# Problem Set 6 for Math 292, April 16, 2013

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April 22, 2013

**1.** Solve the wave equation

$$\frac{\partial^2}{\partial t^2}h(t,x) = 4\frac{\partial^2}{\partial x^2}h(t,x)$$

subject to

$$h(t,0) = h(t,1) = 0 \qquad \text{for all } t\ ,$$

and

$$h(t,x) = h_0(x) := \sin^3(2\pi x) \qquad \text{and} \qquad \frac{\partial}{\partial t}h(0,x) = v_0(x) := \sin^2(\pi x)$$

for all $0 \le x \le 1$.

**(a)** Do this first by first expanding $h_0(x)$ and $v_0(x)$ in the form

$$h_0(x) = \sum_{n=1}^{\infty} \alpha_n \sin(n\pi x) \qquad \text{and} \qquad v_0(x) = \sum_{n=1}^{\infty} \beta_n \sin(n\pi x)\ .$$

The coefficients $\alpha_n$ will be non-zero for only finitely many values of $n$, and there is an easy way to find them. The coefficients $\beta_n$ will be non-zero for infinitely many $n$, but there is an easy way to find them all. Use Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.

**(b)** Next do this by constructing appropriate extensions of the initial data to the whole real line, and using the fact that for $g$ twice continuously differentiable, $g(x - ct)$ and $g(x + tc)$ satisfy the wave equation.

**2.** Find a non-zero function $v(x)$ and a function $q(x)$ so that if $y(x) = v(x)u(x)$,

$$(1 + x^2)y'' - xy' + x^3 y = 0$$

if and only if

$$u'' + qu = 0\ .$$

**3.** Let $y(x)$ solve

$$y''(x) + q(x)y(x) = 0\ ,$$

and suppose that

$$y(2) = y(4) = y(8) = 0$$

and that $y(x) \neq 0$ for any other values of $x$ with $0 \le x \le 16$.

**(a)** Find a number $A$ such that necessarily

$$\min\{\ q(x)\ :\ 0 \le x \le 16\ \} \le A\ .$$

**(b)** Find a number $B$ such that necessarily

$$\max\{\ q(x)\ :\ 0 \le x \le 16\ \} \ge B\ .$$

**4.** Let $\lambda_7$ be the 7th positive number $\lambda$ such that

$$y'' + \left(\lambda + \frac{x^2}{1+x^2}\right) y = 0$$

with $y(0) = y(1) = 0$.

Find numbers $a$ and $b$ such that $a \le \lambda_7 \le b$ with $b - a < 1/10$, and justify your answer. How small can you make $b - a$?