Homework Set 1, Math 292 Spring 2013

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These exercises are due Thursday, January 31.

1. This problem illustrates the use of *reduction of order.* Consider the second order equation

$$tx'' - x' = 3t^2 \ .$$

Introduce a new variable $y(t) = x'(t)$, and notice that the equation becomes a linear first order equation in y. Use this fact to find $y(t)$, and then $x(t)$, given that $x(1) = 1$ and $x'(1) = 2$.

2. Use the method of the previous exercise to find the solution of

$$x'' = -4x \ ,$$

with $x(0) = x_0$ and $x'(0) = y_0$. (Your answer will be a function of t, x_0 and y_0.)

3. **(a)** For $0 < p < 1$, and $x_0 > 0$, consider the equation

$$x' = x(1-x) - px \qquad x(0) = x_0 \ ,$$

which is a variant of the logistic equation. Note that the equation is $x' = v(x)$ with $v(x) = x(1-x) - px$. This is a quadratic polynomial, and so it has two roots, namely $x = 0$ and $x = 1-p$. Factor it, and solve the equation.

(a) Is $v(x)$ Lipschitz continuous on $(1, 1-p)$? justify your answer. For $x_0 \in (0, 1-p)$, determine $\lim_{t\to\pm\infty} x(t)$. Justify your answer.

(b) Solve the equation for $x_0 > 1-p$, Compute $\lim_{t\to\infty} x(t)$, but show that for each $x_0 > 1-p$, there is a value $t_* < 0$ so that $\lim_{t\to t_*} x(t) = \infty$; ie.e, the graph of $x(t)$ has a vertical asymptote at $t = t_*$. Compute $t*$ as a function of x_0.

4. **(a)** Find the solution of the equation

$$x' = 2x^2 + tx^2 \ , \qquad x(0) = 1 \ .$$

(b) On which interval in t around the origin in the solution defined?

(c) What is the minimum value of the solution?

5. We have seen that in solving $x'(t) = v(x(t))$, it is useful to find the *inverse function* $t(x)$ first, and then invert this to obtain $x(t)$.

This approach is also useful for other sorts of equations. Recalling that $t'(x) = 1/x'(t(x))$, or more briefly, $t' = 1/x'$, convert

$$(e^x - 2tx)x' = x^2 \ , \qquad x(1) = 2$$

into a differential equation for $t(x)$, and solve this differential equation. In this case, inverting to find x as a function of t involves solving a transcendental equation, so you cannot do it explicitly for all t, but you can use Newton's method to evaluate $x(t)$ for any t. Use this approach to compute $x(2)$ accurately to 10 decimal places.

6. Use the change of variables $z = x/t$ to find the general solution of

$$tx' = x + 2te^{-x/t} \ .$$