mathcal{C}$ *such that*

$$X = \Psi(X) \ .$$

In other words, on (at least) the interval (a,b)*, there is a unique solution to*

$$\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t), t) \quad \text{with} \quad \mathbf{x}(t_0) \ .$$

Proof. Let the sequence $\{\mathbf{X}_n\}$ in $\mathcal{C}$ be defined above. Fix any $t \in [a,b]$, and consider the sequence $\{\mathbf{x}_n(t)\}$ in $\mathbb{R}^n$. By definition, for all m, n,

$$\|\mathbf{x}_n(t) - \mathbf{x}_m(t)\| \leq d(\mathbf{X}_n, \mathbf{X}_m) \ .$$

By Lemma 8, it follows that $\{\mathbf{x}_n(t)\}$ is a *Cauchy sequence* in $\mathbb{R}^n$.

Since $\mathbb{R}^n$ is *complete*, which means that every Cauchy sequence in $\mathbb{R}^n$ converges to a limit, for each t, $\{\mathbf{x}_n(t)\}$ has a limit that we shall denote by $\mathbf{x}(t)$. That is,

$$\mathbf{x}(t) = \lim_{n\to\infty} \mathbf{x}_n(t) \tag{5.13}$$

exists for each t. This defined a function $\mathbf{X}$ from $[a,b]$ to $\mathbb{R}^n$, Since $\mathbf{x}_n(t_0) = \mathbf{x}_0$ for all n, it is clear that $\mathbf{x}(t_0) = \mathbf{x}_0$.

It remains to show that $\mathbf{x}(t)$ is a *continuous* function of t, so that $\mathbf{X} \in \mathcal{C}$, and then that $\Psi(\mathbf{X}) = \mathbf{X}$. We shall show that this follows from (5.12).

By the continuity of the Euclidean distance function and (5.13)

$$\|\mathbf{x}(t) - \mathbf{x}_m(t)\| = \lim_{n\to\infty} \|\mathbf{x}_n(t) - \mathbf{x}_m(t)\| \ .$$

Combining this with (5.12), we see that

$$\|\mathbf{x}(t) - \mathbf{x}_m(t)\| \le 2^{-m} d(\mathbf{X}_1, \mathbf{X}_0) \ , \tag{5.14}$$

for all $t \in [a, b]$. In other words the rate of convergence of $\mathbf{X}_m$ to $\mathbf{X}$ is *uniform* in t. The continuity of $\mathbf{X}$ follows easily from this.

To show that $\mathbf{X}$ is continuous at $t \in [a, b]$, we must show that for all $\epsilon > 0$, there is a $\delta > 0$ so that $\|\mathbf{x}(s) - \mathbf{x}(t)\| \le \epsilon$ whenever $|s - t| \le \delta$.

Pick $t \in [a, b]$ and $\epsilon > 0$. By (5.14), for *all* $s \in [a, b]$, including t, if we pick m so that

$$2^{-m} d(\mathbf{X}_1, \mathbf{X}_0) \le \frac{\epsilon}{3} \ ,$$

we have

$$\|\mathbf{x}_m(s) - \mathbf{x}(s)\| \le \frac{\epsilon}{3} \ .$$

Then by the triangle inequality in $\mathbb{R}^n$,

$$\begin{aligned} \|\mathbf{x}(s) - \mathbf{x}(t)\| &\le \|\mathbf{x}(s) - \mathbf{x}_m(s)\| + \|\mathbf{x}_m(s) - \mathbf{x}_m(t)\| + \|\mathbf{x}_m(t) - \mathbf{x}(t)\| \\ &= \frac{\epsilon}{3} + \|\mathbf{x}_m(s) - \mathbf{x}_m(t)\| + \frac{\epsilon}{3} \end{aligned}$$

Now since $\mathbf{x}_m$ is continuous, there is a $\delta > 0$ so that whenever $|t - s| \le \delta$, $\|\mathbf{x}_m(s) - \mathbf{x}_m(t)\| \le \epsilon/3$. Putting it all together, whenever $|t - s| \le \delta$, $\|\mathbf{x}(s) - \mathbf{x}(t)\| \le \epsilon$. This shows that $\mathbf{X}$ is continuous at t, and since $t \in [a, b]$ is arbitrary, $\mathbf{X}$ is continuous. □

It is now a simple matter to prove existence for all t provided the Lipschitz condition (5.1) is valid for all $t \in \mathbb{R}$ with the same constant L. In particular, this will be true whenever $\mathbf{v}$ is a time-independent Lipschitz vector field.

Theorem 23 (Global existence)**.** *Let* $\mathbf{v}(\mathbf{x}, t)$ *be a vector field defined on* $\mathbb{R}^n \times \mathbb{R}$ *such that*

$$\|\mathbf{v}(\mathbf{y}, t) - \mathbf{v}(\mathbf{x}, t)\| \le L\|\mathbf{y} - \mathbf{x}\| \tag{5.15}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *and all* $t \in \mathbb{R}$*. Then for each* $\mathbf{x}_0 \in \mathbb{R}^n$ *and each* $t_0 \in \mathbb{R}$*, there exists a solution of*

$$\mathbf{x}'(t) = \mathbf{x}(\mathbf{x}(t), t) \quad \text{with} \quad \mathbf{x}(t_0) = t_0 \ ,$$

defined for all t*, and this solution is unique.*

Proof. We have already proved uniqueness on any time interval. We have also proved existence on at time interval of length at most $1/(2L)$. Dividing the whole line into a sequence of such intervals, taken to be slightly overlapping, we easily piece together a global solution.

□

5.2 Properties of the flow transformation

In this section we consider a vector field $\mathbf{v}(\mathbf{x},t)$ be a vector field defined on $\mathbb{R}^n \times \mathbb{R}$ such that $\|\mathbf{v}(\mathbf{y},t) - \mathbf{v}(\mathbf{x},t)\| \leq L\|\mathbf{y}-\mathbf{x}\|$ for all $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$ and all $t \in \mathbb{R}$.

Then for any $\mathbf{x} \in \mathbb{R}^n$ and $t_0, t_1 \in \mathbb{R}$, we define $\Phi_{t_1,t_0}(\mathbf{x})$ to be $\mathbf{x}(t_1)$ where $\mathbf{X}$ is the unique solution of $\mathbf{x}'(t) = \mathbf{x}(\mathbf{x}(t),t)$ with $\mathbf{x}(t_0) = \mathbf{x}$.

Because that function $t \mapsto \Phi_{t,t_0}(\mathbf{x})$ give the unique solution of $\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t),0)$ passing through $\mathbf{x}$ at time t_0, and since this same solution passes through $\Phi_{t_1,t_0}(\mathbf{x})$ at time t_1, it follows from the definition that for any t_2,

$$\Phi_{t_2,t_0} = \Phi_{t_2,t_1} \circ \Phi_{t_1,t_0}$$

and of course that

$$\Phi_{t,_0 t_0}(\mathbf{x}) = \mathbf{x} \ ,$$

which is to say that $\Phi_{t,_0 t_0}$ is the identity transformation.

Again, the uniqueness we have proved in Chapter 2 is essential for all of this. Furthermore, in the course of proving uniqueness, we have already proved that solutions depend on the initial data in a Lipschitz continuous manner. That is, we have already proved that

$$\|\Phi_{t_1,t_0}(\mathbf{y}) - \Phi_{t_1,t_0}(\mathbf{x})\| \leq e^{|t_1-t_0|L}\|\mathbf{y}-\mathbf{x}\| \ ,$$

so that as a function of the initial data $\mathbf{x}$, $\Phi_{t_1,t_0}(\mathbf{x})$ is Lipschitz continuous.

If we assume a little more about the vector field $\mathbf{v}$, we shall have that $\Phi_{t_1,t_0}(\mathbf{x})$ is a continuously differentiable function of $\mathbf{x}$, and we can even have a formula for the derivative. However, we shall not need this here, and instead turn to an important case in which even more is true – namely that Φ_{t_1,t_0} is a *linear transformation*; i.e., it is given by a matrix $[\Phi_{t_1,t_0}]$.

5.2.1 Non-autonomous linear systems

Let $A(t)$ denote a continuous $n \times n$ matrix values function of t, and consider the vector field $\mathbf{v}(\mathbf{x},t)$ on $\mathbb{R}^n$ given by

$$\mathbf{v}(\mathbf{x},t) = A(t)\mathbf{x} \ .$$

Then

$$\|\mathbf{v}(\mathbf{y},t) - \mathbf{v}(\mathbf{x},t)\| = \|A(t)(\mathbf{y}-\mathbf{x})\| \leq \|A(t)\|_{\mathbf{F}}\|\mathbf{y}-\mathbf{x}\| \ ,$$

it follows that if

$$\max_{t\in[a,b]} \|A(t)\|_{\mathbf{F}} = L < \infty \ ,$$

then $\|\mathbf{v}(\mathbf{y},t) - \mathbf{v}(\mathbf{x},t)\| \leq L\|\mathbf{y}-\mathbf{x}\|$ for all $\mathbf{x},\mathbf{y}$ and all $t \in [a,b]$. Thus, for each $t_)\in [a,b]$ and each $\mathbf{x}_0 \in \mathbb{R}^n$, the system of equation

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t)$$

has a unique solution with $\mathbf{x}(t_0) = \mathbf{x}_0$, and this solution is defined for all $t \in [a,b]$, If $A(t)$ is continuous on any open interval, even all of $\mathbb{R}$, $\|A(t)\|_{\mathbf{F}}$ will be continuous, and therefore bounded on any bounded. bounded closed subinterval. Thus the solution with exist and be unique on any such subinterval. But every point in the original interval belongs to some such subinterval, so the solution is defined on all of any interval on which $A(t)$ is continuous, and it is unique there.

Example 44. *Consider the second order equation for a real function $x(t)$:*

$$x''(t) + p(t)x'(t) + q(t)x(t) = 0 \ .$$

We introduce $y(t) = x'(t)$ and $\mathbf{x}(t) = (x(t), y(t))$ as usual, and have the equivalent system

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) \tag{5.16}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix} \ .$$

Then

$$\|A(t)\|_{\mathbf{F}} = \sqrt{1 + p^2(t) + q^2(t)}$$

will be continuous on any interval on which $p(t)$ and $q(t)$ are continuous. Suppose for example that $p(t)$ and $q(t)$ are continuous on $t > 0$, as would be the case, for, say,

$$p(t) = \frac{1}{t} \qquad \text{and} \qquad q(t) = -\frac{1}{t^2} \ . \tag{5.17}$$

Then, since the Lipschitz condition is satisfied on any bounded, closed subinterval of $(0, \infty)$, it follows that for each $\mathbf{x}_0 = (x_0, y_0)$, and each $t_0 \in (0, \infty)$, there is a unique solution of (5.16) with $\mathbf{x}(t_0) = (x_0, y_0)$,

Equivalently, for each $x_0, y_0 \in \mathbb{R}$ and each $t_0 \in (0, \infty)$ there exists a unique solution of $x''(t) + p(t)x'(t) + q(t)x(t) = 0$ with $x(t_0) = x_0$ and $x'(t_0) = y_0$ and this solution is defined on all of $(0, \infty)$. Of course, the same analysis applies on $(-\infty, 0)$.

What we have seen in Example 44 proves an important theorem:

Theorem 24. *Let (a, b) be an open interval in $\mathbb{R}$. Let $p(t), q(t)$ be continuous on (a, b). Then for each $t_0 \in (a, b)$ and each $x_), y_0 \in \mathbb{R}$, there is a unique solution of*

$$x''(t) + p(t)x'(t) + q(t)x(t) = 0$$

with $x(t_0) = x_0$ and $x'(t_0) = y_0$ and this solution is defined on all of $(a.b)$.

We next prove the flow transformation Φ_{t_1,t_0} corresponding to $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$ is a *linear* transformation of $\mathbb{R}^n$. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and let $\alpha, \beta \in \mathbb{R}$. Then define $\mathbf{x}(t) = \Phi_{t,t_0}(\mathbf{x})$ and $\mathbf{y}(t) = \Phi_{t,t_0}(\mathbf{y})$, so that $\mathbf{x}'(t) = A(t)\mathbf{y}(t)$ and $\mathbf{y}'(t) = A(t)\mathbf{y}(t)$. Then, define $\mathbf{z}(t) = \alpha\mathbf{x}(t) + \beta\mathbf{y}(t)$. Since differentiation is linear,

$$\mathbf{z}'(t) = \alpha\mathbf{x}'(t) + \beta\mathbf{y}'(t) = \alpha A(t)\mathbf{x}(t) + \beta A(t)\mathbf{y}(t) = A(t)(\alpha\mathbf{x}(t) + \beta\mathbf{y}(t)) = A(t)\mathbf{z}(t) \ .$$

Moreover,

$$\mathbf{z}(t_0) = \alpha\mathbf{x}(t_0) + \beta\mathbf{y}(t_0) = \alpha\mathbf{x} + \beta\mathbf{y} \ .$$

Therefore,

$$\mathbf{z}(t) = \Phi_{t,t_0}(\alpha\mathbf{x} + \beta\mathbf{y}) \ .$$

But by definition,

$$\mathbf{z}(t) = \alpha\Phi_{t,t_0}(\mathbf{x}) + \beta\Phi_{t,t_0}(\mathbf{x}) \ .$$

This proves:

Theorem 25 (Linearity of the flow transformation)**.** *Let $A(t)$ be a continuous $n \times n$ matrix valued function on the interval (a, b). for $t_0, t_1 \in (a, b)$, let Φ_{t_1,t_0} be th flow transformation corresponding to the vector field $\mathbf{v}(\mathbf{x}, t) = A(t)\mathbf{x}$. Then Φ_{t_1,t_0} is a linear transformation of $\mathbb{R}^n$.*

5.2.2 Computing the flow transformation for a non-autonomous linear system

Let $A(t)$ be a continuous $n \times n$ matrix valued function on the interval (a, b). Suppose $\mathbf{x}_j(t)$, $t = 1, \ldots, n$, are n curves each satisfying

$$\mathbf{x}_j'(t) = A(t)\mathbf{x}_j(t) \ .$$

Suppose also that for some $t_0 \in (a, b)$, the matrix $[\mathbf{x}_1(t_0), \ldots, \mathbf{x}_n(t_0)]$ is invertible, or, what is the same thing, that $\{\mathbf{x}_1(t_0), \ldots, \mathbf{x}_n(t_0)\}$ is linearly independent.

Form that matrix

$$M(t) = [\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)] \ .$$

Differentiating, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}M(t) = [\mathbf{x}_1'(t), \ldots, \mathbf{x}_n'(t)] = [A(t)\mathbf{x}_1(t), \ldots, A(t)\mathbf{x}_n(t)] = A(t)M(t) \ .$$

Next deine

$$\widehat{M}(t) = M(t)(M(t_0))^{-1} \ .$$

Since the matrix $(M(t_0))^{-1}$ is constant, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{M}(t) = A(t)\widehat{M}(t) \qquad \text{and} \qquad \widehat{M}(t_0) = I \ .$$

Now fix any $\mathbf{x} \in \mathbb{R}^n$, and defined $\mathbf{x}(t) = \widehat{M}(t)\mathbf{x}$. Then

$$\mathbf{x}'(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\widehat{M}(t)\right)\mathbf{x} = A(t)\widehat{M}(t)\mathbf{x} = A(t)\mathbf{x}(t) \qquad \text{and} \qquad \widehat{\mathbf{x}}(t_0) = M(t_0)\mathbf{x} = \mathbf{x} \ .$$

However, by definition, the unique solution of $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$ with $\mathbf{x}(t_0) = \mathbf{x}$ is $\Phi_{t,t_0}(\mathbf{x})$. We conclude that for all $\mathbf{x} \in \mathbb{R}^n$

$$\Phi_{t,t_0}(\mathbf{x}) = \widehat{M}(t) \ .$$

Since the flow transformation is one-to-one, and in fact,

$$\|\Phi_{t,t_0}(\mathbf{y}) - \Phi_{t,t_0}(\mathbf{x})\| \geq e^{-|t-t_0|L}\|\mathbf{y} - \mathbf{x}\| \ ,$$

$M(t)$ is one-to-one for all t. By the Fundamental Theorem of Linear Algebra, it is therefore invertible. That is. the fact that $M(t_0)$ is invertible implies that $M(t)$ is invertible for all t. Put still one more way, for any n solutions $\{\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)\}$ of $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$, linear independence at any one time in (a, b) implies linear independence at all other times t. We have proved:

Theorem 26. *Let $A(t)$ be a continuous $n \times n$ matrix valued function on the interval (a, b). Suppose $\mathbf{x}_j(t)$, $t = 1, \ldots, n$, are n curves each satisfying*

$$\mathbf{x}_j'(t) = A(t)\mathbf{x}_j(t) \ .$$

Suppose also that for some $t_0 \in (a, b)$, the matrix $[\mathbf{x}_1(t_0), \ldots, \mathbf{x}_n(t_0)]$ is invertible. Then for all $t, s \in (a, b)$, Φ_{t,t_0} is the linear transformation whose matrix is given by

$$\Phi_{t,s}(\mathbf{x}) = [\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)][\mathbf{x}_1(s), \ldots, \mathbf{x}_n(s)]^{-1} \ .$$

We see explicitly from this formula that the inverse of $\Phi_{t,s}$ is $\Phi_{s,t}$, though we saw in Chapter 2 that this is always true of flow transformations by a more abstract argument.

To apply Theorem 26 one has to find the n solutions $\{\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)\}$. Threre is no completely general effective method for this. But there is an important case in which there is a general method for producing a second solution out of one particular solution, and for some problems with $n = 2$, one can even find two solutions by inspection.

Example 45. *We have seen in Example 44 that the second order linear equation*

$$x''(t) + p(t)x'(t) + q(t)x(t) = 0$$

can be put in the form $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$ for the 2×2 matrix

$$A = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix} \ .$$

We now considered the specific case

$$x''(t) + \frac{1}{t}x'(t) - \frac{1}{t^2}x(t) = 0$$

which was also introduced there. Since the coefficinets are powers of t, one might try to find solutions of the form $x(t) = Ct^\alpha$, and since the equation is linear, so that any multiple of a solution is again a solution, we may as well take $c = 1$. Inserting $x(t) = t^\alpha$ into our equation,

$$\alpha(\alpha - 1)t^{\alpha-2} + \alpha t^{\alpha-2} - t^{\alpha-2} = (\alpha^2 - 1)t^{2-\alpha}$$

We get solutions with $\alpha = \pm 1$.

Therefore we take $x_1(t) = t$ and $x_2(t) = t^{-1}$. Then

$$\mathbf{x}_1(t) = (x_1(t), x_1'(t)) = (t, 1) \quad \text{and} \quad \mathbf{x}_2(t) = (x_1(t), x_1'(t)) = (t^{-1}, -t^{-2}) \ .$$

We then have

$$M(t) = \begin{bmatrix} t & t^{-1} \\ 1 & -t^{-2} \end{bmatrix} \quad \text{and} \quad M^{-1}(s) = \frac{1}{2}\begin{bmatrix} s^{-1} & 1 \\ s & -s^2 \end{bmatrix} \ .$$

Therefore,

$$[\Phi_{t,s}] = M(t)M^{-1}(s) = \frac{1}{2}\begin{bmatrix} \frac{t^2+s^2}{st} & \frac{t^2-s^2}{t} \\ \frac{t^2-s^2}{st^2} & \frac{t^2+s^2}{t^2} \end{bmatrix} . \tag{5.18}$$

As we have seen in the previous example, the key to computing the flow transformation $\Phi_{t,s}$ associated to $A = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix}$ is to find two linarly independent soluttions of

$$x''(t) + p(t)x'(t) + q(t)x(t) = 0 \ .$$

As we now explain, whenever you can find one non-zero solution, you can find another that is linearly independent. Thus, whenver you can find one non-zero solution, you can compute $\Phi_{t,s}$. We now explain how.

Suppose that $x_1(t)$ is one solution. Let us seek a function $v(t)$ such that

$$x_2(t) = v(t)x_1(t)$$

is also a solution. With this deifnition of $x_2(t)$, we compute

$$x_2'(t) = v'(t)x_1(t) + v(t)x_1'(t) \quad \text{and} \quad x_2'(t) = v''(t)x_1(t) + 2v'(t)x_1'(t) + v(t)x_1''(t) \ .$$

Therefore,

$$\begin{aligned} x_2''(t) + p(t)x_2'(t) + q(t)x_2(t) &= [x_1''(t) + p(t)x_1'(t) + q(t)x_1(t)]v(t) \\ &+ [2x_1'(t) + p(t)x_1(t)]v'(t) + x_1(t)v''(t) \ . \end{aligned} \tag{5.19}$$

Since, by hypothesis, $x_1''(t) + p(t)x_1'(t) + q(t)x_1(t) = 0$,

$$x_2''(t) + p(t)x_2'(t) + q(t)x_2(t) = 0$$

if and only if

$$[2x_1'(t) + p(t)x_1(t)]v'(t) + x_1(t)v''(t) = 0 \ ,$$

which can be written as

$$\frac{v'(t)}{v(t)} = -2\frac{x_1'(t)}{x_1(t)} - p(t) \ .$$

Integrating, we find

$$v'(t) = \frac{1}{x_1^2(t)}e^{-P(x)} \ ,$$

where $P'(t) = p(t)$.

Integrating once more, we have

$$v(t) = \int \frac{1}{x_1^2(t)}e^{-P(t)}\mathrm{d}t + C \ , \tag{5.20}$$

and we may as well set $C = 0$

Our second solution is then

$$x_2(t) = \left(\int \frac{1}{x_1^2(t)}e^{-P(t)}\mathrm{d}t\right) x_1(t) \ .$$

Then the matrix $M(t)$ is given by

$$M(t) = \begin{bmatrix} x_1(t) & \left(\int \frac{1}{x_1^2(t)} e^{-P(x)} \mathrm{d}x\right) x_1(t) \\ x_1'(t) & \left(\int \frac{1}{x_1^2(t)} e^{-P(x)} \mathrm{d}x\right) x_1'(t) + \frac{1}{x_1^2(t)} e^{-P(x)} x_1(t) \end{bmatrix} .$$

Since the determinat function is linear in the columns of a matrix,

$$\begin{aligned} \det(M(t)) &= \det\left(\begin{bmatrix} x_1(t) & \left(\int \frac{1}{x_1^2(t)} e^{-P(x)} \mathrm{d}x\right) x_1(t) \\ x_1'(t) & \left(\int \frac{1}{x_1^2(t)} e^{-P(x)} \mathrm{d}x\right) x_1'(t) \end{bmatrix}\right) + \det\left(\begin{bmatrix} x_1(t) & 0 \\ x_1'(t) & \frac{1}{x_1^2(t)} e^{-P(x)} x_1(t) \end{bmatrix}\right) \\ &= 0 + e^P(t) = e^{P(t)} > 0 . \end{aligned}$$

Since the detemrinant is not zero, $(x_1(t), x_1'(t))$ and $(x_2(t), x_2'(t))$ are linearly independent for all t.

We have proved:

Theorem 27. *Let $p(t)$ and $q(t)$ be consitnuous on (a,b). Suppose that $x_1(t)$ is a solution of*

$$x''(t) + p(t)x'(t) + q(t)x(t) = 0 \tag{5.21}$$

on (a,b) that is not identically zero. Let $P(t)$ be any antiderivative of $p(t)$ on (a,b). Define

$$v(t) := \int \frac{1}{x_1^2(t)} e^{-P(t)} \mathrm{d}t , \tag{5.22}$$

and

$$x_2(t) = v(t)x_1(t) \tag{5.23}$$

Then $x_2(t)$ solves (5.21) on (a,b) and $(x_1(t), x_1'(t))$ and $(x_2(t), x_2'(t))$ are linearly independent for all $t \in (a,b)$.

Example 46. *The second order linear equation*

$$(1-t^2)x''(t) - 2tx'(t) + 2x(t) = 0$$

is the special case of Legendere's equation

$$(1-t^2)x''(t) - 2tx'(t) + r(r+1)x(t) = 0$$

with $r = 1/2$.

It is easy to see that $x_1(t) = t$ is a solution. To find the general solution, we divide through by $(1-t^2)$ to put the equation in standard form:

$$x''(t) - \frac{2t}{1-t^2}x'(t) + \frac{2}{1-t^2}x(t) = 0 . \tag{5.24}$$

The coefficients $p(t)$ and $q(t)$ are continuous on $(-1,1)$. Let us find the general solution on this interval.

Since

$$P(t) = \int p(t)\mathrm{d}t = \int \frac{-2t}{1-t^2}\mathrm{d}t = \ln(1-t^2),$$

we have that $v(t)$ in (5.22) is given by

$$v(t) = \int \frac{1}{t^2(1-t^2)}\mathrm{d}t = \frac{1}{2}\ln\left(\frac{1+t}{1-t}\right) - \frac{1}{t}\ .$$

Therefore,e by (5.23)

$$x_2(t) = \frac{1}{2}t\ln\left(\frac{1+t}{1-t}\right) - 1$$

is a second solution. The general solution therefore has the form

$$c_1 t + c_2\left(\frac{1}{2}t\ln\left(\frac{1+t}{1-t}\right) - 1\right)$$

for arbitrary constant c_1 and c_2.

Example 47. *Consider the second order equation*

$$x''(t) - f(t)x'(t) + [f(t)-1]x(t) = 0$$

where $f(t)$ is any continuous function on $\mathbb{R}$, or some open interval in $\mathbb{R}$. Note that $x_1(t) = e^t$ is a solution. In this case

$$P(t) = \int p(t)\mathrm{d}t = -\int f(t)\mathrm{d}t = -F(t)$$

where $F(t)$ is any antiderivaitve of $f(t)$. Then

$$v(t) = \int e^{-2t-F(t)}\mathrm{d}t\ .$$

Then

$$x_2(t) = \left(\int e^{-2t+F(t)}\mathrm{d}t\right)e^t$$

is a second linearly independent solution, and the general solution is

$$c_1 e^t + c_2\left(\int e^{-2t+F(t)}\mathrm{d}t\right)e^t\ .$$

To make the example more specific, let

$$f(t) = 1 + \frac{1}{t}$$

so that the equation is

$$x''(t) - \frac{t+1}{t}x'(t) + \frac{1}{t}x(t) = 0 \tag{5.25}$$

where the coefficients are continuous on $(0,\infty)$ and $(-\infty,0)$. In this case we find $F(t) = t + \ln(t)$, so that

$$v(t) = \int e^{-t+\ln t}\mathrm{d}t = \int te^{-t}\mathrm{d}t = -(t+1)e^{-t}\ .$$

Thus,

$$x_2(t) = v(t)x_1(t) = -1 - t\ .$$

As you can easily check, this is a second solution.

The corresponding first order system is $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$ *where* $A(t) = \begin{bmatrix} 0 & 1 \\ -1/t & (t+1)/t \end{bmatrix}$*. The two solution found above are* $\mathbf{x}_1(t) = (e^t, e^t)$*, and* $\mathbf{x}_2(t) = (-1-t, -1)$*. Hence we have*

$$M(t) = \begin{bmatrix} e^t & -1-t \\ e^t & -1 \end{bmatrix} ,$$

so that the corresponding flow transformation is given by the matrix

$$[\Phi_{t,s}] = M(t)M^{-1}(s) = \frac{1}{s}\begin{bmatrix} 1+t-e^{t-s} & e^{t-s}(1+s)-t-1 \\ 1-e^{t-s} & e^{t-s}(1+s)-1 \end{bmatrix} ,$$

whihc one finds after routine computation.

5.2.3 The derivative of $\Phi_{t,s}(\mathbf{x})$ in s

We already know, essential by the definition that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{t,s}(\mathbf{x}) = A(t)\Phi_{t,s}(\mathbf{x}) \ .$$

We now work out the derivative in s.

Observe that $\Phi_{t,s}(\mathbf{x}) = \Phi_{t,s+h} \circ \Phi_{s+h,s}$ so that

$$\frac{1}{h}[\Phi_{t,s+h}(\mathbf{x}) - \Phi_{t,s}(\mathbf{x})] = \Phi_{t,s+h}\left(\frac{1}{h}(\mathbf{x} - \Phi_{s+h,s}\mathbf{x})\right) \ .$$

Then since

$$\lim_{h\to 0}\frac{1}{h}\left(\Phi_{s+h,s}\mathbf{x} - \mathbf{x}\right) = A(s)\mathbf{x} \ ,$$

we have proved:

Theorem 28. *Let* $A(t)$ *be a continuous* $n \times n$ *matrix valued function on the interval* (a,b)*. Let* $\Phi_{t,s}$ *be the corresponding flow transformation. Then for all* $t, s \in (a,b)$*, and all* $\mathbf{x} \in \mathbb{R}^n$*,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{t,s}(\mathbf{x}) = A(t)\Phi_{t,s}(\mathbf{x}) \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}s}\Phi_{t,s}(\mathbf{x}) = -\Phi_{t,s}(A(s)\mathbf{x}) \ .$$

5.2.4 Duhamel's formula for non-autonomous systems

In this subsection we study the inhomogeneous equation linear system

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t) \qquad \text{with} \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \ .$$

This is equivalent to

$$\Phi_{t_0,t}(\mathbf{x}'(t) - A(t)\mathbf{x}(t)) = \Phi_{t_0,t}\mathbf{b}(t) \ .$$

However,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi_{t_0,t}\mathbf{x}(t)\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{t_0,t}\right)\mathbf{x}(t) + \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}'(t) = \Phi_{t_0,t}(\mathbf{x}'(t) - A(t)\mathbf{x}(t)) \ .$$

Therefore, our system is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi_{t_0,t}\mathbf{x}(t)\right) = \Phi_{t_0,t}\mathbf{b}(t)\ .$$

Integrating both sides, we see that this is solved by

$$\Phi_{t_0,t}\mathbf{x}(t) - \mathbf{x}_0 = \int_{t_0}^{t} \Phi_{t_0,s}\mathbf{b}(s)\mathrm{d}s\ .$$

Then, since Φ_{t,t_0} is the inverse of $\Phi_{t_0,t}$, and since $\Phi_{t,t_0}\Phi_{t_0,s} = \Phi_{t,s}$, we have

$$\mathbf{x}(t) = \Phi_{t,t_0}(\mathbf{x}_0) + \int_{t_0}^{t} \Phi_{t,s}\mathbf{b}(s)\mathrm{d}s\ .$$

We have proved:

Theorem 29. *Let $A(t)$ be a continuous $n \times n$ matrix valued function on the interval (a,b). Let $\Phi_{t,s}$ be the corresponding flow transformation. Let $\mathbf{b}(t)$ be a continuous $\mathbb{R}^n$ values function on (a,b). Then for all $t,s \in (a,b)$, and all $\mathbf{x}_0 \in \mathbb{R}^n$, there is a unique solution of*

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t) \qquad \text{with} \qquad \mathbf{x}(t_0) = \mathbf{x}_0\ ,$$

and it is given by Duhamel's formula

$$\mathbf{x}(t) = \Phi_{t,t_0}(\mathbf{x}_0) + \int_{t_0}^{t} \Phi_{t,s}\mathbf{b}(s)\mathrm{d}s\ .$$

Example 48. *Let us find the solution to*

$$x''(t) + \frac{1}{t}x'(t) - \frac{1}{t^2}x(t) = t^2$$

Subject to the initial condition $x(t_0) = 1$ and $x'(t_0) = -1$.

The correspnding first order system is

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t) \qquad \text{with} \qquad \mathbf{x}(t_0) = (1,-1)$$

and with

$$A(t) = \begin{bmatrix} 0 & 1 \\ t^{-2} & -t^{-1} \end{bmatrix} \qquad \text{and} \qquad \mathbf{b}(t) = (1, t^2)\ .$$

Having rewritten our equation as a fisr order system, we can apply Duhamel's Formula to solve it.

We have found in Example 45 that the flow transformation for the corresponding homogeneous system is given by the matrix

$$[\Phi_{t,s}] = M(t)M^{-1}(s) = \frac{1}{2}\begin{bmatrix} \frac{t^2+s^2}{st} & \frac{t^2-s^2}{t} \\ \frac{t^2-s^2}{st^2} & \frac{t^2+s^2}{t^2} \end{bmatrix}\ .$$

We compute

$$\int_{t_0}^{t} [\Phi_{t,s}]\mathbf{b}(s)\mathrm{d}s = \frac{1}{2t^2}\int_{t_0}^{t} (t(t^2-s^2)s^2, (t^2+s^2)s^2)\mathrm{d}s$$

To compute $x(t)$*, as opposed to* $(x(t), x'(t))$*, we need only the first component of this, which is*

$$\frac{1}{2t}\left(\frac{1}{3}t^2(t^3 - t_0^3) - \frac{1}{5}(t^5 - t_0^5)\right) = \frac{1}{15}t^4 - \frac{1}{6}tt_0^3 + \frac{1}{10t}t_0^5 \ .$$

Likewise, the first component of $[\Phi_{t,t_0}]\mathbf{x}_0$ *is*

$$\frac{1}{2tt_0}(t^2 + t_0^2 - t_0(t^2 - t_0^2)) = \frac{t}{2t_0} + \frac{t_0}{2t} + \frac{t_0^2}{2t} - \frac{t}{2} \ .$$

Altogether, we have

$$x(t) = \frac{t}{2t_0} + \frac{t_0}{2t} + \frac{t_0^2}{2t} - \frac{t}{2} + \frac{1}{15}t^4 - \frac{1}{6}tt_0^3 + \frac{1}{10t}t_0^5 \ .$$

Chapter 6

BOUNDARY VALUE PROBLEMS

6.1 The vibrating string problem

6.1.1 Derivation of the equations of motion

Consider a string with constant mass density ϱ units of mass per unit length with the ends fixed, or pinned down, a distance of L units of length apart. Let T denote the tension in the string. Think of a violin string, for example stretched quite tight so that the tension is relatively high, and when the string vibrates, its oscillations do not have a large amplitude. Let x denote the position along the string, and let $h(x)$ denote vertical the displacement (in the x, y plane) of the part of the string at distance x from the left fixed point at $x = 0$. When the string is at rest, $h(x) = 0$ for all x. If the string is plucked, or otherwise set in motion, what will that motion be? We can deduce a system of equations for this motion from Newton's Second Law. Pick a large number N, and define

$$x_j = \frac{jL}{N} \qquad \text{and} \qquad \Delta x = \frac{L}{N} \ .$$

Then for $j = 0, \ldots, N$, x_j denotes the position of the the end of a segment of the string of length Δx, and therefore of mass $\varrho \Delta x$. Let $y_j(t)$ denote the vertical displacement (from the rest position) of the center of this segment at time t. Since the ends are fixed,

$$y_0(t) = y_N(t) = 0 \qquad \text{for all } t \ . \tag{6.1}$$

For $1 \le j \le N - 1$, the vertical acceleration of the jth segment is $y_j''(t)$, and so by Newton's Second Law,

$$(\varrho \Delta x) y_j''(t) = F_j(t) \ , \tag{6.2}$$

where $F_j(t)$ is the vertical component of the force acting on the jth segment at time t. The forces would be gravity and the tension in the string. But for a string under high tension, like a violin string, the tension is the only significant force. We therefore neglect gravity. The segment at x_j is tugged on from the right and from the left. At time t, the vertical component of the force from the left is

$$T\frac{1}{\sqrt{\Delta x^2 + (y_{j-1}(t) - y_j(t))^2}}[(x_{j-1}, y_{j-1}) - (x_j, y_j)]$$

and the vertical component of the force from the right is

$$T\frac{1}{\sqrt{\Delta x^2 + (y_{j+1}(t) - y_j(t))^2}}[(x_{j+1}, y_{j+1}) - (x_j, y_j)] \ .$$

Thus,

$$F_j(t) = T\frac{1}{\sqrt{\Delta x^2 + (y_{j-1}(t) - y_j(t))^2}}[(x_{j-1}, y_{j-1}) - (x_j, y_j)] + \\ T\frac{1}{\sqrt{\Delta x^2 + (y_{j+1}(t) - y_j(t))^2}}[(x_{j+1}, y_{j+1}) - (x_j, y_j)] \ . \tag{6.3}$$

We now make a crucial approximation: We assume that, since the tension is high, the amplitude of the vibrations is very small, so that for each j, $|y_{j+1} - y_j|$ is small compared to Δx. Thus in (6.3) we make the replacements

$$\sqrt{\Delta x^2 + (y_{j-1}(t) - y_j(t))^2} \to \Delta x \qquad \text{and} \qquad \sqrt{\Delta x^2 + (y_{j+1}(t) - y_j(t))^2} \to \Delta x \ .$$

This leads to the approximation

$$F_j(t) \approx \frac{T}{\Delta x}(y_{j+1}t) - 2y_j(t) + y_{j-1}(t)) \ . \tag{6.4}$$

Using this in (6.2), we obtain the linear approximation to the system of equations of motion for the vibrating string:

$$(\varrho\Delta x)y_j''(t) = \frac{T}{\Delta x}(y_{j+1}t) - 2y_j(t) + y_{j-1}(t)) \ . \ , \tag{6.5}$$

This equation is to be satisfied for all $1 \le j \le N-1$ with $y_0(t) = y_N(t) = 0$ for all t.

To write this linear system in matrix form, introduce the $\mathbb{R}^{N-1}$ valued functions $\mathbf{y}(t)$ where

$$\mathbf{y}(t) = (y_1(t), \ldots, y_{N-1}(t))$$

and the $(N-1) \times (N-1)$ matrix K given by

$$K_{i,j} = \frac{1}{\Delta x^2}\begin{cases} 2 & i = j \\ -1 & |i-j| = 1 \\ 0 & \text{otherwise} \end{cases} \ .$$

That is, K has the structure

$$K = \frac{1}{\Delta x^2}\begin{bmatrix} 2 & -1 & 0 & 0 & \ldots & 0 \\ -1 & 2 & -1 & 0 & \ldots & 0 \\ 0 & -1 & 2 & -1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & -1 & 2 & -1 \\ 0 & \ldots & 0 & 0 & -1 & 2 \end{bmatrix} \ .$$

Each diagonal entry of K is $2\Delta x^{-2}$, and each entry just above or below the diagonal is $-\Delta x^{-2}$ and all others are zero.

Finally, introduce

$$c := \sqrt{\frac{T}{\varrho}}\ .$$

Then the system (6.5) can be written as

$$\mathbf{y}''(t) = -c^2 K\mathbf{y}(t). \ , \tag{6.6}$$

This is a familiar equation, since the matrix K is not only symmetric; all of its eigenvalues are strictly positive, so that the matrix K is positive definite. To see this, make the simple computation showing that for any $\mathbf{y} = (y_1, \ldots, y_{N-1}) \in \mathbb{R}^{N-1}$,

$$\mathbf{y}\cdot K\mathbf{y} = \frac{1}{\Delta x^2}\left(\sum_{j=0}^{N-1}(y_{j+1}-y_j)^2\right)\ , \tag{6.7}$$

where we put $y_0 = y_N = 0$. As a sum of squares, this is non-negative, and is equal to zero if and only if $y_j{+}1 = y_j$ for all $j = 0, \ldots, N-1$. Then since $y_0 = y_N = 0$, this means that $\mathbf{y} = 0$. In other words, $\mathbf{y}\cdot K\mathbf{y} = 0$ if and only if $\mathbf{y} = \mathbf{0}$, and hence K is positive definite.

We could proceed to analyze this equation by seeking the eigenvectors of K. However, it is in many ways more enlightening to make a further approximation, taking the *continuum limit*. This will lead us to consider a new class of problems involving second order differential equations, namely *boundary value problems*. Furthermore, once we have dealt with the continuum limit, it will be much easier to find the eigenvectors and eigenvalues of K.

Let $h(x,t)$ denote the vertical displacement of the part of the string at horizontal coordinate x and at time t. Form the vector

$$\mathbf{h}(t) = (h(x_1,t), \ldots, h(x_{N-1},t))\ .$$

Then for each $j = 1, \ldots, N-1$,

$$\begin{aligned}(K\mathbf{h}(t))_j &= \frac{1}{\Delta x^2}[-h(x_{j-1},t) + 2h(x_j,t) - h(x_{j+1},t)]\\ &= -\frac{1}{\Delta x}\left[\frac{h(x_j+\Delta x,t)-h(x_j,t)}{\Delta x} - \frac{h(x_j,t)-h(x_j-\Delta x,t)}{\Delta x}\right]\ .\end{aligned}$$

If for each t, $h(x,t)$ is a twice continuously differentiable function of x,

$$\lim_{\Delta x\to 0}\frac{1}{\Delta x}\left[\frac{h(x_j+\Delta x,t)-h(x_j,t)}{\Delta x} - \frac{h(x_j,t)-h(x_j-\Delta x,t)}{\Delta x}\right] = \frac{\partial^2}{\partial x^2}h(x_j,t)\ .$$

The continuum approximation to (6.6) then is to take Δx to zero, and hence N to infinity, and to replace (6.6) with

$$\frac{\partial^2}{\partial t^2}h(x,t) = c^2\frac{\partial^2}{\partial x^2}h(x,t)\ , \tag{6.8}$$

where the equation (6.8) is to hold at each $x \in (0,L)$ and each t, and we also require

$$h(0,t) = h(L,t) = 0 \tag{6.9}$$

for all t. The equation (6.8) is a *partial differential equation* known as the *wave equation.* The theories of partial differential equations and ordinary differential equation are intimately connected, and the theory of ordinary differential equations cannot be developed in isolation from the theory of partial differential equations. First of all, we have seen that the wave equation (6.8) arises as the continuum limit of a system of ordinary differential equations. This is true of many other partial differential equations. Second, as we shall see, the continuum limit often brings along a certain simplicity. In the case at hand, it will be be much easier to find the eigenvectors and eigenvalues of K once we have studied the continuum limit. Third, we shall see that there is a method of solving (6.8) that rests on solving a family of ordinary differential equations.

6.1.2 The wave equation on $\mathbb{R}$

Before considering the wave equation on $[0, L]$ with the boundary condition $h(0,t) = h(L,t) = 0$, let us consider it on the whole real line. We seek functions $h(x,t)$ on $\mathbb{R} \times \mathbb{R}$ such that (6.8) is true for all x, t.

Note that

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right) h(x,t) = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) h(x,t) = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) h(x,t)\ .$$

Therefore, $h(x,t)$ solves the wave equation if either

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) h(x,t) = 0 \qquad \text{or} \qquad \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) h(x,t) = 0\ . \tag{6.10}$$

These first order equations have a simple geometric meaning. Since

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) h(x,t) = (c,1)\cdot\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) h(x,t) = (c,1)\cdot\nabla h(x,t)\ ,$$

where ∇ denotes the gradient in the variables x, t, the first equation in (6.10) is satisfied if and only if the directional derivative of $h(x,t)$ in the direction of $(1,c)$ is zero. That is the case if and only if $h(x,t)$ is constant along each of the lines $x = ct + x_0$, so that $h(x,t)$ only depends on $x_0 = x - ct$. That is, $h(x,t) = g(x-ct)$ for some function g. Conversely, it is easy to check that if g is continuously differentiable, then

$$h(x,t) = g(x - ct)$$

does satisfy the wave equation when g is continuously differentiable.

Considering the second equation in (6.10), the same reason leads to the fact that

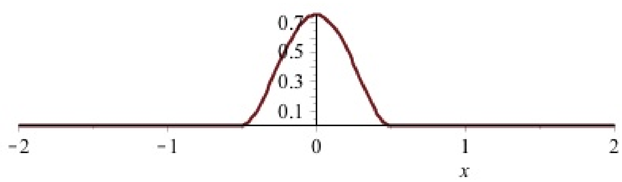
$$h(x,t) = g(x + ct)$$

satisfies the wave equation when g is continuously differentiable. These two solutions describe *traveling waves* moving to the right and to the left, respectively.

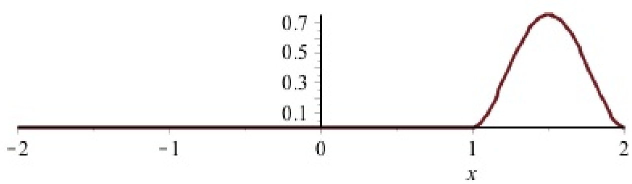
Consider a 'blip' function $g(x)$ of the general form

$$g(x) = \begin{cases} A(x^2 - a^2)^2 & -a \le x \le a \\ 0 & |x| \ge a \end{cases}$$

for some $a > 0$ and some $A \neq 0$. Here is an image of this function for $a = 1/2$ and $A = 6$.



Next, we plot $h(t,x) = g(x - ct)$ for $c = 1$ and $t = 3/2$. The peak of $g(x)$ is located at $x = 0$, so the peak of $g(x - 3/2)$ is located where $x = -3/2 = 0$; i.e., at $x = 3/2$. The 'blip' moves to the right, keeping its shape:



To graph the solution $y = h(x,t) = g(x - ct)$ of the wave equation, one simply shifts the graph of $y = g(x)$ to the right through a distance of ct. This tells us the meaning of c: It is the *speed of wave propagation* for the string. Likewise, the solution $h(x,t) = g(x + ct)$ gives propagation to the left at speed c.

We have deduced an important fact from our analysis of the wave equation: *Small amplitude disturbances will travel along the string at a velocity that is the square root of the mass density ϱ times the tension T.* Of course here we are referring to the special solutions satisfying (6.10), but we shall soon see that all physical solutions are linear combinations of such solutions.

We now wish to solve the initial value problem for (6.8) on $\mathbb{R}$ in which we seek a solution $h(x,t)$ that satisfies

$$h(x,0) = g(x) \qquad \text{and} \qquad \frac{\partial}{\partial t} h(x,t)\bigg|_{t=0} = v(x) \tag{6.11}$$

for all x and for given functions $g(x)$ and $v(x)$.

We will take advantage of the *linearity* of the wave equation. Since differentiation is linear, it is evident that if $h_j(x,t)$, $j = 1,2$ both satisfy the wave equation and a_j, $j = 1,2$ are any two numbers,

$$h(x,t) = a_1 h_1(x,t) + a_2 h_2(x,t)$$

is also a solution of the wave equation.

Therefore, if $h_1(x,t)$ solves the wave equation and satisfies

$$h_1(x,0) = g(x) \qquad \text{and} \qquad \frac{\partial}{\partial t} h_1(x,t)\bigg|_{t=0} = 0 \tag{6.12}$$

for all x, and $h_2(x,t)$ is the solves the wave equation and satisfies

$$h_2(x,0) = 0 \qquad \text{and} \qquad \frac{\partial}{\partial t} h_2(x,t)\bigg|_{t=0} = v(x) \tag{6.13}$$

for all x, then $h(x,t) = h_1(x,t) + h_2(x,t)$ solves the wave equation and satisfies (6.11).

To solve the wave equation subject to the initial data (6.12), we superimpose a wave moving to the left with a wave moving to the right which cancels the initial velocity. That is, we define

$$h_1(x,t) = \frac{1}{2}[g(x-ct) + g(x+ct)] \ . \tag{6.14}$$

It is easy to check that $h(x,t)$ solves the wave equation since it is a linear combination of two solutions, and clearly

$$h_1(x,0) = \frac{1}{2}[g(x) + g(x)] = g(x)$$

and

$$\frac{\partial}{\partial t} h_1(x,t)\bigg|_{t=0} \frac{1}{2}[cg'(x) - cg'(x)] = 0 \ .$$

Therefore, the formula (6.14) gives us a solution of the wave equation that satisfies (6.12).

To get a solution that satisfies (6.12), we make the following observation: Suppose that $\widetilde{h}(x,t)$ is a solution of the wave equation satisfying (6.12) for some function $\widetilde{g}(x)$. Define the function

$$y(x,t) = \frac{\partial}{\partial t}\widetilde{h}(x,t) \ .$$

Then, as an easy consequence of Clairault's Theorem.

$$\begin{aligned}
\frac{\partial^2}{\partial t^2} y(x,t) = \frac{\partial^3}{\partial t^3}\widetilde{h}(x,y) \quad &= \quad \frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial t^2}\widetilde{h}(x,t)\right) \\
&= \quad \frac{\partial}{\partial t}\left(c^2 \frac{\partial^2}{\partial x^2}\widetilde{h}(x,t)\right) \\
&= \quad c^2 \frac{\partial^2}{\partial x^2}\frac{\partial}{\partial t}\widetilde{h}(x,t) \\
&= \quad c^2 \frac{\partial^2}{\partial x^2} y(x,t) \ .
\end{aligned} \tag{6.15}$$

Hence, $y(x,t)$ satisfies the wave equation. By construction,

$$y(x,0) = \frac{\partial}{\partial t} h(x,t)\bigg|_{t=0} = 0$$

and

$$\frac{\partial}{\partial t} y(x,t)\bigg|_{t=0} = \frac{\partial^2}{\partial t^2}\widetilde{h}(x,t)\bigg|_{t=0} = c^2 \frac{\partial^2}{\partial x^2}\widetilde{h}(x,0) = c^2 \widetilde{g}''(x) \ .$$

Therefore, if we choose $\widetilde{g}(x)$ to solve

$$c^2 \widetilde{g}''(x) = v(x) \tag{6.16}$$

for all $x \in (0,L)$, then

$$\widetilde{h}(x,t) = \frac{\partial}{\partial t}\frac{1}{2}[\widetilde{g}(x-ct) + \widetilde{g}(x+ct)] = \frac{c}{2}[\widetilde{g}'(x+ct) - \widetilde{g}'(x-ct)] \tag{6.17}$$

is the solution we seek. We may take

$$\widetilde{g}(x) = \frac{1}{c^2}\int_0^x \left(\int_0^z v(w)\mathrm{d}w\right)\mathrm{d}z \ ,$$

in which case (6.28) gives the solution

$$h_2(x,t) = \frac{c}{2}[\widetilde{g}'(x+ct) - \widetilde{g}'(x-ct)] = \frac{1}{2c}\left(\int_0^{x+ct} v(z)\mathrm{d}z - \int_0^{x-ct} v(z)\mathrm{d}z\right) = \frac{1}{2c}\int_{x-ct}^{x+ct} v(z)\mathrm{d}z \ .$$

Therefore, superimposing the two solutions $h_1(x,t)$ and $h_2(x,t)$

$$h(x,t) = \frac{1}{2}[g(x-ct) + g(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} v(z)\mathrm{d}z \tag{6.18}$$

satisfies the wave equation and (6.11). The formula (6.18) is known as *d'Alembert's Formula.*

Example 49. *Let us use d'Alembert's Formula to find a solution on the wave equation on $\mathbb{R}$ such that*

$$h(x,0) = 0 \qquad \text{and} \qquad \frac{\partial}{\partial t}h(x,t)\bigg|_{t=0} = v(x)$$

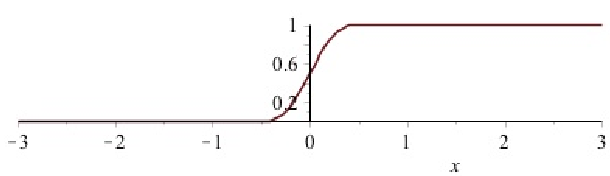
where $v(x)$ is the blip function

$$v(x) = \begin{cases} A(x^2-a^2)^2 & -a \le x \le a \\ 0 & |x| \ge a \end{cases}$$

for $a = 1/2$ and $A = 15$. This is somewhat taller version of the same blip as before, except now it is a velocity blip. Integrating, we find the function $f(x)$ given by

$$f(x) := \int_0^x v(z)\mathrm{d}z = \begin{cases} 0 & x \le -1/2 \\ \frac{1}{2} + \frac{15}{8}x - 5x^3 + 6x^5 & -1/2 \le x \le 1/2 \\ 1 & x \ge 1/2 \ . \end{cases}$$

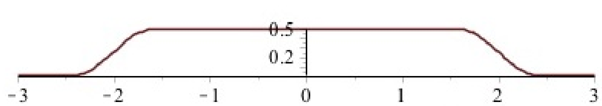
This is a smoothed sort of 'step function', smoothly interpolating between the value 0 for all $x \le -1/2$, and the value 1 for all $x \ge 1/2$. Here is a plot of $f(x)$ for $-3 \le x \le 3$.



By d'Alembert's Formula, with $c = 1$, we have that the solution is

$$h(x,t) = \frac{1}{2}[f(x+t) - f(x-t)] \ .$$

The following plot shows this difference of smooth step functions for $t = 2$:



Notice that the velocity blip spreads out from the center, becoming broader and broader. It is not a traveling wave, but the speed at which it spreads is exactly the speed of the traveling waves, which is natural since it was built out of traveling wave solutions.

Example 50. *It is interesting to see how the traveling wave solution $h(x,t) = g(x-ct)$ comes from d'Alembert's formula. We easily compute that with $h(x,t) = g(x-ct)$*

$$h(x,0) = g(x) \qquad \text{and} \qquad \frac{\partial}{\partial t}h(x,t)\bigg|_{t=0} = -cg'(x)$$

so that we have $v(x) = cg'(x)$ for this solution. But then

$$\frac{1}{2c}\int_{x-ct}^{x+ct} v(z)\mathrm{d}z = -\frac{1}{2}[g(x+ct) - g(x-ct)] \ .$$

Therefore, d'Alembert's formula gives

$$h(x,t) = \frac{1}{2}[g(x+ct)+g(x-ct)] - \frac{1}{2}[g(x+ct)-g(x-ct)] = g(x-ct) \ .$$

6.1.3 Conservation of energy and uniqueness

d'Alembert's Formula gives us a solution of the wave equation on $\mathbb{R}$ that satisfies the initial conditions (6.11). But is this the only such solution? The answer is yes, provided the initial data have *finite energy* as we now explain.

At first sight, there may not seem to be any issue with uniqueness, since we have derived the wave equation as the continuum limit of the system of $N-1$ ordinary differential equations (6.6). We can rewrite this as a system of $2(N-1)$ first order equations,

$$(\mathbf{y}, \mathbf{y}')' = -L(\mathbf{y}, \mathbf{y}')$$

where L is the $2(N-1) \times 2(N-1)$ matrix

$$L = \begin{bmatrix} 0 & I \\ -K & 0 \end{bmatrix}$$

with $-K$ in its lower left $(N-1)\times(N-1)$ block, the $(N-1)\times(N-1)$ identity matrix in its upper right block, and zeros everywhere else.

Like all matrices, L defines a Lipschitz vector field, and so we have a unique solution with

$$(\mathbf{y}(0), \mathbf{y}'((0)) = (\mathbf{g}, \mathbf{v})$$

for any $\mathbf{g}, \mathbf{v} \in \mathbb{R}^{N-1}$. However, the Lipschitz constant diverges to infinity as N tends to infinity, and so it diverges in the continuum limit. To see this note that

$$L\mathbf{e}_1 = \frac{c^2}{\Delta x^2}(\mathbf{e}_{N+1} - 2\mathbf{e}_N)$$

so that $\|L\mathbf{e}_1\| = \sqrt{5}c^2\Delta x^{-2}$ Taking $\mathbf{x} = \mathbf{0}$ and $\mathbf{y} = \mathbf{e}_1$, we have that

$$\|L\mathbf{y} - L\mathbf{x}\| = \sqrt{5}c^2\Delta x^{-2}\|\mathbf{y} - \mathbf{x}\| \ ,$$

and so the Lipschitz constant must be at least as large as $\sqrt{5}c^2\Delta x^{-2}$, which diverges to infinity as Δx goes to zero; i.e., as we take the continuum limit.

Therefore, our main uniqueness theorem for Lipschitz vector fields does not tell us anything in the continuum limit.

However, there is another method for proving uniqueness based on conservation of energy. Introduce the *potential energy function*

$$V(\mathbf{y}) = (\varrho\Delta x)\frac{1}{2}\mathbf{y}\cdot K\mathbf{y}\ .$$

Then $\nabla V(\mathbf{y}) = -c^2 K\mathbf{y}$, so that (6.6) can be written as

$$(\varrho\Delta x)\mathbf{y}''(t) = -\nabla V(\mathbf{y}(t)\ .$$

(We have restored the mass $\varrho\Delta x$ so that our energy has the correct units, though this constant multiple is not important for the conservation property itself.) Therefore, we define the *energy at time t*

$$E_t(\mathbf{y},\mathbf{y}') = \frac{1}{2}(\varrho\Delta x)\|\mathbf{y}'(t)\|^2 + V(\mathbf{y}(t)) = \frac{1}{2}(\varrho\Delta x)\|\mathbf{y}'(t)\|^2 + \frac{1}{2}(\varrho\Delta x)\mathbf{y}\cdot K\mathbf{y}\ , \tag{6.19}$$

and we have that $E_t(\mathbf{y},\mathbf{y}')$ does not depend on t when $\mathbf{y}(t)$ solves (6.6).

We can use the conservation of energy to prove uniqueness since the system (6.6) is linear. Let $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ be two solutions of (6.6). Then $\mathbf{z}(t) = \mathbf{y}_2(t) - \mathbf{y}_1(t)$ is again a solution, and so for any t and t_0,

$$E_t(\mathbf{z},\mathbf{z}') = E_{t_0}(\mathbf{z},\mathbf{z}')\ .$$

Now observe that is $\mathbf{y}_2(t_0) = \mathbf{y}_1(t_0)$, $\mathbf{z}(t_0) = 0$, and therefore $E_{t_0}(\mathbf{z},\mathbf{z}') = 0$. By the conservation of energy, $E_t(\mathbf{z},\mathbf{z}') = 0$ for all t. Writing this out in terms of $\mathbf{y}_1(t),\mathbf{y}_2(t)$, and dropping a constant multiplicative factor,

$$\|\mathbf{y}_1'(t) - \mathbf{y}_2'(t)\|^2 + (\mathbf{y}_2(t) - \mathbf{y}_1(t))\cdot K(\mathbf{y}_2(t) - \mathbf{y}_1(t)) = 0\ .$$

Hence $\mathbf{y}_2'(t) = \mathbf{y}_1'(t)$ for all t, and then since $\mathbf{y}_2(t_0) = \mathbf{y}_1(t_0)$, $\mathbf{y}_2(t) = \mathbf{y}_1(t)$ for all t.

As we now show, conservation of energy still holds in the continuum limit, and gives us a proof of uniqueness whenever the initial data for the wave equation has finite energy.

To take the continuum limit of the energy function (6.19) return to (6.7) to note that

$$\mathbf{y}\cdot K\mathbf{y} = \sum_{j=0}^{N-1}\left(\frac{y_{j+1}-y_j}{\Delta x}\right)^2$$

where we define $y_0 = 0$ and $y_N = 0$,. Letting $y_j(t) = h(x_j,t)$ as before, we see that

$$\lim_{\Delta x\to 0}\varrho\Delta x\,(\mathbf{y}\cdot K\mathbf{y}) = \varrho\sum_{j=1}^{N-2}\left(\frac{h(x_{j+1},t)-h(x_j,t)}{\Delta x}\right)^2\Delta x$$

which is a Riemann sum approximation to

$$\varrho\int_{\mathbb{R}}\left|\frac{\partial}{\partial x}h(x,t)\right|^2\mathrm{d}x\ .$$

Likewise, $\|\mathbf{y}'\|^2\varrho\Delta x$ is a Riemann sum approximation to

$$\varrho\int_{\mathbb{R}}\left|\frac{\partial}{\partial t}h(x,t)\right|^2\mathrm{d}x\ .$$

Therefore, given a continuously differentiable function $h(x,t)$, define the *energy at time* t to be

$$E_t(h,\partial h/\partial t) = \frac{\varrho}{2}\int_{\mathbb{R}}\left|\frac{\partial}{\partial t}h(x,t)\right|^2 \mathrm{d}x + \frac{\varrho c^2}{2}\int_{\mathbb{R}}\left|\frac{\partial}{\partial x}h(x,t)\right|^2 \mathrm{d}x \ .$$

Then, supposing that h solves the wave equation on $\mathbb{R}$ and both h and $\partial h/\partial t$ tend to zero as x tends to infinity, so that we can integrate by parts,

$$\begin{aligned}\frac{1}{\varrho}\frac{\mathrm{d}}{\mathrm{d}t}E_t(h,\partial h/\partial t) &= \int_{\mathbb{R}}\frac{\partial}{\partial t}h(x,t)\frac{\partial^2}{\partial t^2}h(x,t)\mathrm{d}x + \frac{c^2}{2}\int_{\mathbb{R}}\frac{\partial}{\partial x}h(x,t)\frac{\partial^2}{\partial x\partial t}h(x,t)\mathrm{d}x \\ &= \int_{\mathbb{R}}\frac{\partial}{\partial t}h(x,t)\frac{\partial^2}{\partial t^2}h(x,t)\mathrm{d}x - \frac{c^2}{2}\int_{\mathbb{R}}\frac{\partial^2}{\partial x^2}h(x,t)\frac{\partial}{\partial t}h(x,t)\mathrm{d}x \\ &= \int_{\mathbb{R}}\left[\frac{\partial^2}{\partial t^2}h(x,t) - c^2\frac{\partial^2}{\partial x^2}h(x,t)\right]\frac{\partial}{\partial t}h(x,t)\mathrm{d}x = 0 \ . \end{aligned} \tag{6.20}$$

The boundary terms we have discarded in the integration by part are

$$\lim_{x\to\infty}\frac{\partial}{\partial x}h(x,t)\frac{\partial}{\partial t}h(x,t) - \lim_{x\to-\infty}\frac{\partial}{\partial x}h(x,t)\frac{\partial}{\partial t}h(x,t) \ ,$$

so as long as out solutions tend to zero at spatial infinity, there is no problem. In fact, it is not hard to justify the integration by parts only under the condition that the energy is finite.

Now we proceed as before. Let $h_1(x,t)$ and $h_2(x,t)$ be two solutions of the wave equation, and suppose that both satisfy (6.11) for given $g(x)$ and $v(x)$. Suppose furthermore that

$$\frac{1}{2}\int_{\mathbb{R}} g^2(x)\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} v^2(x)\mathrm{d}x < \infty \ .$$

Then both solutions have finite energy. Let $h(x,t) = h_1(x,t) - h_2(x,t)$. Then $h(x,t)$ solves the wave equation, and for each t, it has finite energy. Indeed, using the inequality $(a-b)^2 \le 2(a^2+b^2)$ in the integrals defining the energy, is is simple to see the the difference (or sum) of two finite energy solutions is a finite energy solution.

Hence, the energy is conserved, and since both $h(x,0) = 0$ and $\partial h/\partial t(x,0) = 0$ for all x, it follows that the energy of $h(x,t)$ is zero for all t. That is, for all t,

$$\frac{1}{2}\int_{\mathbb{R}}\left|\frac{\partial}{\partial t}(h_2(x,t)-h_1(x,t))\right|^2 \mathrm{d}x + c^2\frac{1}{2}\int_{\mathbb{R}}\left|\frac{\partial}{\partial x}(h_2(x,t)-h_1(x,t))\right|^2 \mathrm{d}x = 0 \ .$$

Therefore the x,t gradient $h_2(x,t) - h_1(x,t)$ is zero for all x,t, and so $h_2(x,t) - h_1(x,t)$ is constant. Since $h_2(x,0) - h_1(x,0) = g(x) - g(x) = 0$, it follows that $h_2(x,t) = h_1(x,t)$ for all x,t. We have proved:

Theorem 30. *Let $g(x)$ and $v(x)$ be continuous square integrable functions on $\mathbb{R}$. There is exactly one solution to the wave equation (6.8) on $\mathbb{R}$ that satisfies (6.11), and this solution is given by d'Alembert's formula (6.18)*

In particular, we see that every finite energy solution of the wave equation is a linear combination of the special solutions satisfying the first order equations in (6.10), so that these special traveling wave solutions are not special after all.

6.1.4 The wave equation for a pinned string on $[0, L]$

We now return to the problem of solving the wave equation on $[0, L]$ subject to the boundary conditions

$$h(0,t) = h(L,t) = 0 \qquad \text{for all} \quad t \ , \tag{6.21}$$

which represent the condition that the ends of the string are pinned down.

Given functions $g(x)$ and $v(x)$ on $[0, L]$, we see to find a function $h(x,t)$ satisfying the wave equation, the boundary conditions (6.21) and the initial data

$$h(x,t) = g(x) \qquad \text{and} \qquad \left.\frac{\partial}{\partial t}h(x,t)\right|_{t=0} = v(x) \tag{6.22}$$

Because of the boundary conditions,

$$\left.\frac{\partial}{\partial t}h(x,t)\right|_{x=0} = \left.\frac{\partial}{\partial t}h(x,t)\right|_{x=L} = 0 \qquad \text{for all } t \ , \tag{6.23}$$

and consequently, we must have $v(0) = v(L) = 0$. Furthermore, we shall assume that $g(x)$ is continuously differentiable on $(0, L)$ with a derivative that extends continuously to $[0, L]$, and that $v(x)$ is continuous on $[0, L]$.

The energy of such a solution is defined to be

$$E_t(h, \partial h/\partial t) = \frac{\varrho}{2}\int_0^L \left|\frac{\partial}{\partial t}h(x,t)\right|^2 \mathrm{d}x + \frac{\varrho c^2}{2}\int_0^L \left|\frac{\partial}{\partial x}h(x,t)\right|^2 \mathrm{d}x \ .$$

In this case, the boundary terms in the integration by part if (6.20) are clearly zero due to (6.23). Therefore, the energy is constant along solutions of the wave equation.

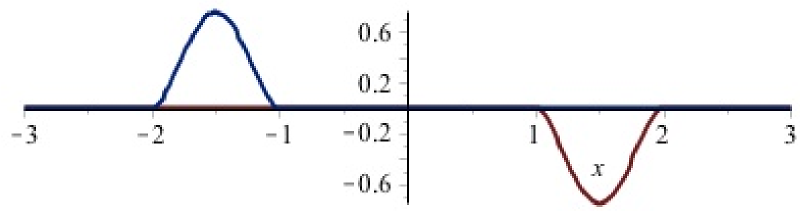
It follows that if $h_1(x,t)$ and $h_2(x,t)$ are two solutions of the wave equation on $[0, L]$ satisfying (6.21) and (6.22), then $h(x,t) = h_1(x,t) - h_2(x,t)$ is a solution whose energy is zero at time $t = 0$, and therefore, for all t. But then, just as on the whole real line, this implies that $h(x,t) = 0$ for all $x \in [0, L]$ and $t \in \mathbb{R}$. Therefore, we know that solutions of the wave equation satisfying (6.21) and (6.22) are unique whenever they exist.

We now show that solutions do exist. The main idea is to extend the initial data to the whole real line in a manner that produces traveling wave that cross each other and 'cancel out' as they cross the boundaries at $x = 0$ and $x = L$.

Before going into the details, let us explain this simple idea more fully. Consider once more our 'blip function' initial data. Let $g_1(x)$ be the blip function shifted so that it is centered on $x = -3/2$. Define

$$g_2(x) = -g_1(-x) \ ,$$

which will be a negative blip; i.e., and 'anti-blip' centered on $x = 3/2$. The next picture shows the superposition of g_1 and g_2.



Now consider the solution of the wave equation given by

$$h(x,t) = g_1(x - ct) + g_2(x + ct) \ .$$

Then the blip on the left moves to the right with speed c, while the anti-blip on the right moves to the left with speed c. By the formula defining g_2,

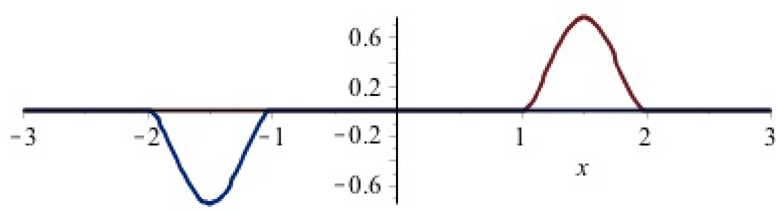
$$g_2(x - ct) = -g_1(ct - x) \ ,$$

and so

$$h(0,t) = g_1(ct) + g_2(ct) = g_1(ct) - g_1(ct) = 0 \ .$$

That is, the blip and the anti-blip cancel each other out at $x = 0$ – but then they pass through each other and reemerge. At the single instant when the blip and anti-blip are exactly on top of each other, the string is completely flat. However, the velocity of the string is not zero, so the blips reemerge.

After they have passed each other, their shapes return and they keep going. Here is a graph of $y = h(x,t)$ for a time t large enough that the blips have cleared one-another. The blip on the right keeps moving to the right, and the anti-blip on the left keeps moving to the left. The condition $h(x,t) = 0$ is satisfied for all t.



Notice that the solution $h(x,t)$ we have plotted has the property that for each t is is *antisymmetric* as a function of x:

$$h(-x,t) = -h(x,t)$$

for all x, t.

To take care of the boundary conditions at both $x = 0$ and $x = L$, we introduce an important class of functions:

Definition 20 (Doubly antisymmetric functions)**.** *A function real valued function g on $\mathbb{R}$ is antisymmetric about $x = 0$ and $x = L$ provided*

$$g(-x) = -g(x) \qquad \text{and} \qquad g(L - x) = -g(L + x)$$

for all $x \in \mathbb{R}$.

Geometrically, a function g is antisymmetric about $x = L$ if upon reflecting the graph $y = g(x)$ about *both* the lines $x = L$ and $y = 0$, the graph is unchanged. For example, the function $g(x) = \sin(x)$ is antisymmetric about both $x = 0$ and $x = \pi$. In fact, the function $g(x) = \sin(x)$ is antisymmetric about both $x = k\pi$ for all $k \in \mathbb{N}$. From here, it is easy to see that each of the functions

$$g_k(x) = \sin(k\pi x/L) \ ,$$

$k \in \mathbb{N}$ is antisymmetric about $x = 0$ and $x = L$.

The relevance of the definition is this: If $g(x)$ is antisymmetric about $x = 0$ and $x = L$, then

$$h(x,t) = \frac{1}{2}[g(x + ct) + g(x - ct)] \tag{6.24}$$

satisfies

$$h(0,t) = \frac{1}{2}[g(ct) + g(-ct)] = 0 \qquad \text{and} \qquad h(L,t) = \frac{1}{2}[g(L + ct) + g(L - ct)] = 0 \ .$$

Since differentiation is linear, any linear combination of solutions of the wave equation is again a solution of the wave equation. Therefore, by what e have noted above, when $g(x)$ is antisymmetric about $x = 0$ and $x = L$, and $h(x,t)$ is defined in therms of $g(x)$ by (6.24), then $h(x,t)$ satisfies the wave equation and the boundary condition $h(0,t) = h(L,t)$ for all t.

Furthermore, it is clear that with this definition,

$$h(x,t) = \frac{1}{2}[g(x) + g(x)] = g(x) \qquad \text{and} \qquad \frac{\partial}{\partial t}h(x,t)\bigg|_{t=0} = \frac{1}{2}[cg'(x) - cg'(x)] = 0 \ .$$

In summary, whenever $g(x)$ is antisymmetric about $x = 0$ and $x = L$, (6.24) defines a solution of the wave equation satisfying the boundary conditions $h(0,t) = h(L,t)$ for all t and with initial data

$$h(x,t) = g(x) \qquad \text{and} \qquad \frac{\partial}{\partial t}h(x,t)\bigg|_{t=0} = 0 \tag{6.25}$$

for all $x \in (0,L)$.

We are now ready to show that we always have a solution of our initial data problem in the special case $v(x) = 0$; i.e., the case in which the initial data id of the form (6.25).

All we need do is to observe that every twice continuously differentiable function $g(x)$ defined on $[0,L]$ such that $g(0) = g(L) = 0$ has a unique extension to all of $\mathbb{R}$ that is antisymmetric about $x = 0$ and $x = L$. This function can be produced by 'repeated reflection', as we now explain:

Since $g(x)$ is required to be antisymmetric about $x = 0$, knowing the values of $g(x)$ for $x \in [0,L]$, we know them also for $x \in [-L,0]$: For $x \in [0,L]$, $g(x) = -g(-x)$. This extends the domain of definition of g from $[0,L]$ to $[-L,L]$.

Next, since $g(x)$ is required to be antisymmetric about $x = L$, knowing the values of $g(x)$ for $x \in [-L, L]$, we know them also for $x \in [-L, 3L]$: For $z \in [0, 2L]$, $g(L + z) = -g(L - z)$. Since $L - z \in [-L, L]$, the right hand side is known, and this gives us the values of $g(L + z)$ for $z\ in[0, 2L]$, or, what is the same, the values of $g(x)$ for $x \in [L, 3L]$. Repeating the procedure extends g to the whole real line.

The extended function is continuously differentiable since it continues across the points of reflection with the same slope. However there may be a 'jump' in the second derivative at the points $x = jL$, j an integer. For instance, if g is concave on $[0, L]$ (i.e., $g''(x)$ is negative on $[0, L]$, then the extension by reflection will be convex on $[L, 0]$: The second derivative will change sign going across the points of reflection. However, if the second derivative of g is zero at $x = 0$ and $x = L$, then the extended function will be twice continuously differentiable on all of $\mathbb{R}$, as is the case, for $\sin(k\pi x/L)$. We have proved:

Lemma 9. *Let $g(x)$ be a twice continuously differentiable function on $(0, L)$ such that*

$$\lim_{x\to 0,L} g(x) = \lim_{x\to 0,L} g''(x) = 0 \ .$$

Let $g(x)$ also denote the unique extension of g to all of $\mathbb{R}$ that is antisymmetric about $x = 0$ and $x = L$. Then $g(x)$ is twice continuously differentiable on $\mathbb{R}$, and

$$h(x, t) = \frac{1}{2}[g(x - ct) + g(x + ct)]$$

satisfies the wave equation (6.8) for all x, t, and the boundary conditions (6.9) for all t, and the initial condition (6.25).

The solutions constructed in Lemma 9 correspond to pulling the string into an initial profile $g(x)$, and then letting the string go, giving it no initial velocity. For example if one plucks a string on a violin, one sets it into motion in this way.

But what about imparting an initial velocity as well? We would also like to solve the wave equation (subject to the boundary conditions) with the initial data

$$h(x, t) = 0 \qquad \text{and} \qquad \frac{\partial}{\partial t}h(x, t)\bigg|_{t=0} = v(x) \tag{6.26}$$

for all $x \in (0, L)$.

Just as on the whole real line, we can leverage Lemma9 by taking the time derivative of the solutions its provides to construct solutions with a specified initial velocity: Suppose that $h(x, t)$ is the solution of the wave equation described in Lemma 9 for some function $g(x)$. As before, define the function

$$y(x, t) = \frac{\partial}{\partial t}h(x, t) \ .$$

Then, as an easy consequence of Clairault's Theorem.

$$\begin{aligned}\frac{\partial^2}{\partial t^2}y(x,t) = \frac{\partial^3}{\partial t^3}h(x,y) &= \frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial t^2}h(x,t)\right)\\ &= \frac{\partial}{\partial t}\left(c^2\frac{\partial^2}{\partial x^2}h(x,t)\right)\\ &= c^2\frac{\partial^2}{\partial x^2}\frac{\partial}{\partial t}h(x,t)\\ &= c^2\frac{\partial^2}{\partial x^2}y(x,t)\ .\end{aligned}$$

Furthermore $y(0,t) = y(L,t) = 0$ since $h(0,t)$ and $h(L,t)$ are constant. Hence $y(x,t)$ satisfies both the wave equation and the boundary conditions (6.9). By construction,

$$y(x,0) = \frac{\partial}{\partial t}h(x,t)\bigg|_{t=0} = 0$$

and

$$\frac{\partial}{\partial t}y(x,t)\bigg|_{t=0} = \frac{\partial^2}{\partial t^2}h(x,t)\bigg|_{t=0} = c^2\frac{\partial^2}{\partial x^2}h(x,0) = c^2 g''(x)\ .$$

Therefore, if we choose $g(x)$ to solve

$$c^2 g''(x) = v(x) \tag{6.27}$$

for all $x \in (0,L)$, and let $h(x,t)$ be the corresponding solution of the wave equation provided by Lemma 9, then its time derivative $y(x,t)$ is the solution we seek.

To apply Lemma 9, we need not only that (6.27) be satisfied, but also that $g(0) = g(L) = 0$.

This is our first encounter with a *boundary value problem* for an ordinary differential equation. We are required to solve (6.27) not for given values $g(0)$ and $g'(0)$, but for given values of $g(0)$ and $g(L)$; i.e., the values of g at at the boundary points of its domain.

(There is one more condition in Lemma 9, which is that $g''(0) = g''(L) = 0$. However, since $v(0) = 0$, we will automatically have this for any solution of (6.27).)

To solve this simple boundary value problem, note that if $g(x)$ is any solution of (6.27), then the function $g(x) - ax - b$ satisfies the same equation for all a and b. Defining $g(x)$ as a double integral, and subtracting off the requisite linear term, we get out solution: Define

$$g(x) = \frac{1}{c^2}\left[\int_0^x\left(\int_0^z v(w)\mathrm{d}w\right)\mathrm{d}z - \frac{x}{L}\int_0^L\left(\int_0^z v(u)\mathrm{d}u\right)\mathrm{d}z\right]\ .$$

Clearly, $g(0) = g(L) = 0$ and (6.27) is satisfied. Let $w(x)$ denote the doubly antisymmetric extension of this function $g(x)$ to all of $\mathbb{R}$, then

$$h(x,t) = \frac{\partial}{\partial t}\frac{1}{2}[w(x-ct) + w(x+ct)] = \frac{c}{2}[w'(x+ct) - w'(x-ct)] \tag{6.28}$$

is the solution we seek.

Altogether we have proved:

Theorem 31. *Let $g(x)$ be a twice continuously differentiable function on $(0, L)$ such that*

$$\lim_{x \to 0, L} g(x) = \lim_{x \to 0, L} g''(x) = 0 \ .$$

Let $g(x)$ also denote the unique extension of g to all of $\mathbb{R}$ that is antisymmetric about $x = 0$ and $x = L$.

Let $v(x)$ be continuous on $[0, L]$ with $v(0) = v(L) = 0$, and define $w(x)$ on $[0, L]$ by

$$w(x) = \frac{1}{c^2}\left[\int_0^x \left(\int_0^z v(w)\mathrm{d}w\right)\mathrm{d}z - \frac{x}{L}\int_0^L \left(\int_0^z v(u)\mathrm{d}u\right)\mathrm{d}z\right] \ ,$$

and we let $w(x)$ also denote the unique extension of w to all of $\mathbb{R}$ that is antisymmetric about $x = 0$ and $x = L$.

Then

$$h(x, t) = \frac{1}{2}[g(x - ct) + g(x + ct)] + \frac{c}{2}[w'(x + ct) - w'(x - ct)]$$

satisfies the wave equation (6.8) for all x, t, and the boundary conditions (6.21) for all t, and the initial condition (6.22), and it is the only such solution.

Example 51. *Fix positive integers m and n, and define*

$$g(x) = \sin(m\pi x/L) \qquad \text{and} \qquad v(x) = \sin(n\pi x/L) \ .$$

Then

$$\int_0^x \left(\int_0^z v(u)\mathrm{d}u\right)\mathrm{d}z = \frac{L}{n\pi}\int_0^x [1 - \cos(n\pi z/L)]\mathrm{d}z = \frac{L}{n\pi}x - \left(\frac{L}{n\pi}\right)^2 \sin(n\pi x/L) \ ,$$

so that

$$w(x) = \frac{1}{c^2}\left[\frac{L}{n\pi}x - \left(\frac{L}{n\pi}\right)^2 \sin(n\pi x/L) - \frac{L}{n\pi}x\right] = \left(\frac{L}{cn\pi}\right)^2 \sin(n\pi x/L)$$

and finally,

$$w'(x) = \frac{L}{c^2 n\pi}\cos(n\pi x/L) \ .$$

The solution then is

$$h(x, t) = \frac{1}{2}[\sin(m\pi(x + ct)/L) + \sin(m\pi(x - ct)/L)] + \frac{L}{2n\pi c}[\cos(n\pi(x + ct)/L) - \cos(n\pi(x - ct)/L)] \ .$$

By the angle addition formulae,

$$\frac{1}{2}[\sin(m\pi(x + ct)/L) + \sin(m\pi(x - ct)/L)] = \sin(m\pi x/L)\cos(m\pi ct/L)$$

and

$$\frac{1}{2}[\cos(n\pi(x + ct)/L) - \cos(n\pi(x - ct)/L)] = \sin(n\pi x/L)\sin(n\pi ct/L) \ .$$

Finally,

$$h(x, t) = \sin(m\pi x/L)\cos(m\pi ct/L) + \sin(n\pi x/L)\frac{L\sin(n\pi ct/L)}{n\pi c} \ .$$

It is clear that $h(x, 0) = \sin(m\pi x/L)$, and differentiating, we find

$$\frac{\partial}{\partial t}h(x, t) = -\frac{cm\pi}{L}\sin(m\pi x/L)\sin(m\pi ct/L) + \sin(n\pi x/L)\cos(n\pi ct/L) \ ,$$

from which it is clear that $\left.\frac{\partial}{\partial t}h(x,t)\right|_{t=0} = \sin(n\pi x/L)$.

Notice that this solution is a superposition of two solutions (one determined by g and the other determined by v) both of which are of th form

$$u(x)v(t)$$

for some function u of x along and some function v of t alone.

6.1.5 Separation of variables

In Example 51, we found solutions of the wave equation of a very special form:

$$h(x,t) = u(x)v(t) \ . \tag{6.29}$$

In such a solution, one might say that the variables *separate*. We now ask: For which functions u and v does (6.29) define a solution of the wave equation?

Computing,

$$\frac{\partial^2}{\partial t^2}u(x)v(t) = u(x)v''(t) \qquad \text{and} \qquad \frac{\partial^2}{\partial x^2}u(x)v(t) = u''(x)v(t) \ .$$

Therefore, (6.29) defines a solution of the wave equation exactly when

$$u(x)v''(t) = c^2u''(x)v(t) \ . \tag{6.30}$$

For x, t such that $u(x) \neq 0$ and $v(t) \neq 0$, this is equivalent to

$$c^{-2}\frac{v''(t)}{v(t)} = \frac{u''(x)}{u(x)} \ . \tag{6.31}$$

Since the left side is independent of x and the right hand side is independent of t, both sides must be constant on the set of all (x,t) such that $u(x) \neq 0$ and $v(t) \neq 0$.

Let λ denote the constant. We seek to solve

$$u''(x) = \lambda u(x) \tag{6.32}$$

subject to the boundary conditions

$$u(0) = u(L) = 0 \ . \tag{6.33}$$

Suppose that $u(x)$ is such a solution. Then

$$\lambda\int_0^L U^2(x)\mathrm{d}x = \int_0^L u''(x)u(x)\mathrm{d}x = -\int_0^L |u'(x)|^2\mathrm{d}x \ .$$

where we have integrated by parts, and there are no boundary terms due to (6.33). The right hand side is strictly negative (unless $u(x) = 0$ for all x), and we conclude that

$$\lambda < 0$$

whenever $u(x)$ is a non-trivial solution.

Now fix $\lambda < 0$, and let us try to solve (6.32) subject to (6.33). The general solution of (6.32) is

$$u(x) = a\sin(\sqrt{|\lambda|}x) + b\cos(\sqrt{|\lambda|}x) \ .$$

The condition $u(0) = 0$ tells us $b = 0$, and then the condition $u(L) = 0$ tells us that

$$\sin(\sqrt{|\lambda|}L) = 0 \ .$$

This in turn tells us that $\sqrt{|\lambda|}L$ is an integer multiple of π. That is,

$$\sqrt{|\lambda|}L = k\pi \ ,$$

where k is a non-negative integer. We therefore define

$$u_k(x) = \sin(k\pi x/L) \qquad k = 1, 2, \dots \ . \tag{6.34}$$

Therefore, there is a non-trivial solution of (6.32) that satisfies (6.33) if and only if $\lambda = -k^2\pi^2$. Let us define

$$\lambda_k = -\frac{k^2\pi^2}{L^2} \qquad k = 1, 2, \dots \ . \tag{6.35}$$

Now let us find the functions $v_k(t)$ such that $u_k(x)v_k(t)$ solves the wave equation. The equation

$$c^{-2}\frac{v''(t)}{v(t)} = \lambda_k$$

is the same as (for t such that $v(t) \neq 0$) the equation

$$v''(t) = -\left(\frac{ck\pi}{L}\right)^2 v(t) \ .$$

We know that the general solution v_k of this equation is $v_k(t) = a\cos(ck\pi t/L) + b\sin(ck\pi t/L)$. Replacing the constant b by the constant $Lb/(ck\pi)$ for convenience later on, we define

$$v_k(t) = a\cos(ck\pi t/L) + b\frac{L\sin(ck\pi t/L)}{ck\pi} \tag{6.36}$$

for some constants $a, b \in \mathbb{R}$.

In conclusion, for each $k = 1, 2, \dots$, let u_k be given by (6.34) and let v_k be given by (6.36) for any $a, b \in \mathbb{R}$. Then

$$h_k(x,t) = u_k(x)v_k(t)$$

solve the wave equation and satisfies the boundary conditions $h(0,t) = h(L,t) = 0$ for all t.

We see that

$$h_k(x,0) = a\sin(k\pi x/L) \qquad \text{and} \qquad \frac{\partial}{\partial t}h_k(x,t)\bigg|_{t=0} = b\sin(k\pi x/L) \ . \tag{6.37}$$

Therefore, by what we have proved above, $h_k(x,t)$ is the unique solution of the wave equation satisfying the boundary condition $h(x,t) = h(L,t) = 0$ for all t and the initial conditions (6.37).

By taking linear combinations of such solutions, we arrive at the unique solution of the wave equation for a very wide class of initial data. Suppose that for some integer N and numbers $a_1, \dots, a_N$ and $b_1, \dots b_N$,

$$g(x) = \sum_{k=1}^{N} a_k\sin(k\pi x/L) \qquad \text{and} \qquad v(x) = \sum_{k=1}^{N} b_k\sin(k\pi x/L) \ . \tag{6.38}$$

Then the unique solutions of the wave equation satisfying the boundary condition $h(x,t) = h(L,t) = 0$ for all t and the initial conditions

$$h_k(x,0) = g(x) \qquad \text{and} \qquad \frac{\partial}{\partial t} h_k(x,t)\bigg|_{t=0} = v(x) \tag{6.39}$$

is

$$h(x,t) = \sum_{k=1}^{N} \sin(k\pi x/L) \left[a_k \cos(ck\pi t/L) + b_k \frac{L \sin(ck\pi t/L)}{ck\pi} \right] . \tag{6.40}$$

This observation is useful because it turns out that every continuous function $f(x)$ on $[0,L]$ satisfying $f(0) = f(L)$ has a *uniformly convergent Fourier series*

$$f(x) = \sum_{k=1}^{\infty} \alpha_k \sin(k\pi x/L) \tag{6.41}$$

where for each k

$$\alpha_k = \frac{2}{\sqrt{L}} \int_0^L f(x) \sin(k\pi x/L) . \tag{6.42}$$

The sequence $\{\alpha_k\}$ is the sequence of *Fourier coefficients* of f.

We shall not prove this in full detail, but part of the story is simple and well-worth explaining here.

Definition 21. *Let f and g be two real-valued continuous functions on $[0,L]$. Let $\varrho(x)$ be a given strictly positive function on $[0,l]$. The*inner product *of f and g with respect to the weight ϱ is the quantity $\langle f, g\rangle$ defined by*

$$\langle f, g\rangle = \int_0^L f(x)g(x)\varrho(x)\mathrm{d}x .$$

(Dince usually the wieght is fixed in advance, we do not make it explicit in our notation.) The L^2-norm *of f is the number $\|f\|_2$ defined by*

$$\|f\|_2 = \sqrt{\langle f, f\rangle} .$$

Two continuous functions f and g on $[0,L]$ are orthogonal *in case*

$$\langle f, g\rangle = 0 .$$

A sequence of continuous functions $\{f_n\}$ on $[0,L]$ is orthonormal *in case*

$$\langle f_n, f_m\rangle = 0 \quad \text{for all} \quad n \neq m \quad \text{and} \quad \|f_n\|_2 = 1 \quad \text{for all} \quad n .$$

To motivate this definition, pick a large $N \in \mathbb{N}$, and let $\Delta x = L/N$. Define $x_j = j\Delta x$. Then the Riemann sum approximation to integral that defines $\langle f, g\rangle$ for this partition of $[0,L]$ is

$$\int_0^L f(x)g(x)\mathrm{d}x \approx \sum_{j=0}^{N-1} f(x_j)g(x_j)\varrho(x_j)\Delta x .$$

Define the vector $\mathbf{f}_N \in \mathbb{R}^N$ by

$$\mathbf{f}_N = \sqrt{\varrho(x_j)\Delta x}(f(x_0), f(x_1), \ldots, f(x_{N-1}))$$

and likewise for $\mathbf{g}_N$. We then have

$$\langle f, g\rangle = \lim_{N\to\infty} \mathbf{f}_N \cdot \mathbf{g}_N \ ,$$

and also

$$\|f\|_2^2 = \lim_{N\to\infty} \mathbf{f}\cdot\mathbf{f} = \lim_{N\to\infty} \|\mathbf{f}_N\|^2 \ .$$

Thus, the inner product is a limiting form of the dot product in $\mathbb{R}^N$, and this justifies our terminology. Furthermore, by the Cauchy-Schwarz inequality in $\mathbb{R}^N$,

$$|\mathbf{f}_N \cdot \mathbf{g}_N| \le \|\mathbf{f}_N\|\|\mathbf{g}_N\| \ .$$

Therefore, taking the limit $N \to \infty$, we obtain the *Cauchy-Schwarz inequality for integrals*:

$$|\langle f, g\rangle| \le \|f\|_2\|g\|_2 \ . \tag{6.43}$$

Likewise, it follows that $\|f+g\|_2 \le \|f\|_2 + \|g\|_2$, and so we can define the *mean square distance* $d_2(f,g)$ between two continuous functions f and g on $[0,L]$ to be the quantity

$$d_2(f,g) = \|f-g\|_2 \ .$$

Theorem 32. *For each $k \in \mathbb{N}$, define the function*

$$u_k(x) = \sqrt{\frac{2}{L}}\sin(k\pi x/L) \ .$$

Then $\{u_k\}$ is orthonormal with respect to the uniform weight $\varrho(x) = 1$.

Proof. By the identity

$$\sin(k\pi x/L)\sin(\ell\pi x/L) = \frac{1}{2}[\cos((k-\ell)\pi x/L) - \cos((k+\ell)\pi x/L)[\ ,$$

we have that for $k \neq \ell$

$$\int_0^L \sin(k\pi x/L)\sin(\ell\pi x/L)\mathrm{d}x = \frac{L}{2\pi}\left[\frac{1}{k-\ell}\sin((k-\ell)x) - \frac{1}{k+\ell}\sin((k+\ell)x)\right]\Big|_0^l = 0$$

and for $k = \ell$,

$$\int_0^L \sin(k\pi x/L)\sin \ell\pi x/L)\mathrm{d}x = \frac{1}{2}\int_0^L [1-\cos(2k\pi x/L)]\mathrm{d}x = \frac{L}{2} \ .$$

□

Given a function a continuous $f(x)$ on $[0,L]$, let us try to approximate $f(x)$ in the mean square sense by a function of the form $\sum_{k=1}^N \alpha_k u_k$.

Theorem 33 (Bessel's theorem)**.** *Let f be a continuous function on $[0,L]$. Let $\{u_n\}$ be any orthonormal sequence with respect to any weight function ϱ. Let $\|\cdot\|_2$ denote the corresponding norm.*

Then for all N and all numbers $\alpha_1,\dots,\alpha_N$,

$$\left\|f - \sum_{k=1}^N \alpha_k u_k\right\|_2^2 = \|f\|_2^2 - \sum_{k=1}^N \langle f, u_k\rangle^2 + \sum_{k=1}^N \alpha_k^2 = \sum_{k=1}^N (\langle f, u_k\rangle - \alpha_k)^2 \ .$$

In particular, among all functions of the form $\sum_{k=1}^{N} \alpha_k u_k$*, the best mean square approximation to* f *is given by*

$$f_N(x) := \sum_{k=1}^{N} \langle f, u_k \rangle u_k(x) \ , \tag{6.44}$$

and

$$\|f - f_N\|_2^2 = \|f\|_2^2 - \sum_{k=1}^{N} \langle f, u_k \rangle^2 \ . \tag{6.45}$$

Proof. By definition,

$$\begin{aligned} \left\| f - \sum_{k=1}^{N} \alpha_k u_k \right\|_2^2 &= \left\langle f - \sum_{k=1}^{N} \alpha_k u_k, f - \sum_{\ell=1}^{N} \alpha_\ell u_\ell \right\rangle \\ &= \langle f, f \rangle - 2 \sum_{k=1}^{N} \alpha_k \langle f, u_k \rangle + \sum_{k,\ell=1}^{N} \alpha_k \alpha_\ell \langle u_k, u_\ell \rangle \\ &= \|f\|_2^2 - 2 \sum_{k=1}^{N} \alpha_k \langle f, u_k \rangle + \sum_{k=1}^{N} \alpha_k^2 \end{aligned}$$

where we have used the orthonormality of $\{u_k\}$ in the last step. Completing the square,

$$-2 \sum_{k=1}^{N} \alpha_k \langle f, u_k \rangle + \sum_{k=1}^{N} \alpha_k^2 = \sum_{k=1}^{N} (\langle f, u_k \rangle - \alpha_k)^2 - \sum_{k=1}^{N} \langle f, u_k \rangle^2 \ ,$$

and this proves the identity. □

As a consequence of (6.45), $\sum_{k=1}^{N} \langle f, u_k \rangle^2 \leq \|f\|_2^2$, and therefore the infinite sum $\sum_{k=1}^{\infty} \langle f, u_k \rangle^2$ is convergent and

$$\sum_{k=1}^{\infty} \langle f, u_k \rangle^2 \leq \|f\|_2^2 \ . \tag{6.46}$$

This is *Bessel's inequality.* It has the following consequence: Let f_N be the Nth partial sum of the Fourier series as in (6.44). Then is is a simple consequence of the orthonormality that for $N > M$,

$$\|f_N - f_M\|_2^2 = \sum_{j=M+1}^{N} \langle f, u_k \rangle^2 \leq \sum_{j=M+1}^{\infty} \langle f, u_k \rangle^2$$

and so by (6.46) for all $\epsilon > 0$ there is an M_ϵ so that

$$\sum_{j=M_\epsilon+1}^{\infty} \langle f, u_k \rangle^2 \leq \epsilon^2 \ .$$

This means that

$$M, N \geq M_\epsilon \Rightarrow \|f_N - f_M\|_2 \leq \epsilon \ .$$

In other words, the sequence $\{f_N\}$ is a Cauchy sequence in the means-square metric. One of the key advantages of the Lebesgue Theory of Integration is that if one extands the space of Reimann suqare-integrable functions to include the large class of Lebesgue square integrable functions, our metric space becomes a *complete* metric space, maening that the Cauchy sequence converges to some square integrale function $\tilde{f}$, which may or may not be the same as f.

However, since

$$\tilde{f} = \sum_{k=1}^{\infty} \langle f, u_k \rangle u_k ,$$

$\langle \tilde{f}, u_k \rangle = \langle f, u_k \rangle$, and so

$$\langle \tilde{f} - f, u_k \rangle = 0$$

for all k. (Here, concerning $\tilde{f}$, we are taking inner products using Lebesgue integrals, but they satisfy all of the formulas we have derived using Reimann integrals.)

In other words, $\tilde{f} - f$ is orthogonal to u_k for every k. This brings us to an inportant definition:

Definition 22 (Complete orthonormal sequensce). *A sequence of functions* $\{u_k\}$ *that is orthonormal with repsect to some weight function* ϱ *on* $[0, L]$ *is* complete *in case the only continuous function* $g(x)$ *satisfying*

$$\int g(x) u_k(x) \varrho(x) \mathrm{d}x = 0 \tag{6.47}$$

for all k *is the zero function* $g(x) = 0$ *for all* x.

It can be shown that when $\{u_k\}$ is complete, the only Lebesgue square integrable functions g

When $\{u_k\}$ is complete, the only Lebesgue sqaure integrable function satisfying (6.47) is the zero functions, and thus $\tilde{f} - f = 0$. That is,

$$f = \lim_{N \to \infty} f_N = \sum_{k=1}^{\infty} \langle f, u_k \rangle u_k \ .$$

Finally, it is not hard to show (this is done in the exercises) that the orthonormal sequences defined in Theorem 32 is complete. Thus, every continuous functions has a convergent Fourier series representation. Hence, one actually has *Paseval's identity*

$$\sum_{k=1}^{\infty} \langle f, u_k \rangle^2 = \|f\|_2^2 \ . \tag{6.48}$$

In conclusion, general initial data for the wave equation can be approximated, to arbitrary accuracy, by initial data of the form (6.38) for which we have a formula for the unique solution. Moreover, we have a simple formula for computing the approximations to the initial data.

In the next section we show that separation of variables strategy leads to similar formulas for the solution of a wide variety of equations.

6.2 Sturm-Liouville Theory

6.2.1 Sturm-Liouville operators

In this section we explain how for a wide class of partial differential equations, the method of separation of variables leads to an ordinary differential equation boundary value problem whose solution yields a complete orthonormal sequence of functions that may be used to express the solution of the partial differential equation as a series of simple product solutions.

Let $\varrho(x)$ and $p(x)$ be strictly positive continuous functions on $[0, L]$. Let $q(x)$ be any continuous function on $[0, L]$.

For any continuous function $u(x)$ on $[0, L]$ that is twice continuously differentiable on $(0, L)$, define the function $\mathcal{L}u(x)$ by

$$\mathcal{L}u(x) = \frac{1}{\varrho(x)}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}}{\mathrm{d}x}u(x)\right) + q(x)u(x)\right] . \tag{6.49}$$

The transformation of the function $u(x)$ in to the new function $\mathcal{L}u(x)$ is often called an *operation*, and then $\mathcal{L}$ itself is called an *operator*. In this specific form, it is the *Strum-Liouville operator*

For different choices of $\varrho(x)$, $p(x)$ and $q(x)$, the Sturm-Lioville operator arrises in connection with important partial differential equations. Here are three examples.

Example 52 (The wighted string)**.** *Consider a string whose mass per unit length depends of the position x, so that instead of a constant density ϱ, the density is function $\varrho(x)$. If the tension in the string is T, the derivation of the wave equation that we made for constant density ϱ now gives, in exactly the same way,*

$$\varrho(x)\frac{\partial^2}{\partial t^2}h(x,t) = T\frac{\partial^2}{\partial x^2}h(x,t) . \tag{6.50}$$

Let us again impose the boundary conditions $h(0,t) = h(L,t) = 0$ for all t, and seek solutions of the form $h(x,t) = u(x)v(t)$. We find

$$\varrho(x)u(x)v''(t) = Tu''(x)v(t) ,$$

so that

$$\frac{v''(t)}{v(t)} = \frac{T}{\varrho(x)}\frac{u''(x)}{u(x)} .$$

Since the left side is independent of x, and the right side is independent of t, both sides must equal some constant λ, and we are left with the two equations

$$v''(t) = \lambda v(t) \tag{6.51}$$

and

$$\frac{T}{\varrho(x)}u''(x) = \lambda u(x) . \tag{6.52}$$

This can be written as

$$\mathcal{L}u(x) = \lambda u(x) \tag{6.53}$$

where $\mathcal{L}$ is the Sturm-Liouville operator with the given weight $\varrho(x)$, and with $p(x) = T$, which is constant, and $q(x) = 0$, again constant.

As we shall see, these is sequence of negative numbers $\{\lambda_n\}$ such $\lim_{n\to\infty}\lambda_n = -\infty$ and such that (6.53) has a solutions satisfying $u(0) = u(L) = 0$ if and only if $\lambda = \lambda_n$ for some n.

The general solution of (6.51) with $\lambda = \lambda_n$ is

$$v(t) = a\cos(\sqrt{|\lambda_n|}t) + \frac{b}{\sqrt{|\lambda_n|}}\sin(\sqrt{|\lambda_n|}t)$$

If we then let $u_n(x)$ denote the solution to $\mathcal{L}u = \lambda_n u$ subject to the boundary conditions $u(0) = u(L) = 0$,

$$h_n(x,t) = \left[a\cos(\sqrt{|\lambda_n|}t) + \frac{b}{\sqrt{|\lambda_n|}}\sin(\sqrt{|\lambda_n|}t)\right]u_n(x)$$

is the solution of our equation subject to our boundary conditions and satisfying

$$h(x,0) = au_n(x) \qquad \text{and} \qquad \frac{\partial}{\partial t}h(x,0) = bu_n(x)\ .$$

By expanding initial data as linear combinations of the u_n, we arrive at the general solution of the equation.

In a later example, we shall explicitly compute all of the λ_n and u_n for the particular case $\varrho(x) = (1+x)^{-2}$.

Example 53 (The heat equation)**.** *Consider a metal rod of length L, and let x be the coordinate marking position along the rod form one end ($x = 0$) to the other ($x = L$).*

Let $h(x,t)$ denote the temperature at x at time t. Heat, which is thermal energy, flows along the rod. The rate of flow depends on the local difference in temperature: heat flows from hot to cold.

The rate at which heat flows from right to left across x at time t is

$$p(x)\frac{\partial}{\partial x}h(x,t)$$

where $p(x)$ is a proportionality constant called the thermal conductivity*. Note that if the derivative is positive, the metal rod is hotter to the right of x than it is to the left, and there will be positive heat flow from right to left, so that $p(x) \geq =$. It is equal to zero if and only if the metal rod is a perfect thermal insulator at x, so that no heat flows.*

The amount of thermal energy entering the part of the rod between x and $x + \Delta x$ in a time Δt Is what flows in across the boundary of this segment in time Δt. By what we have explained above, this is

$$p(x+\Delta x)\frac{\partial}{\partial x}h(x+\Delta x,t) - p(x)\frac{\partial}{\partial x}h(x,t)\ .$$

The quantity of heat in the segment is (proportional to) the temperature times the length. Thus

$$[h(x,t+\Delta t) - h(x,t)]\Delta x$$

is the change in thermal energy in the segment, in the limit in which Δx is so smaller that h is essentially constant on the interval. Thus equating the change in thermal energy to the flux across the boundary, we have

$$[h(x,t+\Delta t) - h(x,t)]\Delta x = p(x+\Delta x)\frac{\partial}{\partial x}h(x+\Delta x,t) - p(x)\frac{\partial}{\partial x}h(x,t)\ ,$$

absorbing an additional proportionality constant into $p(x)$. Dividing by $\Delta x\Delta t$ and taking both to zero, we obtain

$$\frac{\partial}{\partial t}h(x,t) = \frac{\partial}{\partial x}\left(p(x)\frac{\partial}{\partial x}h(x,t)\right)\ . \tag{6.54}$$

This can be written as

$$\frac{\partial}{\partial t}h(x,t) = \mathcal{L}h(x,t)\ . \tag{6.55}$$

where $\mathcal{L}$ is the Sturm-Liouville operator with $\varrho(x) = 1$ and $q(x) = 0$, and $p(x)$ as given.

If the rod is insulated, there is no heat flux at the ends of the rod, and so, since heat flux is proportional to the temperature derivative, the natural boundary conditions are

$$\frac{\partial}{\partial x}h(0,t) = \frac{\partial}{\partial x}h(L,t) = 0 \ . \tag{6.56}$$

Let us try to solve the heat equation by separation of variables. We seek solutions of the form $h(x,t) = u(x)v(t)$. We find

$$u(x)v'(t) = (\mathcal{L}u(x))v(t) \ ,$$

so that

$$\frac{v'(t)}{v(t)} = \frac{\mathcal{L}u(x)}{u(x)} \ .$$

Once again, since the right hand side depends only on x and the left hand side only on t, both sides are constant, and so for some λ,

$$v'(t) = \lambda v(t) \qquad \text{and} \qquad \mathcal{L}u(c) = \lambda u(x) \ .$$

Once again, we are led to the problem of solving an eigenvalue equation *for a Sturm-Liouville operators; i.e., $\mathcal{L}u(c) = \lambda u(x)$, but this time subject to the* Neumann boundary conditions

$$u'(0) = u'(L) = 0 \ . \tag{6.57}$$

As we shall see, when p is continuously differentiable and strictly positive on $[0, L]$, there is a sequence of numbers λ_n tending to $-\infty$ as n increases so that $\mathcal{L}u(c) = \lambda u(x)$ has a solutions satisfying (6.57) if and only if $\lambda = \lambda_n$ for some n, and for each n, there is exactly one such solution (up to constant multiples) which we may call u_n.

Since $v'(t) = \lambda_n v(t)$ is solved uniquely (up to a constant multiple) by $v(t) = e^{t\lambda_n}$, we will then have that

$$h_n(x,t) = a_n e^{t\lambda_n} u_n(x)$$

is a solution of the heat equation for any choice of the constant a_n. It will turn out that the $\{u_n\}$ will be a complete orthonormal sequence so that we can readily express the solution for essentially arbitrary initial data as a linear combination of these special solutions. Finally, as explained in the exercises, energy methods can be used to prove a uniqueness theorem for the heat equation.

6.2.2 The Sturm-Liouville eigenvalue problem

Given a Sturm-Liouville operator $\mathcal{L}$ that acts on twice continuously differentiable functions $u(x)$ on $[0, L]$, the *Sturm-Liouville eigenvalue problem* is to find all numbers λ so that there is a nontrivial (i.e., not identically zero) function $u(x)$ such that

$$\mathcal{L}u(x) = \lambda u(x)$$

on $[0, L]$ and such that either the *Dirichlet boundary conditions*

$$u(0) = u(L) = 0 \tag{6.58}$$

or the *Neuman boundary conditions*

$$u'(0) = u'(L) = 0 \tag{6.59}$$

are satisfied. The values of λ for which such a solution exists are called the *eigenvalues* of $\mathcal{L}$, and the corresponding non-trivial solutions are called the *eigenfunctions* of $\mathcal{L}$.

We assume that $\mathcal{L}$ is given by (6.49) where ϱ and p are strictly positive and continuous on $[0, L]$, and p is continuously differentiable on $(0, L)$.

We have seen how the wave equation and the heat equation may be solved by solving a Sturm-Liouville eigenvalue problem that arrises through separation of variables. Now we turn to solving the Sturm-Liouville eigenvalue problem itself, and explaining the nature of the eigenfunctions that this leads too. The first important fact is that we always get an orthonormal set of eigenfuctions.

Lemma 10. *Consider a Sturm-Liouville operator with either Neuman or Dirichlet boundary conditions imposed. Let u_1, u_2 be two eigenfunctions of $\mathcal{L}$ corresponding to different eigenvalues λ_1, λ_2. Then*

$$\int_0^L u_1(x)u_2(x)\varrho(x)\mathrm{d}x = 0\ . \tag{6.60}$$

In other words, u_1 and u_2 are orthogonal with respect to the weight ϱ. Moreover every eigenvalue λ of $\mathcal{L}$ satisfies

$$\lambda \leq \min_{x\in[0,L]} \{q(x)/\varrho(x)\}\ . \tag{6.61}$$

Proof. We compute

$$\lambda_1 u_1(x)u_2(x)\varrho(x) = (\mathcal{L}u(x))u_2(x)\varrho(x)\ .$$

Integrating both sides in x, and then integrating by parts.

$$\begin{aligned} \lambda_1 \int_0^L u_1(x)u_2(x)\varrho(x)\mathrm{d}x &= \int_0^L [(p(x)u_1'(x))' + q(x)u_1(x)]u_2(x)\mathrm{d}x \\ &= [p(x)u_1'(x)]u_2(x)\Big|_{x=0}^{x=L} - \int_0^L [p(x)u_1'(x)u_2'(x) + q(x)u_1(x)]u_2(x)\mathrm{d}x \\ &= -\int_0^L [p(x)u_1'(x)u_2'(x) + q(x)u_1(x)]u_2(x)\mathrm{d}x \end{aligned}$$

since the boundary terms vanish under either Dirichlet or Neuman boundary conditions.

The right hand side is symmetric in u_1 and u_2. Therefore, we get the same result interchanging the roles of 1 and 2, so that

$$\lambda_1 \int_0^L u_1(x)u_2(x)\varrho(x)\mathrm{d}x = \lambda_2 \int_0^L u_1(x)u_2(x)\varrho(x)\mathrm{d}x\ .$$

Since $\lambda_1 \neq \lambda_2$, this means that (6.60) is true.

Next, replacing u_2 with u_1, the same calculation shows that

$$\begin{aligned}\lambda_1 \int_0^L |u_1(x)|^2 \varrho(x)\mathrm{d}x &= -\int_0^L [p(x)|u_1'(x)|^2 + q(x)|u_1(x)|^2 \mathrm{d}x \\ &\leq -\int_0^L q(x)|u_1(x)|^2 \mathrm{d}x \\ &= -\int_0^L \frac{q(x)}{\varrho(x)}|u_1(x)|^2 \varrho(x)\mathrm{d}x \\ &\leq -\min_{x\in[0,L]}\{q(x)/\varrho(x)\} \int_0^L |u_1(x)|^2 \varrho(x)\mathrm{d}x \ .\end{aligned}$$

This proves (6.61). □

Next, let us seek the eigenvalues. The point of writing the Sturm-Liouville operator in the form we wrote it is that this form makes it easy to prove the orthogonality of the previous lemma, and, more important, the coefficient functions in this form often have a physical meaning. For example, the coefficient $p(x)$ may represent a *thermal conductivity function* in a heat equation problem.

However, for other purposes, it is preferable to write the equation in a different form. The eigenvalue equation can be written as

$$\varrho(x)\mathcal{L}u(x) = \lambda\varrho(x)u(x)$$

and we can expand this as

$$u''(x) + \frac{p'(x)}{p(x)}u'(x) + \frac{1}{p(x)}[q(x) - \lambda\varrho(x)]u(x) = 0 \ .$$

This in turn can be written as

$$u''(x) + P(x)u'(x) + Q(x)u(x) = 0 \tag{6.62}$$

which is the second order linear equation that we have studied in the previous chapter.

We know that if $u_1(x)$ and $u_2(x)$ are *any* two linearly independent solutions of this equation, the general solution has the form

$$au_1(x) + bu_2(x)$$

for arbitrary constants a and b. Our goal is to choose the constants a and b so that our boundary conditions (either Dirichlet or Neuman) are satisfied. In general, this will only be possible for certain particular values of λ.

The reason is this: We know that as long as $P(x)$ and $Q(x)$ are continuous on $[0, L]$, there is a unique solution of (6.62) for any specified values of $u(0)$ and $u'(0)$. Furthermore, by the uniqueness and the linearity, if $u(x)$ is the solution with

$$u(0) = 0 \qquad \text{and} \qquad u'(0) = 1 \ , \tag{6.63}$$

and $a \neq 0$, defining $v(x)$ by $v(x) = au(x)$, we see that $v(x)$ is the solution of the same equation with

$$v(0) = 0 \qquad \text{and} \qquad v'(0) = a \ .$$

clearly, $u(L) = 0$ if and only if $v(L) = 0$. Therefore, if we want to satisfy both $u(0) = 0$ and $u(L) = 0$, we see that this can be done if and only if $u(L) = 0$ where $u(x)$ is the solution satisfying (6.63).

Now, either this solution satisfies $u(L) = 0$ or it does not. The eigenvalues are the special values of λ that 'adjust' $Q(x)$ to make this happen. The solution is quite analogous for Neuman boundary condition in place of dirichlet boundary conditions; se the next example.

We see from this discussion that finding eigenvalues of a Sturm-Liouville problem is closely connecting with the question of determining the set of *zeros* of solutions of (6.62); i.e., the set of points at which the solution satisfies $u(x) = 0$.

Example 54. *Consider the case $\mathcal{L}u(x) = u''(x)$ with Neumann boundary conditions on $[0, L]$. Since $q(x) = 0$, all of the eigenvalues will be non-positive. For $\lambda < 0$, the general solution of*

$$u'' = \lambda u(x)$$

is

$$u(x) = a\cos(\sqrt{|\lambda|}x) + b\sin(\sqrt{|\lambda|}x)\ .$$

Then

$$u'(x) = \sqrt{|\lambda|}(-a\sin(\sqrt{|\lambda|}x) + b\cos(\sqrt{|\lambda|}x))\ ,$$

and we have $u'(0) = 0$ if and only if $b = 0$. In this case

$$u'(L) = b\sqrt{|\lambda|}\cos(\sqrt{|\lambda|}L)\ ,$$

and so for $b \neq 0$, so we do not have the trivial solution, $u'(L) = 0$ if and only if

$$\cos(\sqrt{|\lambda|}L) = 0\ ,$$

and this means that

$$\sqrt{|\lambda|}L = k\pi - \frac{\pi}{2}$$

for some positive *integer k. The corresponding eigenfunction is, up to a multiple, uniquely given by*

$$u_k(x) = \sin\left(\left(k\pi - \frac{\pi}{2}\right)x\right)\ .$$

There is one more eigenvalue: We know $\lambda \leq 0$, but we have so far found only the strictly negative eigenvalues. Is $\lambda = 0$ an eigenvalue? To determine this, note that the general solution of $u''(x) = 0$ is $u(x) = ax + b$, so that $u'(x) = a$. To satisfy $u'(0) = 0$, we must take $a = 0$, and then $u'(L) = 0$ automatically. Hence,

$$\lambda_0 = 0$$

is an eigenvalue, and the corresponding eigenfunction is

$$u_0(x) = 1\ .$$

The general solution of the heat equation

$$\frac{\partial}{\partial t}h(x,t) = \frac{\partial^2}{\partial x^2}h(x,t)$$

with Neumann boundary conditions on $[0, L]$ is therefore of the form

$$h(x,t) = \sum_{k=0}^{\infty} \alpha_k e^{t\lambda_k t} u_k(x) \ .$$

Since $\lambda_k < 0$ for $k > 0$,

$$\lim_{t\to\infty} h(t,x) = \alpha_0 \ .$$

That is the temperature tends to a constant. It is easy to see that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^L h(x,t)\mathrm{d}x = 0$$

and so

$$\alpha_0 L = \lim_{t\to\infty} \int_0^L h(x,t)\mathrm{d}x = \int_0^L h(x,0)\mathrm{d}x \ .$$

That is

$$\alpha_0 = \frac{1}{L} \int_0^L h(x,0)\mathrm{d}x \ .$$

Therefore, if the initial temperature profile is given by $h(x,0) = g(x)$, we can see that

$$\lim_{t\to\infty} h(t,x) = \frac{1}{L} \int_0^L g(x)\mathrm{d}x \ .$$

That is, the temperature converges to its average value as the heat diffuses through the metal rod. Notice that this happens exponentially fast: The least negative non-zero eigenvalue is

$$\lambda_1 = -\frac{\pi^2}{4L^2} \ .$$

All of the other terms in the solution decay away at least as fast as $e^{-t\pi^2/4L^2}$.

Example 55. *Consider the case $\mathcal{L}u(x) = \frac{1}{\varrho(x)} u''(x)$ with Dirichlet boundary conditions on $[0, L]$. Since $q(x) = 0$, all of the eigenvalues will be non-positive. This arrises in the wave equation for a weighted string with mass density $\varrho(x)$. To carry out explicit computations, we fix the choice*

$$\varrho(x) = (1+x)^{-2} \ .$$

We seek the general solution of

$$u''(x) = \lambda(1+x)^{-2} u(x) \ .$$

It is natural to look for solutions of the form

$$u(x) = (1+x)^{\alpha}$$

for some undetermined α. Inserting this into our equation, we find

$$\alpha(\alpha - 1) = \lambda$$

This quadratic equation has the roots

$$\alpha = \frac{1}{2}(1 \pm \sqrt{1+4\lambda}) \ .$$

For $\lambda \neq -1/4$, we get two distinct roots and hence two linearly independent solutions.

For $-1/4\lambda \leq 0$, $\sqrt{1+4\lambda}$ is real, and so the general solution is

$$u(x) = a(1+x)^{\frac{1}{2}(1+\sqrt{1+4\lambda})} + b(1+x)^{\frac{1}{2}(1-\sqrt{1+4\lambda})} \ .$$

Then, $u(0) = 0$ forces $b = -a$. But then

$$u(L) = a\left((1+L)^{\frac{1}{2}(1+\sqrt{1+4\lambda})} - (1+L)^{\frac{1}{2}(1-\sqrt{1+4\lambda})}\right) \ ,$$

and this is not zero unless $a = 0$. Hence for $-1/4\lambda \leq 0$, the only solution is the trivial solution, and there are no eigenvalues in this range

For $\lambda = 1/4$, we have only the single solution $u_1(x) = \sqrt{1+x}$. We get a second solution by multiplying by

$$\int \frac{1}{u_1^2(x)} \mathrm{d}x = \ln(1+x) \ ,$$

where we have use the fact that $P(x) = 0$ in this case. Hence our second solution is

$$u_2(x) = \sqrt{1+x}\ln(1+x) \ ,$$

and the general solution is

$$u(x) = \sqrt{1+x}(a + b\ln(1+x)) \ .$$

To satisfy $u(0) = 0$, we must have $a = 0$, and then $u(L) = b\sqrt{1+L}\ln(1+L)$, which is zero only if $b = 0$; i.e., only for the trivial solution. Hence $\lambda = -1/4$ is not an eigenvalue.

Things are different when $\lambda < -1/4$. In this case

$$\sqrt{1+4\lambda} = i\sqrt{4|\lambda|-1}$$

is pure imaginary. Thus,

$$\begin{aligned} (1+x)^{\frac{1}{2}(1+\sqrt{1+4\lambda})} &= \sqrt{1+x}e^{i\frac{1}{2}\sqrt{4|\lambda|-1}\ln(1+x)} \\ &= \sqrt{1+x}\left(\cos(\tfrac{1}{2}\sqrt{4|\lambda|-1}\ln(1+x)) + i\sin(\tfrac{1}{2}\sqrt{4|\lambda|-1}\ln(1+x))\right) \ . \end{aligned}$$

The real and imaginary parts give two independent solutions, and so the general solution is

$$u(x) = \sqrt{1+x}\left(a\cos(\tfrac{1}{2}\sqrt{4|\lambda|-1}\ln(1+x)) + b\sin(\tfrac{1}{2}\sqrt{4|\lambda|-1}\ln(1+x))\right) \ .$$

Then $u(0) = 0$ requires $a = 0$, in which case

$$u(L) = b\sqrt{1+x}\sin(\tfrac{1}{2}\sqrt{4|\lambda|-1}\ln(1+L)) \ ,$$

and for $b \neq 0$, this is satisfied if and only if

$$\sqrt{|\lambda| - 1/4}L = k\pi$$

for some positive integer k. The solution of this is

$$\lambda_k = -\frac{k^2\pi^2}{(\ln(1+L)^2} - \frac{1}{4} \ .$$

The corresponding eigenfunctions are

$$u_k(x) = \sqrt{1+x}\sin(\tfrac{1}{2}\sqrt{4|\lambda_k|-1}\ln(1+x)) \ .$$

In the last two examples, we have found infinite sequences of eigenvalues and eigenfunctions by direct computation. It is not always possible to make these computation explicitly, but it turns out that once can show the existence of such sequences in great generality by indirect means, and can even say write a lot about the properties of the eigenvalues and eigenfunctions. This is done through an investigation of the zeros of solutions of the second order linear equation. As we have seen in the last example, when there are plenty of zeros, due to oscillation, we have a chance to match both boundary conditions. Our investigation of the zeros will make frequent use of the *Wronskian* of two functions.

Definition 23 (Wronskian)**.** *Let $u_1(x)$ and $u_2(x)$ be two continuously differentiable functions. Their* Wronskian *is the function*

$$W_{u_1,u_2}(x) = u_1(x)u_2'(x) - u_1'(x)u_2(x) = \det\begin{bmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{bmatrix} .$$

Lemma 11. *Let $u_1(x)$ and $u_2(x)$ be two solutions of (6.62) on an interval $[x_1, x_2]$. Then*

$$W_{u_1,u_2}(x_2) = e^{\int_{x_1}^{x_2} P(x)\mathrm{d}x} W_{u_1,u_2}(x_1) .$$

In particular, if $W_{u_1,u_2}(x_1) > 0$, then $W_{u_1,u_2}(x_2) > 0$ as long as the solutions are defined on the interval bounded by x_1 and x_2.

Proof. Differentiating,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} W_{u_1,u_2}(x) &= u_1(x)u_2''(x) - u_1''(x)u_2(x) \\
&= u_1(x)[-P(x)u_2'(x) - Q(x)u_2(x)] - [-P(x)u_1'(x) - Q(x)u_1(x)]u_2(x) \\
&= -P(x)[u_1(x)u_2'(x) - u_1'(x)u_2(x)] \\
&= -P(x)W_{u_1,u_2}(x)
\end{aligned} \tag{6.64}$$

and this first order linear equation has the solution

$$W_{u_1,u_2}(x) = e^{-\int_{x_1}^{x} P(z)\mathrm{d}z} W_{u_1,u_2}(x_1) .$$

Since the exponential function is strictly positive, if $W_{u_1,u_2}(x_1) \neq 0$, then for all x, $W_{u_1,u_2}(x) \neq 0$ and has the same sign as $W_{u_1,u_2}(x_1)$. □

Theorem 34. *Let $u_1(x)$ and $u_2(x)$ be two linearly independent solutions of (6.62). Then their zeros interlace. That is, the zeros are isolated, and between any two successive zeros of u_1, there is exactly one zero of u_2.*

Proof. First we claim that the zeros of any nontrivial solution of (6.62) are isolated, in that around any zero there is an open interval containing no other zeros. This is because if $u(x)$ is a non-trivial solution, and $u(x_0) = 0$, then $u'(x_0) \neq 0$ since the only solution with both $u(x_0) = 0$ and $u'(x_0) = 0$ is the trivial solution. But if $u'(x_0) \neq 0$, the solution passes though through $x = x_0$ with a non-zero slope, and so there is an open interval in which x_0 is the only zero. Hence, it makes sense to speak of successive zeros.

Let x_1 and x_2 be successive zeros of $u_1(x)$. Since u_1 and u_2 are linearly independent, $u_2(x_1) \neq 0$ and $u_2(x_2) \neq 0$: The vectors $\big(u_1(x_j), u_1'(x_j)\big)$ and $\big(u_2(x_j), u_2'(x_j)\big)$ are linearly independent for $j = 1, 2$.

Then, since $u_2(x_1) = u_2(x_2) = 0$,

$$W_{u_1,u_2}(x_1) = -u_1'(x_1)u_2(x_1) \qquad \text{and} \qquad W_{u_1,u_2}(x_2) = -u_1'(x_2)u_2(x_2) \ .$$

Since the Wronskian does not change sign, $u_1'(x_1)u_2(x_1)$ has the same sign as $u_1'(x_2)u_2(x_2)$.

We may suppose, replacing either u_1 or u_2 by $-u_1$ or $-u_2$ as necessary, which does not affect the location of zeros, that both u_1 and u_2 are not only non-zero, but strictly positive on (x_1, x_2). Then since $u_2(x_1) \neq 0$ and $u_2(x_2) \neq 0$, u_2 is strictly positive on $[x_1, x_2]$.

Since $u_1(x)$ is positive on (x_1, x_2) and zero at the boundary of this interval, the slope must be strictly positive at x_1, and strictly negative at x_2, since the slope of u_1 cannot be zero at a zero of u_1.

Therefore, $u_1'(x_1) > 0$ and $u_1'(x_2) < 0$, so that $u_1'(x_1)$ and $u_1'(x_2)$ have the opposite sign. The same holds in the same way u_1 is negative on (x_1, x_2), But then it must be that case that $u_2(x_1)$ and $u_2(x_2)$ have opposite signs, and then by the Intermediate Value Theorem, $u_2(x) = 0$ for some $x \in (x_1, x_2)$.

Thus, there is a zero of u_2 between each pair of successive zero of u_1. But then there cannot be any more zeros of u_2 in $[x_1, x_2]$, since otherwise u_1 would fail to have any zeros in between two successive zero of u_2, and that cannot happen since, switching the roles of u_1 and u_2, we have just proved that between any to zero of u_2, there must be a zero of u_1. □

We now show that to study the zeros of solutions of (6.62), we need only study the zeros of solutions of a simpler equation, namely

$$y''(x) + V(x)y(x) = 0 \ . \tag{6.65}$$

Here is why.

Lemma 12. *Let $u(x)$ be twice continuously differentiable. Define*

$$v(x) = e^{\frac{1}{2}\int P(x)\mathrm{d}x} \ , \tag{6.66}$$

and define

$$V(x) = Q(x) - \frac{1}{4}P^2(x) - \frac{1}{2}P'(x) \ . \tag{6.67}$$

Then

$$y(x) = u(x)v(x)$$

satisfies (6.65) if and only if $u(x)$ solves $u''(x) + P(x)u'(x) + Q(x)u(x) = 0$.

Proof. Let $y(x)$ and $w(x)$ be any two twice continuously differentiable functions, and define $u(x) = w(x)y(x)$. Then

$$u' = w'y + wy' \qquad \text{and} \qquad u'' = w''y + 2w'y' + wy'' \ .$$

Therefore,

$$\begin{aligned} u'' + Pu' + Qu &= [w''y + 2w'y' + wy''] + P[w'y + wy'] + Qy \\ &= wy'' + [2w' + Pw]y' + [w'' + Pw' + w'']y = 0 \ . \end{aligned}$$

To eliminate the coefficient of y', we require that

$$2w' + Pw = 0$$

and this is a first oder linear equation that is solved by

$$w = e^{-\frac{1}{2}\int P \mathrm{d}x} \ .$$

Making this choice for $w(x)$, we see that $u(x) = w(x)y(x)$ satisfies (6.62) if and only if $y(x)$ satisfies (6.65). Then, with $v(x) = 1/w(x)$, $y(x) = v(x)u(x)$ and $v(x)$ is given by (6.66).

□

The point of this lemma is that since the exponential function is never zero, the functions $y(x)$ and $u(x)$ have the same zeros. Therefore, studying the properties of the set of zeros of solutions of the equation (6.62) is the same as studying the properties of the set of zeros of solutions of the equation (6.65) when $V(x)$ and $Q(x)$ are related by (6.67).

Therefore, in what follows, we study properties of the set of zeros of the equation (6.65), knowing that this tells us about the properties of the set of zeros of the equation (6.62) when $V(x)$ and $Q(x)$ are related by (6.67).

Example 56 (Bessel's equation and its standard form)**.** *Let $a \geq 0$, and consider the equation*

$$x^2u''(x) + xu'(x) + (x^2 - \alpha^2)u(x) = 0 \tag{6.68}$$

on the interval $(0, \infty)$. This is Bessel's equation with parameter a. *dividing through by x^2, we find*

$$u''(x) + \frac{1}{x}u'(x) + \left(1 - \frac{\alpha^2}{x^2}\right)u(x) = 0 \ .$$

In this case $P(x) = 2/x$, and so

$$v(x) = e^{\frac{1}{2}\int P(x)\mathrm{d}x} = e^{\ln x} = x \ .$$

Computing $V(x)$, we find that the standard form is

$$y''(x) + \left(1 + \frac{1 - 4\alpha^2}{4x^2}\right)y(x) \ . \tag{6.69}$$

Notice that for $\alpha = 1/2$, this reduces to $y'' + y = 0$ which is solved by $y(x) = a\cos x + b\sin x$. It follows that

$$u(x) = a\frac{\sin x}{x} + b\frac{\cos x}{x}$$

is the general solution of Bessel's equation for $\alpha = 1/2$.

Notice that for $\alpha = 1/2$, every solution of Bessel's equation has infinitely many zeros. As we shall soon see, this is true for all values of α. To show this, we need an alternative to direct calculation of the the solution.

Theorem 35. *If $y(x)$ is a non-trivial solution of $y''(x)+V(x)y(x)=0$ and $V(x)<0$ for all x, then $u(x)$ has at most one zero.*

Proof. Suppose that there is at least one zero. Let x_0 be such that $y(x_0)=0$. Since $y(x)$ is a non-trivial solution, $y'(x_0)\neq 0$, and we may assume that $y'(x_0)>0$. Suppose there is another zero of $y(x)$ to the right of x_0. Let x_1 be the first such zero. Then $y(x)>0$ on (x_0,x_1), and so $(y'(x))'=-V(x)y(x)>0$ on (x_0,x_1). That is, $y'(x)$ is strictly increasing on (x_0,x_1) and then since $y'(x_0)>0$, $y'(x)>0$ on (x_0,x_1). It follows that

$$y(x_1)=y(x_0)+\int_{x_0}^{x_1} y'(x)\mathrm{d}x \geq \int_{x_0}^{x_1} y'(x_0)\mathrm{d}x = y'(x_0)(x_1-x_0)>0\ .$$

This contradicts $y(x_1)=0$. Therefore, there can be no zero of $y(x)$ to the right of x_0.

A similar argument shows that there can be no zero of $y(x)$ to the left of x_0. □

Theorem 36. *If $y''(x)+V(x)y(x)=0$ and for some x_0, $V(x)>0$ for all $x\geq x_0$ and*

$$\int_{x_0}^{\infty} V(x)\mathrm{d}x=\infty\ ,$$

then $u(x)$ has infinitely many zeros.

Proof. If there are only finitely many zeros, then there is a last one, and we may suppose that $u(x)>0$ for all $x>x_0$. Define

$$v(x)=-\frac{u'(x)}{u(x)}\ .$$

Then

$$v'(x)=V(x)+v^2(x)\geq V(x)\ .$$

Therefore,

$$v(x)\geq \int_{x_0}^{x} V(z)\mathrm{d}z+v(x_0)\ .$$

Taking x large enough, we see that for all x sufficiently large $v(x)>0$, and $u'(x)<0$.

But since $u'(x)$ is decreasing on $[x_0,\infty)$ since $(u'(x))'=-V(x)u(x)$ is negative there, the slope is always less than some strictly negative number from some point on. Hence, for all sufficiently large x, the graph of $u(x)$ lies below a line with negative slope. Hence $u(x)$ must have another zero. This shows there can be no last zero. □

Example 57. *We see that for all values of α, there is an x_0 so that*

$$V(x)=\left(1+\frac{1-4\alpha^2}{4x^2}\right)>0$$

and

$$\int_{x_0}^{\infty} V(x)\mathrm{d}x=\infty\ .$$

It now follows that every solution of $y''(x)+V(x)y(x)=0$ has infinitely many zeros. But every solution of Bessel's equation with parameter α is a non-zero multiple of such a solutions. Hence, every solution of Bessel's equation with parameter α has infinitely many zeros.

Theorem 37 (Sturm Comparison Theorem)**.** *Suppose*

$$y''(x) + V(x)y(x) = 0 \qquad \text{and} \qquad z''(x) + U(x)z(x) = 0$$

on an interval $[a, b]$ *and that*

$$V(x) > U(x)$$

on $[a, b]$. *Then* y *has at least one zero between any two zero of* z.

Proof. Suppose that x_1, x_2 are successive zeros of $z(x)$ in $[a, b]$ and that $y(x)$ has no zeros on (x_1, x_2). Then without loss of generality, we may suppose that $z(x), y(x) > 0$ for all $x \in (x_1, x_2)$. Now,

$$\frac{\mathrm{d}}{\mathrm{d}x} W_{y,z}(x) = y(x)z''(x) - z(x)y''(x) = (V(x) - U(x))y(x)z(x) > 0 \ .$$

Thus,

$$y(x_2)z'(x_2) = W_{y,z}(x_2) > W_{y,z}(x_1) = y(x_1)z'(x_1) \ .$$

But since $z(x) > 0$ on (x_1, x_2) and $z(x_1) = z(x_2) = 0$, we have $z'(x_1) > 0$ and $z'(x_2) < 0$. But then $y(x_2)z'(x_2) \leq 0$ and $y(x_2)z'(x_2) \geq 0$. This is impossible so $y(x)$ must have a zero in (x_1, x_2). □

Theorem 38 (Bounds on separation of zeros)**.** *Let* V *be a continuous function on* $[a, b]$ *satisfying*

$$0 < m^2 < V(x) < M^2$$

for all $x \in [a.b]$. *Let* $y(x)$ *be any nontrivial solutions of* $y''(x) + V(x)y(x) = 0$ *on* $[a, b]$. *Then if* x_1 *and* x_2 *are successive zeros of* $y(x)$ *in* $[a, b]$,

$$\frac{\pi}{M} \leq x_2 - x_1 \leq \frac{\pi}{m} \ .$$

Moreover, if $y(a) = y(b) = 0$, *and* $y(x) = 0$ *at exactly* $n - 1$ *points in* (a, b), *then*

$$m < \frac{n\pi}{b - a} < M \ .$$

Proof. The function

$$z(x) = \sin(m(x - x_1))$$

satisfies

$$z''(x) + m^2(x) = 0 \qquad \text{and} \qquad z(x_1) = 0 \ .$$

The next zero of $z(x)$ is at $x_1 + \pi/m$. By the Sturm Comparison Theorem, $y(x)$ must have a zero in $(x_1, x_1 + \pi/m)$. Since x_2 is the successive zero, $x_2 < x_1 + \pi/m$. This proves the first upper bound.

Next, the function

$$z(x) = \sin(M(x - x_1))$$

satisfies

$$z''(x) + M^2(x) = 0 \qquad \text{and} \qquad z(x_1) = 0 \ .$$

The next zero of $z(x)$ is at $x_1 + \pi/M$. By the Sturm Comparison Theorem, $z(x)$ must have a zero in (x_1, x_2). Hence $x_1 + \pi/M < x_2$. This gives the lower bound. $x_2 < x_1 + \pi/m$. This proves the first upper bound.

For the second part, the zeros of $y(x)$ divide $[a, b]$ into exactly n intervals bounded by successive zeros of $y(x)$. The average length of these intervals is

$$\frac{b-a}{n} \ .$$

They could all be exactly this length, but in any case the minimum is no greater than the average, and the maximum is no less.

Hence, focusing first on the minimum, there exist successive zeros x_1 and x_2 of $y(x)$ with

$$x_2 - x_1 \leq \frac{b-a}{n} \ .$$

But then, by the above

$$\frac{\pi}{M} < x_2 - x_1 \leq \frac{b-a}{n} \qquad \text{and hence} \qquad M > \frac{n\pi}{b-a} \ .$$

A similar argument proves the other bound. □

Theorem 39. *Consider a Sturm-Liouville operator $\mathcal{L}$ on $[0, L]$ satisfying the continuity and positivity conditions imposed above. Then there is a sequence $\{\lambda_n\}$ of strictly decreasing numbers with $\lim_{n\to\infty} \lambda_n = -\infty$ such that*

$$\mathcal{L}u(x) = \lambda u(x) \qquad \text{with} \qquad u(0) = u(L) = 0$$

has a nontrivial solutions if and only if $\lambda - \lambda_n$ for some n. The corresponding solution $u_n(x)$ is zero at exactly $n - 1$ points in $(0, L)$.

Proof. We can convert the equation $\mathcal{L}u(x) = \lambda u(x)$ into standard form

$$y''(x) + V_\lambda(x)y(x) = 0 \ ,$$

where V_λ has the form

$$V_0(x) - \lambda\frac{\varrho(x)}{p(x)}.$$

Decreasing λ increases both the minimum and maximum of V_λ on $[0, L]$.

Now consider the solution of $y''(x) + V_\lambda(x)y(x) = 0$ with $y(0) = 0$ and $y'(0) = 1$. As λ is decreased, the solutions become more and more oscillatory. More and more zeros 'move in' to the interval $[0, L]$. Every time a new zero enters through the right, we get a new eigenvalue.

Let M_λ be the square root of maximum of V_λ on $[0, L]$. We must have

$$M_\lambda \geq \frac{n\pi}{L} \ .$$

This requires λ_n to be very negative for large n. This is the basis of the claim that

$$\lim_{n\to\infty} \lambda_n = -\infty \ .$$

However, we can say more: Let m_λ be the square root of the minimum of V_λ. Then m_λ cannot be larger than $n\pi/L$, and so λ_n cannot be *too* negative.

□

Example 58 (Estimation of eigenvalues)**.** *Consider the Sturm-Liouville operator*

$$\mathcal{L}u(x) = ((1+x)u'(x))'$$

with Dirichlet boundary conditions on $[0, L]$*. Then the eigenvalue equation* $\mathcal{L}u(x) = \lambda u(x)$ *can be written as*

$$u'' + \frac{1}{1+x}u' - \frac{\lambda}{1+x}u = 0 \ .$$

The standard form of this equation is $y'' + V(x)y = 0$ *where*

$$V(x) = \frac{-\lambda}{1+x} + \frac{1}{4}\frac{1}{(1+x)^2} \ .$$

We know all eigenvalues are non-positive, so we only consider $\lambda \leq 0$*. Differentiating, we see that* $V(x)$ *is monotone decreasing so that*

$$\frac{-\lambda}{1+L} + \frac{1}{4}\frac{1}{(1+L)^2} = V(L) \leq V(x) \leq -\lambda + \frac{1}{4} = V(0)$$

for all $x \in [0, L]$*. Therefore, with*

$$m_\lambda^2 = \frac{-\lambda}{1+L} + \frac{1}{4}\frac{1}{(1+L)^2} \qquad \text{and} \qquad M_\lambda^2 = -\lambda + \frac{1}{4} \ ,$$

we have that the condition of Theorem 38 are satisfied. Since λ_n *has exactly* $n-1$ *zeros,*

$$m_{\lambda_n}^2 \leq \frac{n^2\pi^2}{L^2} \leq M_{\lambda_n}^2 \ .$$

Therefore,

$$\frac{-\lambda_n}{1+L} + \frac{1}{4}\frac{1}{(1+L)^2} \leq \frac{n^2\pi^2}{L^2} \leq -\lambda_n + \frac{1}{4} \ .$$

Rearranging terms,

$$-(L+1)\frac{n^2\pi^2}{L^2} + \frac{1}{4}\frac{1}{L+1} \leq \lambda_n \leq -\frac{n^2\pi^2}{L^2} + \frac{1}{4} \ .$$

Chapter 7

GREEN'S FUNCTIONS FOR SECOND ORDER EQUATIONS

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7.1 Inverting Sturm-Liouville operators

7.1.1 The advantages of the Sturm-Liouville form

We have already studied the inhomogeneous second-order linear differential equation

$$u''(x) + P(x)u'(x) + Q(x)u(x) = g(x) \ . \tag{7.1}$$

We have seen that provided P, Q and g are continuous and bounded on some open interval containing a, then for each $\gamma, \delta \in \mathbb{R}$, there exists a unique solution of (7.1) satisfying

$$u(a) = \gamma \qquad \text{and} \qquad u'(a) = \delta \ . \tag{7.2}$$

In this chapter we are concerned with corresponding boundary value problem, in which we seek to solve (7.1) on some interval (a, b) subject to

$$u(a) = \gamma \qquad \text{and} \qquad u(b) = \delta \ . \tag{7.3}$$

As we shall see, the key to this, and to an even more general problem, is to focus first on the special case

$$u(a) = 0 \qquad \text{and} \qquad u(b) = 0 \ . \tag{7.4}$$

As we have seen, on any interval on which P and Q are continuous and bounded, there are two linearly independent $u_1(x)$ and $u_2(x)$ solutions of the homogeneous equation

$$u''(x) + P(x)u'(x) + Q(x)u(x) = 0 \ . \tag{7.5}$$

For any such pair of solutions, define

$$M(x) = \begin{bmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{bmatrix} .$$

Then the flow transformation for the corresponding first order system is

$$\begin{aligned}
\Phi_{x,y} &= M(x)M^{-1}(y) = \frac{1}{u_1(y)u_2'(y) - u_2(y)u_1'(y)} \begin{bmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{bmatrix} \begin{bmatrix} u_2'(y) & -u_2(y) \\ -u_1'(y) & u_1(y) \end{bmatrix} \\
&= \frac{1}{u_1(y)u_2'(y) - u_2(y)u_1'(y)} \begin{bmatrix} u_1(x)u_2'(y) - u_2(x)u_1'(y) & u_2(x)u_1(y) - u_1(x)u_2(y) \\ u_1'(x)u_2'(y) - u_2'(x)u_1'(y) & u_2'(x)u_1(y) - u_1'(x)u_2(y) \end{bmatrix}
\end{aligned} \quad (7.6)$$

Then the unique solution $u(x)$ of (7.1) subject to (7.3) is given, together with is derivative, by

$$(u(x), u'(x)) = \Phi_{x,a}(\gamma, \delta) + \int_a^x \Phi_{x,y}(0, g(y))\mathrm{d}y \ . \quad (7.7)$$

On the other hand, it turns out that there may be *no solution* to (7.1) subject to (7.4). It will be easiest to see when there is, and is not, such a solution by writing our equation in a different form.

Define $p(x) = e^{\int_a^x P(z)\mathrm{d}z}$. Since $p(x)$ is never zero, the equation we get by multiplying (7.1) through by $p(x)$ has the same set of solutions as the original. Since $p'(x) = P(x)p(x)$, the new equation is

$$(p(x)u'(x))' + p(x)Q(x)u(x) = p(x)g(x) \ .$$

Therefore, let us define

$$q(x) = p(x)Q(x) \qquad \text{and} \qquad f(x) = p(x)g(x) \ ,$$

and the Sturm-Liouville operator $\mathcal{L}$ by

$$\mathcal{L}u(x) = (p(x)u'(x))' + q(x)u(x) \ .$$

Then the equation (7.1) is equivalent to

$$\mathcal{L}u(x) = f(x) \ . \quad (7.8)$$

This is the *Sturm-Liouville form of the inhomogeneous equation (7.1).*

We will prove the following theorem:

Theorem 40. *Let p and q be continuous on $[a, b]$ with a and b finite. Suppose also that p is strictly positive and continuously differentiable on $[a, b]$. Let $\mathcal{L}$ denote the Sturm-Liouville operator $\mathcal{L}u = (pu')' + qu$. Then, if there is no non-trivial solution of*

$$\mathcal{L}u = 0 \qquad \text{with} \qquad u(a) = u(b) = 0 \ , \quad (7.9)$$

there is a unique solution of

$$\mathcal{L}u = f \qquad \text{with} \qquad u(a) = u(b) = 0 \quad (7.10)$$

for every continuous function f on $[a, b]$.

If there does exist a nontrivial solution $\tilde{u}$ of (7.9), then then all such solutions are multiples of one another, and in this case there exist infinitely many solutions of (7.10) if and only if

$$\int_a^n \tilde{u}(x)f(x)\mathrm{d}x = 0 \ . \tag{7.11}$$

Otherwise, if (7.11) is not satisfied, there does not exist any solution of (7.10).

This Theorem bears a strong resemblance to a theorem from linear algebra which we now quote for comparison:

Theorem 41. *Let A be an $n \times n$ matrix. Suppose also that A is symmetric; i.e., $A = A^t$. Suppose that $A\mathbf{x} = \mathbf{0}$ has no solutions except the zero solution. Then*

$$A\mathbf{x} = \mathbf{b}$$

has a unique solution for all $\mathbf{b} \in \mathbb{R}^n$.

However, if there exist non-zero vectors $\tilde{\mathbf{x}}$ such that $A\tilde{\mathbf{x}} = \mathbf{0}$, then $A\mathbf{x} = \mathbf{b}$ never has a unique solution: It will always have either infinitely many or none, and it has infinitely many precisely when

$$\tilde{\mathbf{x}} \cdot \mathbf{b} = 0$$

for all $\tilde{\mathbf{x}}$ such that $A\tilde{\mathbf{x}} = \mathbf{0}$.

Proof. if $A\mathbf{x} = \mathbf{0}$ has only the zero solution, the linear transformation represented by A is one-to-one. Since this is a transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$, the Fundamental Theorem of Linear Algebra tells us that A is invertible, and so the unique solution of $A\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x} = A^{-1}\mathbf{b}$.

So far, we have made no use of the symmetry of A; the first part of the theorem is true without this hypothesis. Now suppose that $A\tilde{\mathbf{x}} = \mathbf{0}$, but $\tilde{\mathbf{x}} \cdot \mathbf{b} \neq 0$. Then if $A\mathbf{x} = \mathbf{b}$,

$$\tilde{\mathbf{x}} \cdot \mathbf{b} = \tilde{\mathbf{x}} \cdot A\mathbf{x} = (A\tilde{\mathbf{x}}) \cdot \mathbf{x} = 0 \ .$$

Therefore, whenever $A\mathbf{x} = \mathbf{b}$ has solutions $\mathbf{x}$, those solutions must be orthogonal to every vector $\tilde{\mathbf{x}}$ such that $A\tilde{\mathbf{x}} = \mathbf{0}$. It remains to be seen that when $\tilde{\mathbf{x}} \cdot \mathbf{b} = 0$ for all $\tilde{\mathbf{x}}$ such that $A\tilde{\mathbf{x}} = \mathbf{0}$, then solutions do exist. This will be left to the exercises. □

In some sense, the linear algebra theorem is more complicated: There can exist several linearly independent solutions of $\tilde{\mathbf{x}}$ of $A\mathbf{x} = \mathbf{0}$, and all of them must be taken into account. As we shall see, there is at most one solution $\tilde{u}$, up to constant multiples, of $\mathcal{L}u = 0$.

In the linear algebra theorem, the symmetry of the matrix of A came into play when we showed that $\tilde{\mathbf{x}} \cdot \mathbf{b} = 0$ for all $\tilde{\mathbf{x}}$ such that $A\tilde{\mathbf{x}} = \mathbf{0}$ was necessary for solutions to exist.

We have already introduced the idea that the inner product

$$\langle u, v\rangle = \int_a^b ux)v(x)\mathrm{d}x$$

is an analog of the dot product. Also, the transformation sending u to $\mathcal{L}u$ is linear. We now show that

$$\langle \mathcal{L}u(x), v(x)\rangle = \langle \mathcal{L}u(x), \mathcal{L}v(x)\rangle$$

which corresponds to the symmetry condition

$$\tilde{\mathbf{x}} \cdot A\mathbf{x} = (A\tilde{\mathbf{x}}) \cdot \mathbf{x}$$

used in the proof given above.

Lemma 13 (Symmetry of the Sturm-Liouville operator)**.** *For all twice continuously differentiable functions u, v on (a, b) with*

$$u(a) = v(a) = u(b) = v(b) = 0 \ , \tag{7.12}$$

$$\int_a^b (\mathcal{L}u(x))v(x)\mathrm{d}x = \int_a^b u(x)(\mathcal{L}v(x))\mathrm{d}x \ .$$

Proof. By definition,

$$\int_a^b (\mathcal{L}u(x))v(x)\mathrm{d}x = \int_a^b (p(x)u'(x))'v(x)\mathrm{d}x + \int_a^b q(x)u(x)v(x)\mathrm{d}x \ .$$

We integrate by parts twice on the first term:

$$\begin{aligned}
\int_a^b (p(x)u'(x))'v(x)\mathrm{d}x &= (p(x)u'(x))'v(x)\Big|_a^b - \int_a^b (p(x)u'(x))v'(x)\mathrm{d}x \\
&= -\int_a^b u'(x)(p(x)v'(x))\mathrm{d}x \\
&= -u(x)(p(x)v'(x))\Big|_a^b + \int_a^b u(x)(p(x)u'(x))\mathrm{d}x \\
&= \int_a^b u(x)(p(x)u'(x))\mathrm{d}x \ ,
\end{aligned}$$

where the boundary terms are zero because of (7.12). Now combining the terms, we obtain the desired identity. □

Just as in the linear algebra case, this gives us a necessary condition for $\mathcal{L}u = f$ to have a solution.

Lemma 14 (Necessary condition for the solution of $\mathcal{L}u = f$)**.** *Suppose that there exists a non-trivial solution $\tilde{u}$ of $\mathcal{L}\tilde{u} = 0$ satisfying $\tilde{u}(a) = \tilde{u}(b) = 0$. Let f be continuous and not identically zero on $[a, b]$. Then unless*

$$\int_a^b \tilde{u}(x)f(x)\mathrm{d}x = 0 \ , \tag{7.13}$$

there is no solution of $\mathcal{L}u = f$ satisfying $u(a) = u(b) = 0$.

Proof. Suppose that there does exist a solution of $\mathcal{L}u = f$ satisfying $u(a) = u(b) = 0$. Then

$$\int_a^b \tilde{u}(x)f(x)\mathrm{d}x = \int_a^b \tilde{u}(x)(\mathcal{L}f(x))\mathrm{d}x = \int_a^b (\mathcal{L}\tilde{u}(x))u(x)\mathrm{d}x = 0 \ ,$$

where we have used Lemma 13 in the last line. □

We will see later that this condition is also necessary for the solution to exist. We now turn to another advantage of the Sturm-Liouville form of our equation:

By (7.6) and (7.7), the unique solution $u(x)$ of (7.1) with $u(a) = u(b) = 0$ is

$$u(x) = \int_a^x \frac{1}{u_1(y)u_2'(y) - u_2(y)u_1'(y)}[u_2(x)u_1(y) - u_1(x)u_2(y)]g(y)\mathrm{d}y \ .$$

Then with $f(x) = g(x)/p(x)$, the unique solution $u(x)$ of $\mathcal{L}u = f$ with $u(a) = u(b) = 0$ is

$$u(x) = \int_a^x \frac{1}{[u_1(y)u_2'(y) - u_2(y)u_1'(y)]p(y)}[u_2(x)u_1(y) - u_1(x)u_2(y)]f(y)\mathrm{d}y \ .$$

This formula now simplifies since, as we show next, $[u_1(y)u_2'(y) - u_2(y)u_1'(y)]p(y)$ is constant.

Lemma 15. *Define*

$$\widetilde{W}(y) = [u_1(y)u_2'(y) - u_2(y)u_1'(y)]p(y) = u_1(y)[p(y)u_2'(y)] - u_2(y)[p(y)u_1'(y)] \ .$$

We compute the derivative:

$$\begin{aligned} \widetilde{W}'(y) &= u_1'(y)[p(y)u_2'(y)] + u_1(y)[p(y)u_2'(y)]' - u_2'(y)[p(y)u_1'(y)] - u_2(y)[p(y)u_1'(y)]' \\ &= u_1(y)[p(y)u_2'(y)]' - u_2(y)[p(y)u_1'(y)]' \\ &= -u_1(y)q(y)u_2(y) + u_2(y)q(y)u_1(y) = 0 \end{aligned} \tag{7.14}$$

where we have used the $\mathcal{L}u_j = 0$ for $j = 1, 2$.

Therefore, let us define the constant

$$C := [u_1(a)u_2'(a) - u_2(a)u_1'(a)]p(a) \ . \tag{7.15}$$

Notice that with p defined by $p(x) = e^{\int_a^x P(z)\mathrm{d}z}$, we have $p(a) = 1$, so that

$$C = [u_1(a)u_2'(a) - u_2(a)u_1'(a)] \ .$$

By Lemma 15, our formula for the unique solution $u(x)$ of $\mathcal{L}u = f$ with $u(a) = u(b) = 0$ simplifies to

$$u(x) = \frac{1}{C}\int_a^x [u_2(x)u_1(y) - u_1(x)u_2(y)]f(y)\mathrm{d}y \ .$$

We get the general solution to $\mathcal{L}u = f$ by adding this particular solution to the general solution of the homogeneous equation $\mathcal{L}u = 0$, which is

$$\alpha u_1(x) + \beta u_2(x)$$

for arbitrary constant α and β. We summarize:

Theorem 42. *With $\mathcal{L}$ defined as above and f continuous on $[a, b]$, the general solution of $\mathcal{L}u = f$ is*

$$u(x) = \alpha u_1(x) + \beta u_2(x) + \frac{1}{C}\int_a^x [u_2(x)u_1(y) - u_1(x)u_2(y)]f(y)\mathrm{d}y \ .$$

Note that advantage of the Sturm-Liouville form: The denominator in the usual formula one gets from Duhamel's formula is constant, and comes outside the integral.

7.1.2 The Green's functions as the inverse of $\mathcal{L}$

Our goal in this subsection is to prove that when $\mathcal{L}u = 0$ has no non-trivial solution with $u(a) = u(b) = 0$, then there exists a unique solution to $\mathcal{L}u = f$ satisfying $u(a) = u(b) = 0$ for every continuous functions f on $[a, b]$. Moreover, we shall find an explicit formula for the solution. This amounts to computing the inverse of the operator $\mathcal{L}$.

Our strategy is to use the fact that we have an explicit formula for the solution of the *initial value problem*. We seek to match one of these solutions to our boundary conditions.

That is, we seek to choose α and β so that $u(a) = u(b) = 0$. This leads to the system of equations

$$\begin{aligned} \alpha u_1(a) + \beta u_2(a) &= 0 \\ \alpha u_1(b) + \beta u_2(b) &= -\frac{1}{C}\int_a^b [u_2(b)u_1(y) - u_1(b)u_2(y)]f(y)\mathrm{d}y \end{aligned} \tag{7.16}$$

This system of equations has a unique solution if and only if the matrix $\begin{bmatrix} u_1(a) & u_2(a) \\ u_1(b) & u_2(b) \end{bmatrix}$ is invertible. Define

$$D = \det\left(\begin{bmatrix} u_1(a) & u_2(a) \\ u_1(b) & u_2(b) \end{bmatrix}\right) = u_1(a)u_2(b) - u_2(a)u_1(b) \ . \tag{7.17}$$

Let us assume for the moment that $D \neq 0$. Then the unique solution of (7.16) is

$$\begin{aligned} \alpha &= \frac{1}{CD}u_2(a)\int_a^b [u_2(b)u_1(y) - u_1(b)u_2(y)]f(y)\mathrm{d}y \\ \beta &= -\frac{1}{CD}u_1(a)\int_a^b [u_2(b)u_1(y) - u_1(b)u_2(y)]f(y)\mathrm{d}y \end{aligned} \tag{7.18}$$

Altogether, the unique solution of $\mathcal{L}u = f$

$$\begin{aligned} &\frac{1}{CD}[u_1(x)u_2(a) - u_2(x)u_1(a)]\int_a^b [u_2(b)u_1(y) - u_1(b)u_2(y)]f(y)\mathrm{d}y \\ &\qquad + \frac{1}{C}\int_a^x [u_2(x)u_1(y) - u_1(x)u_2(y)]f(y)\mathrm{d}y \ . \end{aligned} \tag{7.19}$$

Breaking the integral for a to b into two pieces, one from a to x and the other from x to b, we can writ this is

$$\begin{aligned} u(x) &= \frac{1}{CD}\int_x^b [u_1(x)u_2(a) - u_2(x)u_1(a)][u_2(b)u_1(y) - u_1(b)u_2(y)]f(y)\mathrm{d}y \\ &+ \frac{1}{CD}\int_a^x [u_1(x)u_2(a) - u_2(x)u_1(a)][u_2(b)u_1(y) - u_1(b)u_2(y)]f(y)\mathrm{d}y \\ &+ \frac{1}{CD}\int_a^x [u_1(a)u_2(b) - u_2(a)u_1(b)][u_2(x)u_1(y) - u_1(x)u_2(y)]f(y)\mathrm{d}y \ . \end{aligned}$$

Noting that

$$\begin{aligned} &[u_1(x)u_2(a) - u_2(x)u_1(a)][u_2(b)u_1(y) - u_1(b)u_2(y)] + [u_1(a)u_2(b) - u_2(a)u_1(b)][u_2(x)u_1(y) - u_1(x)u_2(y)] \\ &\qquad = [u_1(y)u_2(a) - u_2(y)u_1(a)][u_2(b)u_1(x) - u_1(b)u_2(x)] \ . \end{aligned}$$

Therefore, we define the *Green's function* $G(x,y)$ for $\mathcal{L}$ on (a,b) by

$$G(x,y) = \frac{1}{CD}\begin{cases}[u_1(x)u_2(a) - u_2(x)u_1(a)][u_2(b)u_1(y) - u_1(b)u_2(y)] & y \geq x \\ [u_1(y)u_2(a) - u_2(y)u_1(a)][u_2(b)u_1(x) - u_1(b)u_2(x)] & x \geq y\ .\end{cases} \tag{7.20}$$

We have proved that, provided $D \neq 0$, so that the Green's function is defined, for all continuous and bounded functions f on $[a,b]$, there is a unique solution of $\mathcal{L}u = f$ satisfying $u(a) = u(b) = 0$, and it is given by

$$u(x) = \int_a^b G(x,y)f(y)\mathrm{d}y\ .$$

We summarize our results:

Theorem 43. *With $\mathcal{L}$ as above, let u_1 and u_2 be two linearly independent solutions of $\mathcal{L}u = 0$. Suppose that*

$$D = u_1(a)u_2(b) - u_2(a)u_1(b) \neq 0\ . \tag{7.21}$$

Then there exists a unique solution to $\mathcal{L}u = f$ satisfying $u(a) = u(b) = 0$ for every continuous functions f on $[a,b]$, and this solution is given by $u(x) = \int_a^b G(x,y)f(y)\mathrm{d}y$ where the Green's function *$G(x,y)$ is given by (7.20).*

Now note that if $D = 0$, the matrix $\begin{bmatrix} u_1(a) & u_2(a) \\ u_1(b) & u_2(b)\end{bmatrix}$ is not one-to-one, and so there exists a non-zero (α,β) such that

$$\begin{bmatrix} u_1(a) & u_2(a) \\ u_1(b) & u_2(b)\end{bmatrix}(\alpha,\beta) = (0,0)\ .$$

Then, with this choice of α and β,

$$\tilde{u}(x) = \alpha u_1(x) + \beta u_2(x)$$

satisfies $\mathcal{L}\tilde{u}(x) = 0$ and $\tilde{u}(x) = \tilde{u}(b) = 0$. Up to a multiple, any such solution that is non-trivial satisfies $\tilde{u}(a) = 0$ and $\tilde{u}'(a) = 1$, and since there is a unique solution of the equation with these initial conditions, all such solutions are the same up to a multiple.

We have already proved that unless $\int_a^b \tilde{u}(x)f(x)\mathrm{d}x = 0$, there can be no solutions of $\mathcal{L}u = f$. On the other hand, suppose $\int_a^b \tilde{u}(x)f(x)\mathrm{d}x = 0$. Since we may take $u_1$and u_2 to be any two linearly independent solutions of $\mathcal{L}u = 0$, we may take $u_1 = \tilde{u}$. Then from Theorem 42, one solution to $\mathcal{L}u = f$ is given by

$$u(x) = \frac{1}{C}\int_a^x [u_2(x)u_1(y) - u_1(x)u_2(y)]f(y)\mathrm{d}y\ .$$

Evidently this solution satisfies $u(a) = 0$. Next, we see that

$$\begin{aligned} u(b) &= \frac{1}{C}\int_a^b [u_2(b)u_1(y) - u_1(b)u_2(y)]f(y)\mathrm{d}y \\ &= \frac{1}{C}u_2(b)\int_a^b u_1(y)f(y)\mathrm{d}y - \frac{1}{C}u_1(b)\int_a^b u_2(y)f(y)\mathrm{d}y = 0 \end{aligned}$$

where we use the facts that

$$\int_a^b u_1(y)f(y)\mathrm{d}y = 0 \qquad \text{and} \qquad u_1(b) = 0\ .$$

Thus we have at least one solution. But adding any multiple of u_1 to this solution gives us another solutions. Also, the difference of any two solutions satisfies $\mathcal{L}u = 0$ and $u(a) = u(b) = 0$, and is therefore a multiple of u_1. Thus, the general solution has the form

$$\gamma u_1(x) + \frac{1}{C}\int_a^b [u_2(b)u_1(y) - u_1(b)u_2(y)]f(y)\mathrm{d}y$$

for an arbitrary constant γ.

Putting it all together, we have now proved Thoerem 40. Moreover, we have found explicit formulae for the solutions when they exist, provide we can find two linearly independent soltiions of the homogeneous equation $\mathcal{L}u = 0$.