Challenge Problem Set for Math 292, April 25, 2013

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This challenge problem set is about the Calculus of Variations and *Minimial Surfaces*. Given two points (x_1, y_1) and (x_2, y_2) with $x_2 > x_1$ and $y_1, y_2 > 0$, Let $\mathcal{K}$ denote the set of continuously differentiable functions y on $[x_1, x_2]$ such that $y(x_1) = y_1$, $y(x_2) = y_2$, and $y(x) > 0$ for all $x_1 \leq x \leq x_2$.

Now consider the surface obtained by rotating the curve $y = y(x)$ about the x-axis for $x_1 \leq x \leq x_2$. Call this surface of revolution $\mathcal{S}_y$. Then the surface area of $\mathcal{S}_y$ equals $I[y]$ where

$$I[y] := 2\pi \int_{x_1}^{x_2} y\sqrt{1 + (y')^2}\mathrm{d}x \ .$$

For now, let us fix

$$(x_1, y_1) = (0, R) \qquad \text{and} \qquad (x_2, y_2) = (L, R) \tag{0.1}$$

where $R, L > 0$.

(1.) Consider the curves $y_n(x)$ where

$$y(x) = \frac{R}{L^{2n}}(2x - L)^{2n}$$

Note that for our chosen endpoints, $y \in \mathcal{K}$ for each non-negative integer n.

Compute $I[y_0]$, and compute $\lim_{n\to\infty} I[y_n]$. Hint: to do this, you do not need to compute $I[y_n]$ as a function of n, which would be very messy. Instead show that the surface of revolution produced by rotating y_n about the x-axis is for large n, essentially two disks of radius R at $x = 0$ and $x = L$, perpendicular to the x-axis, and connected by a very narrow "neck", almost a line. You can then figure out the limiting area of this.

Also, compute $I[y_1]$, which involves rotating a parabola. Which of the example you computed does best?

Now let us try to find the optimal curve by solving the corresponding Euler-Lagrange equation. Since the variable x is missing from $f(x, y, z) = 2\pi y\sqrt{1 + z^2}$, the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial}{\partial y'}f(x, y, y')\right) - \frac{\partial}{\partial y}f(x, y, y') = 0$$

reduces to

$$\frac{\partial}{\partial y'} f(x, y, y') y' - f = c_1 = \text{constant} \ .$$

(2.) Show that the Euler-Lagrange equation reduces to

$$c_1 y' = \sqrt{y^2 - c_1^2} \ .$$

(3.) Separate variables to deduce that

$$x(y) = c_1 \ln\left(\frac{y + \sqrt{y^2 - c_1^2}}{c_1}\right) + c_2 \ .$$

(4.) Solve for $y(x)$ to deduce

$$y(x) = c_1 \cosh\left(\frac{x - c_2}{c_1}\right) \ .$$

(5.) The equations determining c_1 and c_2 are

$$\frac{R}{c_1} = \cosh\left(\frac{-c_2}{c_1}\right)$$

and

$$\frac{R}{c_1} = \cosh\left(\frac{L - c_2}{c_1}\right) \ .$$

Show that $c_2 = L/2$, and if we define

$$a = \frac{L}{2c_1} \ ,$$

then

$$\frac{2R}{L} = \frac{\cosh(a)}{a} \ .$$

Show that if R/L is too small, this equation has no solution, and that there is a unique value of R/L where it has exactly one solution, and for all larger values of R/L, is has exactly two solutions.

(6.) For $R = 2L$, the two solutions of $4a = \cosh(a)$ are approximately $a = 0.258$ and $a = 3.259$. Show that both values of a give a curve $y \in \mathcal{K}$, and compute to see which one is the best of the two. (You may use Maple, Mathematica, or a calculator to do numerical integrals.)

(7.) Show that if R/L is sufficiently small, the problem has no minimizer in $\mathcal{K}$. Can you find the greatest lower bound to $I[y]$ for $y \in \mathcal{K}$ is this situation?