Challenge Problem Set for Math 292, April 11, 2013

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This challenge problem set is about using symmetry — especially scaling symmetry – to find changes of variables that make it possible to solve differential equations, at least in implicit form.

Consider the differential equation

$$y'(x) = f(x,y) \ . \tag{0.1}$$

The solutions of this equation describe curves $y = y(x)$ in the x, y plane.

One way to "solve" this equation is to find a function $F(x,y)$ on the x,y plane such that:

(1) The function F is constant on the curves $(x, y(x))$ where $y(x)$ is a solution of (0.1), and

(2) $\nabla F(x,y) \neq (0,0)$ for any (x,y) so that the equation $F(x,y) = F(x_0,y_0)$ defines, via the Implicit Function Theorem, a curve

$$y = y(x) \qquad \text{with} \qquad y(x_0) = y_0 \qquad \text{and} \qquad y'(x) = f(x, y(x))$$

for all x on some interval $(x_0 - a, x_0 + a)$.

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(1) Let F be continuously differential on an open set $U \subset \mathbb{R}^2$. Suppose that $\nabla F(x,y) \neq (0,0)$ for any $(x,y) \in U$, so that there is a unique level curve of F passing through each point in U.

Suppose that $\nabla F(x,y)$ is orthogonal to $(1, f(x,y))$ at each (x,y). Show that a function $y = y(x)$, defined for x such that $(x, y(x)) \in U$, satisfies $y'(x) = f(x, y(x))$ if and only if

$$\frac{\mathrm{d}}{\mathrm{d}t} F(t, y(t)) = 0 \ . \tag{0.2}$$

Conversely, show that if (0.2) is true for each solution of $y = y(x)$, then $\nabla F(x,y)$ is orthogonal to $(1, f(x,y))$ at each (x,y).

(2) Now let $h(x,y) = (u(x,y), v(x,y))$ be a continuously differentiable function from $U \subset \mathbb{R}^2$ to $V \subset \mathbb{R}^2$ that is one to one and onto, and hence invertible. Define

$$G(u,v) = F \circ h^{-1}(u,v) \ .$$

Let $\mathbf{x}(t) = (x(t), y(t))$ be a continuously differentiable curve in in $\mathbb{R}^2$. Let $\mathbf{u}(t) = h(\mathbf{x}(t)) = (u(t), v(t))$. That is,

$$(u(t), v(t)) = h(x(t), y(t)) \ .$$

Show that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}G(\mathbf{u}(t)) &= \left[\nabla F(h^{-1}(\mathbf{u}(t))\right] \cdot \frac{\mathrm{d}}{\mathrm{d}t}h^{-1}(\mathbf{u}(t)) \\ &= \left[\nabla F(h^{-1}(\mathbf{u}(t))\right] \cdot [J_{h^{-1}}(\mathbf{u}(t))]\,\mathbf{u}'(t)\ .\end{aligned}$$

Of course, we also have

$$\frac{\mathrm{d}}{\mathrm{d}t}G(\mathbf{u}(t)) = \nabla G(\mathbf{u}(t)) \cdot \mathbf{u}'(t)\ .$$

Put this together to deduce that transformation rule for gradients under a coordinate change:

$$\nabla G(u,v) = [J_{h^{-1}}(u,v)]^t\,\nabla F(h^{-1}(u,v))\ .$$

where t denote the transpose.

We now seek to choose the coordinate change h so that the level curves of the transformed function $G = F \circ h^{-1}$ satisfy an *autonomous* first order equation

$$v'(u) = w(v(u))\ , \tag{0.3}$$

for some function $w(v)$. This is helpful, because we know how to solve such equations.

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(3) Recall the fundamental formula for solving (0.3): We can write they verse function $u(v)$ in the form

$$u(v) - u_0 = \int_{v_0}^{v} \frac{1}{w(z)}\mathrm{d}z\ .$$

Define a function $G(u,v)$ by

$$G(u,v) = u - \int_{v_0}^{v} \frac{1}{w(z)}\mathrm{d}z\ .$$

Show that the level curves of G are integral curves of (0.3), and in particular that

$$\nabla G(u,v) \cdot (1, w(v)) = 0\ . \tag{0.4}$$

Show that the solution of (0.3) with $v(u_0) = v_0$ lies on the curve $G(u,v) = u_0$.

Thus, we may not be able to do the algebra to invert $u(v)$ and find v as a function of u, but we can always, at least in integral form, find a function $G(u,v)$ wholes level curves are the solutions of the autonomous equation $v' = w(v)$.

We now seek, by change of variables, to do the same for certain non-autonomous equations.

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(4) Let $v(u)$ be a solution of the autonomous equation (0.3), and define a curve $\mathbf{x}(t)$ by

$$\mathbf{x}(t) = h^{-1}(t, v(t)) \tag{0.5}$$

and a function

$$F(x,y) = G \circ h(x,y) \tag{0.6}$$

where G is determined by the autonomous equation as above.

Show that for a solution $y(x)$ of $y' = f(x,y)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} F(t, y(t)) = 0$$

if and only if

$$[J_{h^{-1}}(u,v)]\,(1, w(v)) = a(u,v)(1, f(h^{-1}(u,v)))$$

for some non-zero function $a(u,v)$, and therefore, if we can find a change of variables h such that

$$[J_{h^{-1}}(u,v)]^{-1}\,(1, f(h^{-1}(u,v))) = \frac{1}{a(u,v)}(1, w(v)) \ ,$$

the curve (0.5), written in the form $y = y(x)$, will be a solution of $y' = f(x,y)$.

In class we have seen how when $f(x,y)$ has the form

$$f(x,y) = \sum_{j=1}^{m} a_j x^{p_j} y^{q_j}$$

with power, which need not be integer, such that for some α and β

$$\alpha p_j + \beta q_j = \beta - \alpha$$

for all j, then the change of variables

$$h(x,y) = \left(\frac{1}{\alpha} \ln x \ , \ x^{-\beta/\alpha} y \right) \ ,$$

will have the desires property, and have seen how this property follows from *generalized scale symmetry* of the equation $y' = f(x,y)$.

(5) Apply this method to find a function $F(x,y)$ whose level curves satisfy

$$y' = y^2 + xy^3 \ .$$