Challenge Problem Set for Math 292, March 14, 2013

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Let $K = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$. The object of this challenge problem set is to find and study the solution of

$$\mathbf{x}''(t) = -K\mathbf{x}(t) + \mathbf{f}(t) \qquad \text{with} \quad \mathbf{x}(0) = (1,2) \quad \text{and} \quad \mathbf{x}'(0) = (1,1)\ , \qquad (*)$$

where

$$\mathbf{f}(t) = \sum_{j=1}^{4} \mathbf{f}_j(t)$$

and

$$\begin{aligned} \mathbf{f}_1(t) &= (1,3)\cos(\omega_1 t) \\ \mathbf{f}_2(t) &= (1,-1)\sin(\omega_2 t) \\ \mathbf{f}_3(t) &= (3,-1)\sin(\omega_3 t) \\ \mathbf{f}_4(t) &= (1,0)\cos(\omega_4 t) \end{aligned}$$

1: Solve the homogeneous system

$$\mathbf{x}''(t) = -K\mathbf{x}(t) \qquad \text{with} \quad \mathbf{x}(0) = (1,2) \quad \text{and} \quad \mathbf{x}'(0) = (1,1)\ .$$

2: The next part is to be done in parallel:

(a) Task for team 1: Solve the inhomogeneous system

$$\mathbf{x}''(t) = -K\mathbf{x}(t) + \mathbf{f_4}(t) \qquad \text{with} \quad \mathbf{x}(0) = (0,0) \quad \text{and} \quad \mathbf{x}'(0) = (0,0)\ .$$

(a) Task for team 2: Solve the inhomogeneous system

$$\mathbf{x}''(t) = -K\mathbf{x}(t) + \mathbf{f_4}(t) \qquad \text{with} \quad \mathbf{x}(0) = (0,0) \quad \text{and} \quad \mathbf{x}'(0) = (0,0)\ .$$

(a) Task for team 3: Solve the inhomogeneous system

$$\mathbf{x}''(t) = -K\mathbf{x}(t) + \mathbf{f_4}(t) \qquad \text{with} \quad \mathbf{x}(0) = (0,0) \quad \text{and} \quad \mathbf{x}'(0) = (0,0)\ .$$

(a) Task for team 4: Solve the inhomogeneous system

$$\mathbf{x}''(t) = -K\mathbf{x}(t) + \mathbf{f_4}(t) \qquad \text{with} \quad \mathbf{x}(0) = (0,0) \quad \text{and} \quad \mathbf{x}'(0) = (0,0) \ .$$

3: Put the results together, and write down the solutions to $(*)$. For which values of ω_1, ω_2, ω_3 and ω_4 does the system exhibit resonance?

4: Let $\omega_1 = 1$, $\omega_2 = 1.65$, $\omega_3 = 2.65$ and $\omega_4 = 3.75$. Then one of the five functions that were added up to obtain the solution is much larger (for most t) than the others, and the solution is well approximated by keeping only this main term. What is the main term that gives this approximate solution?

5: Now we examine the effects of friction. Let us consider the dingle variable equation

$$x''(t) + \gamma x'(t) + \omega_0^2 x(t) = F_0 \cos(\omega t)$$

where γ, ω and ω_0 are positive constants.

Show that

$$x_p(t) = \frac{F_0}{\sqrt{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2}} \cos(\omega t - \delta)$$

where

$$\tan(\delta) = \frac{\gamma\omega}{\omega_0^2 - \omega^2}$$

is a particular solution. You can either verify this, or derive it using the variation of constants formula. For that you need the homogeneous solutions, as we've done in class, but you need those next anyhow.

Next, show that for *any* solution $x(t)$,

$$\lim_{t\to\infty} |x(t) - x_p(t)| = 0 \ .$$

To do this note that $x(t) - x_p(t)$ solves the homogeneous equation, so it is sufficient to show that every solution of that converges to zero as t tends to infinity. Thus, $x_p(t)$ describes the *steady sate.*

The amplitude of the steady state oscillation is

$$A(\omega) = \frac{F_0}{\sqrt{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2}} \ .$$

Sketch a plot of the graph of $A(\omega)$, and show that is has a maximum at

$$\omega_{\max} = \omega_0 \sqrt{1 = \frac{\gamma^2}{2\omega_0^2}} \ .$$

Thus increasing friction *lowers* the frequency at which the steady state oscillations have the largest amplitudes.