Homework Set 4, Math 292 Spring 2013

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These exercises are due Wednesday, .March 6

In these exercises, M and A are the matrices

$$M = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} 31 & 32 \\ 32 & 40 \end{bmatrix} .$$

All of the exercises below are closely related, and are steps along the way to the solution of the equation

$$M\mathbf{x}'' = -A\mathbf{x} + \mathbf{f} \qquad \text{with} \qquad \mathbf{x}(0) = \mathbf{x}_0 \ , \quad \mathbf{x}'(0) = \mathbf{v}_0 \ .$$

1. Show that M and A are positive definite, and find the positive definite matrix $M^{1/2}$ so that $(M^{1/2})^2 = M$. Finally, compute $K := M^{-1/2}AM^{-1/2}$.

2. (a) Find vectors $\mathbf{y}_0$ and $\mathbf{w}_0$ so that if $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are two twice continuously differentiable curves, $\mathbf{x}(t)$ satisfies

$$M\mathbf{x}'' = -A\mathbf{x} \qquad \text{with} \qquad \mathbf{x}(0) = \mathbf{x}_0 \ , \quad \mathbf{x}'(0) = \mathbf{v}_0 \tag{0.1}$$

if and only if $\mathbf{y}(t)$ satisfies

$$\mathbf{y}'' = -L\mathbf{y} \qquad \text{with} \qquad \mathbf{y}(0) = \mathbf{y}_0 \ , \quad \mathbf{y}'(0) = \mathbf{w}_0 \tag{0.2}$$

(b) Find an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ of $\mathbb{R}^2$ consisting of eigenvectors of K.

(c) Let $\mathbf{y}(t)$ be the solution of (0.2). Using the orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ found above, write

$$\mathbf{y}(t) = w_1(t)\mathbf{u}_1 + w_2(t)\mathbf{u}_2 \ .$$

Find the equations that w_1 and w_2 must satisfy, and then solve these equations.

(d) Find the solutions to equations (0.4) and (0.2).

3. (a) Now consider the *driving force* $\mathbf{f} = 3\cos(\omega t)(1, 2)$, where $\omega > 0$.

Using the method of normal modes together with Duhamel's formula, find the solution to

$$M\mathbf{x}'' = -A\mathbf{x} + \mathbf{f} \qquad \text{with} \qquad \mathbf{x}(0) = \mathbf{0} \ , \quad \mathbf{x}'(0) = \mathbf{0} \ . \tag{0.3}$$

For which values of ω will resonance occur?

(b) Using the superposition principle, use the previous results to find the general solution to

$$M\mathbf{x}'' = -A\mathbf{x} + \mathbf{f} \ . \tag{0.4}$$