Driven Oscilations, Math 292 Spring 2013

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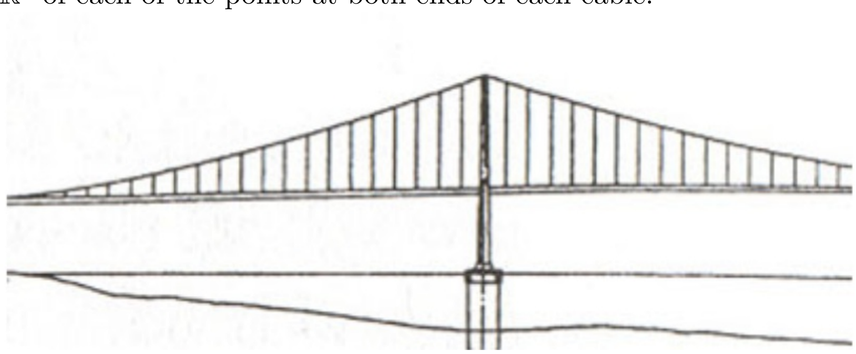
February 21, 2013

1 Introduction

The methods we have been studying provide an effective means for analyzing mechanical systems near equilibrium, but being driven by periodic driving forces.

The systems we shall study typically require a large number of coordinates for their description. For instance, we might be considering a suspension bridge, and then we would like to keep track of the position in $\mathbb{R}^3$ of each of the points at both ends of each cable.



As you see, this will require a large number of coordinates. We can combine them all into one vector $\mathbf{x} \in \mathbb{R}^n$, for large n. The vector $\mathbf{x}$ describes the *configuration* of the system, in this case a bridge. In the absence of any external driving forces, when the bridge is in its steady-state rest configuration, all of the forces are balanced. The total forces can be written as the gradient of a potential energy function $V(\mathbf{x})$, so that the equilibrium position $\mathbf{x}_\star$ is a solution of

$$\nabla V(\mathbf{x}) = \mathbf{0} \ ,$$

and the system will be stable under small disturbances provided $\mathbf{x}_\star$ is strict local minimum of V.

If the configuration at time $t = 0$ is not $\mathbf{x}_\star$, there will be motion described by Newton's equations

$$M\mathbf{x}''(t) = -\nabla V(\mathbf{x}(t) \qquad \text{and} \qquad \mathbf{x}(0) = \mathbf{x}_0 \ . \tag{1.1}$$

Here M is a "mass matrix", which you can think of as being a diagonal $n \times n$ matrix with masses associated to the various coordinate points on the diagonal. More generally, for systems described by *Lagrangian dynamics*, M is a symmetric $n \times n$ matrix all of whose eigenvalues are strictly positive. We shall not go into Lagrangian dynamics here, which would take us far afield into physics, but shall take the applicability of the equation (1.1) for granted, and shall study its solutions, together with those of a related equation that includes the effects of periodic external driving forces.

Now introduce a new variable $\mathbf{z}(t) = \mathbf{x}(t) - \mathbf{x}_\star$, which measures the displacement from equilibrium. Then since $\mathbf{x}_\star$ is constant,

$$\mathbf{z}''(t) = \mathbf{x}''(t) = -\nabla V(\mathbf{x}(t)) = -\nabla V(\mathbf{z}(t) + \mathbf{x}_\star) \ .$$

Linearizing about $\mathbf{x}_\star$, we have

$$\nabla V(\mathbf{z}(t) + \mathbf{x}_\star) \approx [\mathrm{Hess}_V(\mathbf{x}_\star)]\mathbf{z}(t) \ ,$$

where $[\mathrm{Hess}_V(\mathbf{x}_\star)]$ is the Hessian matrix of V at $\mathbf{x}_\star$.

For the equilibrium position to be steady, we require that $\mathbf{x}_\star$ be a strict local minimum of V. This implies in particular that all eigenvalues of $[\mathrm{Hess}_V(\mathbf{x}_\star)]$ are non-negative. We shall generally require a more robust form of stability, namely, that all of the eigenvalues of $[\mathrm{Hess}_V(\mathbf{x}_\star)]$ are *strictly* positive. This brings us to a basic definition:

1 DEFINITION (The mechanical vibration equation)**.** Let M and A be $n \times n$ symmetric matrices, each with the property that all of its eigenvalues are strictly positive. The *mechanical vibration equation* is the second order linear equation in $\mathbb{R}^n$

$$M\mathbf{z}''(t) = -A\mathbf{z}(t) \ . \tag{1.2}$$

It is in *reduced form* if $M = I_{n\times n}$, the $n \times n$ identity matrix.

We have explained how such equation arise: The matrix A will be the Hessian of a potential energy function at a local minimum $\mathbf{x}_\star$, and $\mathbf{z}$ specifies the difference between the current configuration, and the local minimum $\mathbf{x}_\star$. To proceed, we need a few simple facts about symmetric matrices with strictly positive eigenvlaues. These follow easily from the Spectral Theorem, which we have proved last semester, and which says that given any symmetric $n \times n$ matrix, there is an orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ of $\mathbb{R}^n$ consisting of eigenvecotrs of that matrix.

1.1 Positive definite matrices and their square roots

2 DEFINITION (Positive definite matrix)**.** An $n \times n$ matrix A is *positive define* in case it is symmetric, and for all $\mathbf{x} \neq \mathbf{0}$,

$$\mathbf{x} \cdot A\mathbf{x} > 0 \ .$$

The next theorem provides a means to check whether a matrix is positive definite – by computing its eigenvalues – and identifies the matrices in the mechanical vibration equation as positive definite.

3 Theorem. *A symmetric $n \times n$ matrix M is positive definite if an only if each of its eigenvalues is positive.*

Proof. Suppose M is positive definite. Let $\mathbf{u}$ be an eigenvector of M so that $M\mathbf{u} = \mu\mathbf{u}$ for some $\mu \in \mathbb{R}$. We must show that $\mu > 0$. Since M is positive definite, and since $\mathbf{u} \neq \mathbf{0}$ by virtue of being an eigenvector, $\mathbf{u} \cdot M\mathbf{u} > 0$. But

$$\mathbf{u} \cdot M\mathbf{u} = \mathbf{u} \cdot \mu\mathbf{u} = \mu\|\mathbf{u}\|^2 > 0$$

and so $\mu > 0$ as was to be shown.

Next, suppose that each of the eigenvalues of M is strictly positive. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvecotrs of M; we shall write $A\mathbf{u}_j = \mu_j \mathbf{u}_j$, $j = 1, \ldots, n$. Next, for any $\mathbf{x} \in \mathbb{R}^n$,

$$\mathbf{x} = \sum_{j=1}^{n} (\mathbf{u}_j \cdot \mathbf{x})\mathbf{u}_j \ .$$

Thus,

$$\mathbf{x} \cdot M\mathbf{x} = \sum_{j=1}^{n} \mu_j (\mathbf{u}_j \cdot \mathbf{x})^2 \geq 0 \ ,$$

and since each μ_j is stricly positive, the sum is zero if and only if $\mathbf{u}_j \cdot \mathbf{x} = 0$ for each j, and this means $\mathbf{x} = \mathbf{0}$. □

4 LEMMA. *Let A and B be positive definite $n \times n$ matrices. Then $A + B$ is positive definite.*

Proof. Since A and B are symmetric, so is $A + B$. Moreover, for any $\mathbf{x} \neq \mathbf{0}$,

$$\mathbf{x} \cdot (A + B)\mathbf{x} = \mathbf{x} \cdot A\mathbf{x} + \mathbf{x} \cdot B\mathbf{x} > 0 \ .$$

□

5 LEMMA. *Let A be a positive definite $n \times n$ matrix. Then A is invertible. Moreover, if C is an other invertible $n \times n$ matrix, then $C^t AC$ is positive definite.*

Proof. Suppose $A\mathbf{x} = 0$. Then $\mathbf{x} \cdot A\mathbf{x} = 0$, and this means that $\mathbf{x} = 0$. Thus, the only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$, and then by the Fundamental Theorem of Linear Algebra, A is invertible.

Next, if A is positive definite and C is invertible, then for any $\mathbf{x} \neq \mathbf{0}$,

$$\mathbf{x} \cdot (C^t AC)\mathbf{x} = (C\mathbf{x}) \cdot A(C\mathbf{x}) > 0$$

since $C\mathbf{x} \neq \mathbf{0}$. □

The next theorem tells us that positive define matrices have a positive define *square root for matrix multiplication*, and tells us how to compute this matrix square root.

6 Theorem. *Let M be a positive definite $n \times n$ matrix. Then there is a unique positive definite matrix $M^{1/2}$ such that $(M^{1/2})^2 = M$. Moreover, let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be any orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of M, let $U = [\mathbf{u}_1, \ldots, \mathbf{u}_n]$ and let $D^{1/2}$ be the $n \times n$ diagonal matrix whose jth diagonal entry is $\sqrt{m_j}$ where $M\mathbf{u}_j = m_j \mathbf{u}_j$. Then*

$$M^{1/2} = UD^{1/2}U^t \ . \tag{1.3}$$

7 Example. Let $M = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$. The characteristic polynomial of M is $(t - 25)(t - 9)$ so that the eigenvalues are $m_1 = 25$ and $m_2 = 9$, and thus, M is positive definite. The corresponding normalized eigenvectors are

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}(1, 1) \qquad \text{and} \qquad \mathbf{u}_1 = \frac{1}{\sqrt{2}}(1, -1) \ .$$

Therefore

$$D^{1/2} = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \qquad \text{and} \qquad U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} .$$

Therefore,

$$M^{1/2} = UD^{1/2}U^t = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} .$$

As you can check,

$$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}^2 = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix} ,$$

so we have indeed computed a matrix square root. Acoording to the theorem, this is the only matrix square root that is itself positive definite.

Proof of Theorem 6. By the Spectral Theorem, there does exist an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of M, Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be such a basis, and then define $M^{1/2} = UD^{1/2}U^t$ as in the statement of the theorem.

Let us show that $(M^{1/2})^2 = M$. It suffices to show that for all $\mathbf{x} \in \mathbb{R}^n$, $(M^{1/2})^2\mathbf{x} = M\mathbf{x}$. By the rules of matrix multiplication,

$$U^t\mathbf{x} = (\mathbf{u}_1, \cdot\mathbf{x}, \ldots, \mathbf{u}_n \cdot \mathbf{x}) .$$

Therefore,

$$D^{1/2}U^t\mathbf{x} = (\sqrt{m_1}\mathbf{u}_1 \cdot \mathbf{x}, \ldots, \sqrt{m_n}\mathbf{u}_n \cdot \mathbf{x}) .$$

Finally,

$$M^{1/2}\mathbf{x} = UD^{1/2}U^t\mathbf{x} = \sum_{j=1}^{n} \sqrt{m_j}(\mathbf{u}_j \cdot \mathbf{x})\mathbf{u}_j .$$

Repeating the procedure, e find

$$(M^{1/2})^2\mathbf{x} = \sum_{j=1}^{n} m_j(\mathbf{u}_j \cdot \mathbf{x})\mathbf{u}_j .$$

On the other hand, since $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is an orthonormal basis,

$$\mathbf{x} = \sum_{j=1}^{n} (\mathbf{u}_j \cdot \mathbf{x})\mathbf{u}_j .$$

Thus,

$$M\mathbf{x} = \sum_{j=1}^{n} (\mathbf{u}_j \cdot \mathbf{x})M\mathbf{u}_j = \sum_{j=1}^{n} m_j(\mathbf{u}_j \cdot \mathbf{x})\mathbf{u}_j .$$

This proves $(M^{1/2})^2 = M$.

Now let A be any positive definite matrix such that $A^2 = M$. For any positive number μ,

$$(A + \sqrt{\mu}I_{n\times n})(A - \sqrt{\mu}I_{n\times n}) = M - \mu I_{n\times n} .$$

Since by Lemma 4, $(A+\sqrt{\mu}I_{n\times n})$ is positive definite and therefore invertible, for all unit vectors $\mathbf{u}$, $(A-\sqrt{\mu}I_{n\times n})\mathbf{u} = \mathbf{0}$ if and only if $(M-\mu I_{n\times n})\mathbf{u} = \mathbf{0}$. That is, $\mathbf{u}$ is an eigenvector of M with eigenvalue μ if and only if $\mathbf{u}$ is an eigenvector of A with eigenvalue $\sqrt{\mu}$.

Now using the orthonormal basis $\{\mathbf{u}_1,\dots,\mathbf{u}_n\}$ that we have used to define $M^{1/2}$, we take any $\mathbf{x}\in\mathbb{R}^n$, and compute

$$A\mathbf{x} = A\left(\sum_{j=1}^n (\mathbf{u}_j\cdot\mathbf{x})\mathbf{u}_j\right) = \left(\sum_{j=1}^n (\mathbf{u}_j\cdot\mathbf{x})A\mathbf{u}_j\right) = \left(\sum_{j=1}^n \sqrt{m_j}(\mathbf{u}_j\cdot\mathbf{x})\mathbf{u}_j\right) = M^{1/2}\mathbf{x}\ .$$

since $\mathbf{x}$ is arbitrary, this shows $A = M^{1/2}$. □

1.2 Reduction to reduced form

We now explain how every equation of the form $M\mathbf{z}''(t) = -A\mathbf{z}(t)$ with M and A positive definite may be put in reduced form.

Now multiply both sides of $M\mathbf{z}''(t) = -A\mathbf{z}(t)$ by $M^{-1/2}$ to obtain

$$M^{1/2}\mathbf{z}''(t) = -M^{-1/2}A\mathbf{z}(t)\ . \tag{1.4}$$

Next, since $M^{-1/2}M^{1/2}$ is the identity matrix, we may freely insert $M^{-1/2}M^{1/2}$ between A and $\mathbf{z}$ on the right. We obtain

$$M^{1/2}\mathbf{z}''(t) = -[M^{-1/2}AM^{-1/2}]M^{1/2}\mathbf{z}(t)\ . \tag{1.5}$$

Now let us define

$$\mathbf{y}(t) = M^{1/2}\mathbf{z}''(t) \qquad \text{and} \qquad K = [M^{-1/2}AM^{-1/2}]\ . \tag{1.6}$$

Then we have

$$\mathbf{y}''(t) = -K\mathbf{y}(t)\ . \tag{1.7}$$

Furthermore, all of the transformations we have made are invertible, and so (1.7) is equivalent to (1.2). Finally, by Lemma 5, K is positive definite.

We summarize:

8 Theorem. *Let M and A be positive definite $n\times n$ matrices. Let $\mathbf{x}(t)$ be a twice continuously differentiable curve, and define $L = M^{-12/}AM^{-1/2}$ and $\mathbf{y}(t) = M^{1/2}\mathbf{x}(t)$. Then*

$$M\mathbf{x}''(t) = -A\mathbf{x}(t) \qquad \Longleftrightarrow \qquad \mathbf{y}''(t) = -K\mathbf{y}(t)\ .$$

9 Example. Consider the system

$$\begin{aligned} 4x_1''(t) &= -8x_1(t) - 4x_2(t) \\ x_2''(t) &= -4x_1(t) - 5x_2(t) \end{aligned}$$

with the initial conditions

$$x_1(0) = -1\ ,\ x_2(0) = 1\ ,\ x_1'(0) = 1/2 \quad \text{and} \quad x_2'(0) = 2\ .$$

Introducing

$$M = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} 8 & 4 \\ 4 & 5 \end{bmatrix} ,$$

and $\mathbf{x}(t) = (x_1(t), x_2(t))$, we can write this system as

$$M\mathbf{x}''(t) = -A\mathbf{x}(t) \ .$$

In this case, it is easy to see that $M^{1/2} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, and so

$$K = M^{-1/2}AM^{-1/2} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \ .$$

Then with $\mathbf{y}(t) = M^{1/2}\mathbf{x}(t) = (2x_1(t), x_2(t))$, we have

$$\mathbf{y}''(t) = -K\mathbf{y}(t) \ ,$$

and the initial conditions transform to

$$\mathbf{y}(0) = (-2, 1) \quad \text{and} \quad \mathbf{y}'(0) = (1, 2) \ .$$

In the next subsection, we shall see how to solve systems of this type.

1.3 Normal modes

We are now in a position to decouple the system $\mathbf{y}'' = -K\mathbf{y}$ into n independent one-variable equations. To do this, let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of K. We denote the jth eigenvalue of K by κ_j; i.e.,

$$K\mathbf{u}_j = \kappa_j \mathbf{u}_j \ ,$$

for each $j = 1, \ldots, n$, and since K is positive definite, $\kappa_j > 0$ for each j.

Suppose that $\mathbf{y}''(t) = -K\mathbf{y}(t)$. Let us take the dot product of both sides with $\mathbf{u}_j$, On the left we get

$$\mathbf{u}_j \cdot \mathbf{y}''(t) = (\mathbf{u}_j \cdot \mathbf{y}(t))'' \ .$$

On the right we get, using the transpose identity $\mathbf{x} \cdot B\mathbf{y} = B^t\mathbf{x} \cdot \mathbf{y}$ and the fact that K is symmetric,

$$-\mathbf{u}_j \cdot K\mathbf{y}(t) = -(K\mathbf{u}_j) \cdot \mathbf{y}(t) = -\kappa_j(\mathbf{u}_j \cdot \mathbf{y}(t)) \ .$$

Thus, if we define

$$w_j(t) = \mathbf{u}_j \cdot \mathbf{y}(t) \ , \tag{1.8}$$

we have shown that

$$w_j''(t) = -\kappa_j w_j(t) \ , \tag{1.9}$$

Conversely, suppose that $\{w_1(t), \ldots, w_n(t)\}$ are such that (1.9) is satisfied for each $j = 1, \ldots, n$, Then, defining

$$\mathbf{y}(t) = \sum_{j=1}^{n} w_j(t)\mathbf{u}_j \ , \tag{1.10}$$

we have

$$\mathbf{y}''(t) = -K\mathbf{y}(t) \tag{1.11}$$

and

$$\mathbf{y}(0) = \sum_{j=1}^{n} w_j(0)\mathbf{u}_j \qquad \text{and} \qquad \mathbf{y}'(0) = \sum_{j=1}^{n} w_j'(0)\mathbf{u}_j \ . \tag{1.12}$$

Therefore, if we can solve the one-variable equation

$$w''(t) = -\kappa w(t) \qquad \text{with} \qquad w(0) = a \quad \text{and} \quad w(0) = b \ , \tag{1.13}$$

for $\kappa > 0$ and a and b arbitrary, we can solve

$$\mathbf{y}''(t) = -K\mathbf{y}(t) \qquad \text{with} \qquad \mathbf{y}(0) = \mathbf{a} \quad \text{and} \quad \mathbf{y}(0) = \mathbf{b} \ , \tag{1.14}$$

for K positive define and $\mathbf{a}$ and $\mathbf{b}$ arbitrary, and we obtain the solution in the form of a sum (1.10). The special solutions that are the terms of this sum are called *normal modes*, and the decomposition of the solution into a sum of these special solutions is called a *normal mode decomposition*. We refer to (1.13) as the *normal mode equation*.

Now let us solve the normal mode equation. We reduce it to a first order system, defining $v(t) = w'(t)$ and $\mathbf{w}(t) = (w(t), v(t))$, so that (1.13) is equivalent to

$$\mathbf{w}'(t) = \begin{bmatrix} 0 & 1 \\ -\kappa & 0 \end{bmatrix} \mathbf{w}(t) \qquad \text{with} \qquad \mathbf{w}(0) = \mathbf{w}_0 := (a, b) \ . \tag{1.15}$$

The characteristic polynomial of the matrix

$$L := \begin{bmatrix} 0 & 1 \\ -\kappa & 0 \end{bmatrix}$$

is $t^2 + \kappa = 0$. This has the roots $\pm i\sqrt{\kappa}$. and so we find that

$$(1, i\sqrt{\kappa}) \ ,$$

is an eigenvector of L with eigenvalue $i\sqrt{\kappa}$. Thus,

$$e^{i\sqrt{\kappa}}(1, i\sqrt{\kappa})$$

is a complex solution of our equation, and the real and imaginary parts give us the two real solutions we need to compute e^{tL}. Carrying out the simple computations, we find

$$e^{tL} = \begin{bmatrix} \cos(\sqrt{\kappa}t) & \frac{1}{\sqrt{\kappa}}\sin(\sqrt{\kappa}t) \\ -\sqrt{\kappa}\sin(\sqrt{\kappa}t) & \cos(\sqrt{\kappa}t) \end{bmatrix} \ . \tag{1.16}$$

Therefore, the solution of (1.13) is

$$w(t) = a\cos(\sqrt{\kappa}t) + \frac{b}{\sqrt{\kappa}}\sin(\sqrt{\kappa}t) \ . \tag{1.17}$$

10 Example. Let $K = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$. We shall use a normal modes decomposition to solve

$$\mathbf{y}''(t) = -K\mathbf{y}(t) \qquad \text{with} \qquad \mathbf{y}(0) = (-2,1) \quad \text{and} \quad \mathbf{y}'(0) = (1,2) \ .$$

To find the normal modes, we first compute the eigenvalues. The characteristic polynomial is $t^2 - 7t + 6 = (t-6)(t-1)$ so the eigenvalues are $\mu_1 = 1$ and $\mu_2 = 6$. we compute:

$$K - I_{2\times 2} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \ .$$

Any vector orthogonal orthogonal to the rows of this matrix is an eigenvector with eigenvalue 1. Let us choose the unit vector

$$\mathbf{u}_1 = \frac{1}{\sqrt{5}}(-2,1) \ .$$

Since K is symmetric, the orthogonal unit vector

$$\mathbf{u}_2 = \frac{1}{\sqrt{5}}(1,2)$$

is an eigenvector with eigenvalue 6.

In the normal mode expansion

$$\mathbf{y}(t) = w_1(t)\mathbf{u}_1 + w_2(t)\mathbf{u}_2 \ ,$$

$w_1(t)$ is the solution of

$$w_1''(t) = -w_1(t) \qquad \text{with} \qquad w_1(0) = \mathbf{u}_1 \cdot \mathbf{y}(0) = \sqrt{5} \qquad \text{and} \qquad \mathbf{u}_1 \cdot \mathbf{y}'(0) = 0 \ ,$$

and $w_2(t)$ is the solution of

$$w_2''(t) = -6w_2(t) \qquad \text{with} \qquad w_2(0) = \mathbf{u}_2 \cdot \mathbf{y}(0) = 0 \qquad \text{and} \qquad \mathbf{u}_1 \cdot \mathbf{y}'(0) = \sqrt{5} \ ,$$

Then from (1.17) we have

$$w_1(t) = \sqrt{5}\cos(t) \qquad \text{and} \qquad w_2(t) = \sqrt{\frac{5}{6}}\sin(\sqrt{6}t) \ .$$

Finally,

$$\begin{aligned} \mathbf{y}(t) &= w_1(t)\mathbf{u}_1 + w_2(t)\mathbf{u}_2 \\ &= \cos(t)(-2,1) + \sqrt{6}\sin(\sqrt{6}t)(1,2) \\ &= (\sin(\sqrt{6}t) - 2\cos(t) \ , \ 2\sin(\sqrt{6}t) + \cos(t)) \end{aligned}$$

Now recall that the system we have solved in this example is the reduced form of the system introduced in Example 9. To concert back to the original variables $(x_1(t), x_2(t))$, all we need do is use

$$\mathbf{x}(t) = M^{-1/2}\mathbf{y}(t)$$

where M is the mass matrix $M = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$ from Example 9. We find

$$(x_1(t), x_2(t)) = (\sin(\sqrt{6}t)/2 - \cos(t) \ , \ 2\sin(\sqrt{6}t) + \cos(t)) \ .$$

2 Driven oscillations

We now suppose that there is a time-dependent external force $\mathbf{f}(t)$ acting our mechanical system. Now the equations of motion become

$$M\mathbf{x}''(t) = -\nabla V(\mathbf{x}(t)) + \mathbf{f}(t) \ .$$

If we linearize about a local minimum $\mathbf{x}_\star$ of V, and let A denote the Hessian of V at $\mathbf{x}_\star$, we obtain that $\mathbf{z}(t0 = \mathbf{x}(t) - \mathbf{x}_\star$ satisfies

$$M\mathbf{z}''(t) = -A\mathbf{z}(t) + \mathbf{f}(t) \ .$$

Because $\mathbf{x}_\star$ is a local minimum, all of the eigenvlaues of A are non-negative, but we shall assume a little more, namely that all of the eigenvalues are strictly positive.

Multiplying through by $M^{-1/2}$ as before, we obtain

$$M^{1/2}\mathbf{z}''(t) = -[M^{-1/2}AM^{-1/2}]M^{1/2}\mathbf{z}(t) = M^{1/2}\mathbf{f}(t) \ .$$

Therefore, defining

$$\mathbf{y}(t) := M^{-1/2}\mathbf{z}(t) \ , \quad K := M^{-1/2}AM^{-1/2} \quad \text{and} \quad \mathbf{g}(t) = M^{1/2}\mathbf{f}(t) \ ,$$

our equation is equivalent to

$$\mathbf{y}''(t) = -K\mathbf{y}(t) + \mathbf{g}(t) \ . \tag{2.1}$$

We shall reduce to normal modes, just as in the previous section. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be an orthonormal basis of eigenvectors of K with $K\mathbf{u}_j = \kappa_j \mathbf{u}_j$ for $j = 1, \dots, n$. Then, just as in the previous section, we can represent the solution to (2.1) in the form

$$\mathbf{y}(t) = \sum_{j=1}^n w_j(t)\mathbf{u}_j \tag{2.2}$$

where $w_j(t)$ solves

$$w_j''(t) = -\kappa_j w_j(t) + g_j(t) \tag{2.3}$$

where

$$g_j(t) = \mathbf{u}_j \cdot \mathbf{g}(t) \ .$$

To solve the inhomogenous equation

$$w''(t) = -\kappa w(t) + g(t) \qquad \text{with} \quad w(0) = a \quad \text{and} \quad w'(0) = b \ , \tag{2.4}$$

we reduce to a first order system as before, writing $v'(t) = w'(t)$ and $\mathbf{w}(t) = (w(t), v(t))$. Then (2.4) becomes

$$\mathbf{w}'(t) = L\mathbf{w}(t) + (0, g(t)) \qquad \text{with} \qquad \mathbf{w}(0) = \mathbf{w}_0 := (a, b) \ , \tag{2.5}$$

where, as before,

$$L := \begin{bmatrix} 0 & 1 \\ -\kappa & 0 \end{bmatrix} \ .$$

To solve this, regroup the terms and multiply through by e^{-tL} to obtain

$$(e^{-tL}(\mathbf{w}'(t) - L\mathbf{w}(t)) = (e^{-tL}(0, g(t)) \ .$$

That is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-tL}\mathbf{w}(t)\right) = (e^{-tL}(0, g(t)) \ .$$

Integrating, we find

$$e^{-tL}\mathbf{w}(t) - \mathbf{w}(0) = \int_0^t (e^{-sL}(0, g(s))\mathrm{d}s \ .$$

Thus,

$$\mathbf{w}(t) = e^{tL}\mathbf{w}_0 + e^{tL}\left(\int_0^t (e^{-sL}(0, g(s))\mathrm{d}s\right) \ .$$

We can simplify the right hand side: Since the variable of integration is s and not t, e^{tL} is a constant matrix as far as the integration is concerned, and so we may bring it under the integral sign, and then use the fact that $e^{tL}e^{-sL} = e^{(t-s)L}$ to obtain

$$\mathbf{w}(t) = e^{tL}\mathbf{w}_0 + \int_0^t e^{(t-s)L}(0, g(s))\mathrm{d}s \ . \tag{2.6}$$

Recall that we real want to know $w(t)$, from which we can recover the rest of $\mathbf{w}(t) = (w(t), v(t))$ by differentiation in any case. Using the explicit from of e^{tL} that we have computed earlier, we find

$$w(t) = a\cos(\sqrt{\kappa}t) - \frac{b}{\sqrt{\kappa}}\sin(\sqrt{\kappa}t) + \frac{1}{\sqrt{\kappa}}\int_0^t \sin(\sqrt{\kappa}(t-s))g(s)\mathrm{d}s \ . \tag{2.7}$$

This formula is as far as we can go without specifying the forcing function $g(t)$.

2.1 Periodic forcing

Now suppose that

$$g(t) = \alpha\cos(\omega t + \phi_0) \tag{2.8}$$

where α is the *amplitude*, ω is the *frequency* and ϕ_0 is the *phase shift*.

Recall that $\sin(A+B) = sin(A)\cos(B) + \sin(B)\cos(A)$. From this we obtain

$$\sin(A)\cos(B) = \frac{1}{2}[\sin(A+B) + \sin(A-B)] \ .$$

Thus,

$$\sin(\sqrt{\kappa}(t-s))\cos(\omega s + \phi_0) = \frac{1}{2}[\sin(\sqrt{\kappa}t + \phi_0 - (\sqrt{\kappa} - \omega)s) + \sin(\sqrt{\kappa}t - \phi_0 - (\sqrt{\kappa} + \omega)s]$$

We can now easily do the integration:

$$\begin{aligned}\int_0^t \sin(\sqrt{\kappa}(t-s))\cos(\omega s + \phi_0)\mathrm{d}s &= \frac{1}{2}\frac{1}{\sqrt{\kappa} - \omega}[\cos(\omega t + \phi_0) - \cos(\sqrt{\kappa}t + \phi_0)] \\ &+ \frac{1}{2}\frac{1}{\sqrt{\kappa} + \omega}[\cos(\omega t + \phi_0) - \cos(-\sqrt{\kappa}t + \phi_0)] \ ,\end{aligned}$$

where we assume for the time being that $\omega \neq \pm\sqrt{\kappa}$. (We shall remove this condition below.) To simplify the result, define

$$\eta := \frac{\sqrt{\kappa}+\omega}{2} \qquad \text{and} \qquad \xi := \frac{\sqrt{\kappa}-\omega}{2} \ .$$

Then

$$\begin{aligned} \cos(\omega t + \phi_0) - \cos(\sqrt{\kappa}t + \phi_0) &= \cos([\eta t + \phi_0] - \xi t) - \cos([\eta t + \phi_0] + \xi t) \\ &= 2\sin(\eta t + \phi_0)\sin(\xi t) \end{aligned}$$

where we have used

$$\cos(A - B) - \cos(A + B) = 2\sin(A)\sin(B) \ .$$

Therefore,

$$\frac{1}{2}\frac{1}{\sqrt{\kappa}-\omega}[\cos(\omega t + \phi_0) - \cos(\sqrt{\kappa}t + \phi_0)] = 2\sin(\eta t + \phi_0)\frac{\sin(\xi t)}{\xi} \ .$$

Likewise,

$$\begin{aligned} \cos(\omega t + \phi_0) - \cos(-\sqrt{\kappa}t + \phi_0) &= \cos([\xi t + \phi_0] - \eta t) - \cos([\xi t + \phi_0] + \eta t) \\ &= 2\sin(\phi_0 - \xi t)\sin(\eta t) \end{aligned}$$

Therefore,

$$\frac{1}{2}\frac{1}{\sqrt{\kappa}+\omega}[\cos(\omega t + \phi_0) - \cos(-\sqrt{\kappa}t + \phi_0)] = 2\sin(\phi_0 - \xi t)\frac{\sin(\eta t)}{\eta} \ .$$

Altogether, from these calculations, (2.7) and (2.8)

$$\begin{aligned} w(t) &= a\cos(\sqrt{\kappa}t) - \frac{b}{\sqrt{\kappa}}\sin(\sqrt{\kappa}t) \\ &+ \frac{2\alpha}{\sqrt{\kappa}}\left[\sin(\phi_0 - \xi t)\frac{\sin(\eta t)}{\eta} + \sin(\eta t + \phi_0)\frac{\sin(\xi t)}{\xi}\right] \end{aligned} \tag{2.9}$$

2.2 Beats and resonance

Now let us consider the solution formula (2.9) in the case in which ω is very close to $\sqrt{\kappa}$. Then ξ will be very small, and so

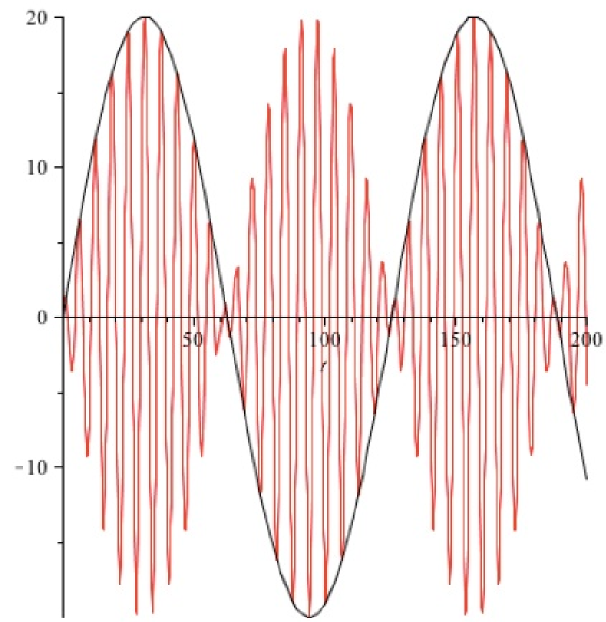
$$\sin(\eta t + \phi_0)\frac{\sin(\xi t)}{\xi}$$

becomes large.

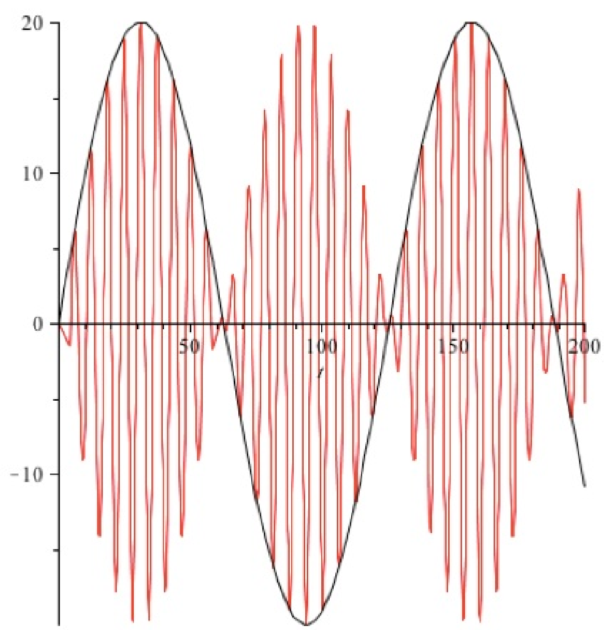
Let us look at some plots of $w(t)$ where we take $a = b = 0$, $\phi_0 = \pi/2$, and $\alpha = \sqrt{\kappa}/2$ in (2.9) so that

$$w(t) = \cos(\xi t)\frac{\sin(\eta t)}{\eta} + \cos(\eta t)\frac{\sin(\xi t)}{\xi} \ .$$

The next plot shows $w(t)$ for $\xi = 1/20$ and $\eta = 1$, for $0 \leq t \leq 200$:

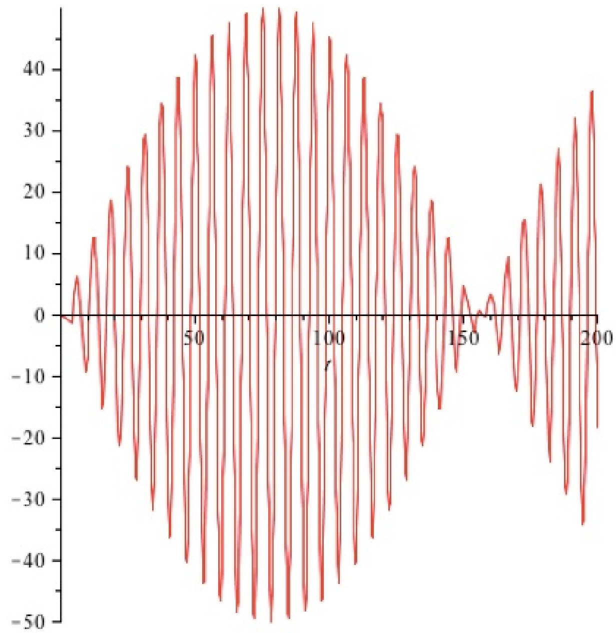


Also shown in the plot is the "envelope curve" $\dfrac{1}{20}\sin\left(\dfrac{1}{20}t\right)$. The second term in $w(t)$ dominates the first one, and if we leave the latter out, the plot is essentially unchanged:

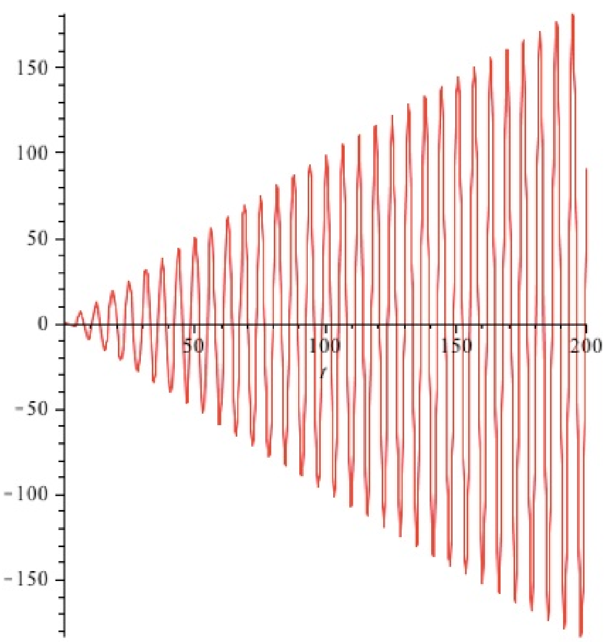


This is an illustration of the phenomenon known as *beats*: The beats are a high-frequency (rapid) oscillation, modulated by a low frequency amplitude oscillation. If you listened to this pattern of vibrations played with the fast oscilations played in the audible range, you would hear this recency at a sinusoidally varying volume, with the volume modulations at the longer period. The volume modulations are called beats.

If we lower ξ further, both the amplitude and period of the beats increases. Here is the plot for $\eta = 1$ and $\xi = 1/50$:



Here is the plot for $\eta = 1$ and $\xi = 1/300$:



What we see in the last plot is very close to *resonance*. Resonance occurs when the driving frequency ω (or $-\omega$) is exactly equal to $\sqrt{\kappa}$. As ω approaches $\sqrt{\kappa}$, ξ approaches zero and η approaches $\sqrt{\kappa}$. Thus, by l'Hospital's rule,

$$\lim_{\omega \to \sqrt{\kappa}} [\cos(\xi t)\frac{\sin(\eta t)}{\eta} + \cos(\eta t)\frac{\sin(\xi t)}{\xi}] = \frac{\sin(\sqrt{\kappa} t)}{\sqrt{\kappa}} + t\cos(\sqrt{\kappa} t) \ .$$

The amplitude of the oscillations now grows linearly, without bound.

In an actual mechanical system, once the oscillation become large enough, some sot of breakdown will occur. For example, if you take a wine glass, and tap it with a fork, say, you will hear a "ping". The frequency of this ping is the *natural frequency* $\sqrt{\kappa}$ of the wine glass. If you have perfect pitch, you would know what this is.

If you now turned on an oscillator producing sound at exactly (or nearly exactly) this frequency, the glass would begin to vibrate with larger and larger oscillations, and eventually shatter.

Suspension bridge failures have been caused by resonance. The first example on record occurred with the Broughton Suspension Bridge that was built in 1826 across the River Irwell near Manchester, England. This was among the first suspension bridges built anywhere in Europe. On April 12, 1831, troops of the $60th$ rifle Corps were marching across the bridge four abreast and marching in step – and unfortunately, the frequency of their march was one of the natural frequencies of the bridge. Resonance occurred, the oscillations built up – which the soldiers found amusing – and then, spoiling the amusement, the bridge collapsed, dumping 40 men into the river. Since this event, soldiers everywhere break cadence when crossing bridges.

A similar collapse occurred with the Tacoma Narrows Bridge November 7, 1940. In this case, wind caused some of the cables to vibrate at a natural frequency of certain twisting motions of the bridge. The resonance produced larger and larger twists, eventually destroying the bridge.

11 Example. Let $K = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$ and let $\mathbf{f}_0 = (-2, 1)$. Consider the differential equation

$$\mathbf{y}''(t) = -K\mathbf{y}(t) + \cos(\omega t)\mathbf{f}_0 \ ,$$

with

$$\mathbf{y}(0) = \mathbf{0} \qquad \text{and} \qquad \mathbf{y}'(0) = (1.2) \ .$$

Let us find the solution for all values of ω and see if there are any values of ω at which resonance occurs.

This is the same matrix we have treated in our example on normal modes, so we already have the normal mode decomposition: Let

$$\mathbf{u}_1 = \frac{1}{\sqrt{5}}(-2, 1) \qquad \text{and} \qquad \mathbf{u}_2 = \frac{1}{\sqrt{5}}(1, 2) \ .$$

Then $K\mathbf{u}_1 = \mathbf{u}_1$ and $K\mathbf{u}_2 = 6\mathbf{u}_2$.

We now compute the forcing terms for each of the normal modes. For the first mode

$$\mathbf{f}(t) \cdot \mathbf{u}_1 = \sqrt{5}\cos(\omega t) \qquad \text{and} \qquad \mathbf{f}(t) \cdot \mathbf{u}_2 = 0 \ .$$

Thus, the driven normal mode equation we must solve are

$$w_1''(t) = -w_1(t) + \sqrt{5}\cos(\omega t) \qquad \text{with} \qquad w_1(0) = w_1'(0) = 0$$

and

$$w_2''(t) = -6w_1(t) + \sqrt{5}\cos(\omega t) \qquad \text{with} \qquad w_1(0) = w_1'(0) = \sqrt{5}$$

These are now easily solved by the formulas we have derived. Notice that there is resonance only for $|\omega| = 1$; the driving force "does not couple" to the mode with the natural frequency $\sqrt{6}$.