Challenge Problem Set 2, Math 292 Fall 2013

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This problems set concerns differentiability for the flow transformation associated to a non-autonomous but uniformly Lifschitz equations in one variables.

That is we consider the equation

$$x'(t) = v(t, x(t)) \qquad \text{with} \qquad x(0) = x_0 \tag{0.1}$$

(Here we are using x for the dependent variable, and t for the independent variable, as opposed to y for the dependent variable and x for the independent variable. That is, we solve for $x(t)$ instead of $y(x)$ as in Simmons.)

We assume that for some finite constant L,

$$|v(t,x) - v(t,y)| \leq L|x-y| \qquad \text{for all} \qquad x, y, t \in \mathbb{R}\ . \tag{0.2}$$

In fact, we shall assume the slightly stronger condition

$$\left|\frac{\partial}{\partial x} v(t,x)\right| \leq L \qquad \text{for all} \qquad x, t \in \mathbb{R}\ . \tag{0.3}$$

Let $x(t, x_0)$ be the solution of (0.1). We want to show that this is differentiable as a function of x_0, and to compute the derivative.

We have seen that

$$x(t, x_0) = \lim_{n\to\infty} x_n(t, x_0) \tag{0.4}$$

where

$$x_0(t, x_0) = x_0 \qquad \text{and} \qquad x_n(t, x_0) = x_0 + \int_0^t v(s, x_{n-1}(s, x_0))\mathrm{d}s \quad \text{for} \quad n \geq 1\ . \tag{0.5}$$

(1) Show by induction that for each n and t, $x_n(t, x_0)$ is continuously differentiable as a function of x_0, and if we define

$$z_n(t, x_0) = \frac{\partial}{\partial x_0} x_n(t, x_0)\ , \tag{0.6}$$

we have

$$z_n(t, x_0) = 1 + \int_0^t \frac{\partial}{\partial x} v(t, x_{n-1}(s, x_0)) z_{n-1}(s, x_0)\mathrm{d}s\ . \tag{0.7}$$

(2) Based on what we have seen, it is natural to guess that

$$\lim_{n\to\infty} z_n(t,x_0) = z(t,x_0)$$

exists, and that the convergence is uniform on any bounded interval $[0,T]$.

Assuming this, which is true, show that

$$z(t,x_0) = 1 + \int_0^t \frac{\partial}{\partial x} v(s,x(s,x_0))z(s,x_0)\mathrm{d}s \ . \tag{0.8}$$

(3) Building on part **(2)**, define

$$a(t,x_0) := \frac{\partial}{\partial x} v(t,x(t,x_0)) \ .$$

Show that

$$z(t,x_0) = \exp\left(\int_0^t a(s,x(s,x_0))\mathrm{d}s\right) \ ,$$

where $x(t,x_0)$ solves(0.1).

(4) Show that if v is autonomous; i.e., independent of t, so that

$$a(t,x_0) := \frac{\partial}{\partial x} v(x(t,x_0)) \ .$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln(v(x(t,x_0))) = a(t,x_0)) \ ,$$

and therefore that

$$z(t,x_0) = \frac{v(x(t,x_0))}{v(x_0)} \ ,$$

as we found before in the autonomous case.

(5) Let

$$v(t,x) = \sqrt{\frac{1+x^2}{1+t^2}} \ .$$

Compute the solution $x(t,x_0)$ of (0.1), and the, by direct computation, compute

$$z(t,z_0) = \frac{\partial}{\partial x_0} x(t,x_0) \ .$$

(6) Extra Credit Returning to **(2)**, show that

$$\lim_{n\to\infty} z_n(t,x_0) = z(t,x_0)$$

exists, and that the convergence is uniform on any bounded interval $[0,T]$.