Homework Set 2, Math 292 Spring 2013

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These exercises are due Monday, February 11.

1. Consider the equation

$$x' = x + x^2 \ .$$

(a) Find the general solution; i.e., the solution for general initial data $x(0) = x_0$, by treating this equation as a Bernouli equation.

(b) Find the general solution by writing the equation as $x' = v(x)$ with $v(x) = x + x^2$, and using the formula $t(x) = t_0 + \int_{x_0}^{x} (1/v(z))\mathrm{d}z$.

(c) For each x_0, there is a time $t(x_0)$ such that $\lim_{t \to t(x_0)} x(t) = +\infty$. Find a formula for $t(x_0)$.

2. Consider the Riccati equation

$$x' = x^2 - x - 2 \ .$$

(a) Find a particular solution of the form $x(t) = Ct^{\alpha}$,

(b) Find the general solution; i.e., the solution for general initial data, using the Riccati reduction method. (Since the equation is autonomous, you could also use the $t(x)$ formula, but use the reduction method instead.)

(c) find the solution with $x(0) = -1$.

3. Find the general solution of the Riccati equation

$$x' = (x - t)^2 + 1 \ .$$

4. Let $U(x, y) = x^2 + 2y^2 + 2xy + x - 5y$. Consider the system of equations

$$\mathbf{x}'(t) = \nabla U^{\perp}(\mathbf{x}(t)) \qquad \text{with} \qquad \mathbf{x}(0) = (1, 1) \ .$$

Sketch a plot of the solution curve. Also, compute

$$\min\{y(t) \ : \ t \in \mathbb{R}\} \qquad \text{and} \qquad \min\{y(t) \ : \ t \in \mathbb{R}\} \ .$$

5. In this problem, we shall see how to use reduction of order and inversion of dependence to solve

$$x'' = -4x \ ,$$

with $x(0) = x_0$ and $x'(0) = y_0$. Both of these techniques were on the previous problem set, and now we shall put them together to get another way to solve this equation.

(a) Introduce $y := x'$. The equation becomes $y' = -4x$ where the prime denotes differentiation by t. Now note that by the chain rule

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}$$

where we use the function $t(x)$ to regard y as a function of x. (In a slight abuse of notation, we write $y(x) = y(t(x))$.) Show that

$$y\frac{\mathrm{d}y}{\mathrm{d}x} = -4x \ .$$

(b) Now separate variables, and find the general solution of the original equation.