# Challenge Problem Set 1, Math 292 Spring 2011

Eric A. Carlen[1]
Rutgers University

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## 1 Introduction

This challenge problem set concerns small amplitude oscillations of the spherical pendulum. The spherical pendulum consists of a mass moving on the surface of a sphere under the influence of gravity. For example, think of a steel cable of length $\ell$ with one end fixed at the point $(0,0,0)$ in three dimensional Euclidean space, and the other end attached to a bob of mass $m$ on it. Think of the cable as being perfectly flexible but inextensible and as having a mass that is negligible compared to the mass of the bob.

If this is set swinging, then at any given time the blob of mass will be at some point $\mathbf{x} = (x, y, z)$ on the centered sphere of radius $\ell$, given by

$$x^2 + y^2 + z^2 = \ell^2 \ ,$$

which is the equation of the sphere of radius $R$ centered at $(0,0,R)$.

We now derive the equations of motion. Let $\mathbf{x}(t)$ be a twice differentiable curve on the sphere of radius $\ell$. Let $\mathbf{v}(t) = \dot{\mathbf{x}}(t)$ and $\mathbf{a}(t) = \dot{\mathbf{v}}(t)$ be it velocity and acceleration respectively. The outward unit normal $\mathbf{n}$ to the sphere at any point $\mathbf{x}$ on the sphere is

$$\mathbf{n} = \frac{1}{\ell}\mathbf{x} \ .$$

In the calculations that follow, we suppress the $t$ in our notation for simplicity. Let us decompose the acceleration at time $t$ into its components parallel and orthogonal to $\mathbf{n}$:

$$\mathbf{a} = \mathbf{a}_\| + \mathbf{a}_\perp \qquad \text{where} \qquad \mathbf{a}_\| = (\mathbf{a}\cdot\mathbf{n})\mathbf{n} \ .$$

Then $\mathbf{a}_\|$ is the component of the acceleration that is normal to the sphere, and $\mathbf{a}_\perp$ is the component of the acceleration that is tangent to the sphere.

We now claim that for any twice differentiable curve of the sphere of radius $\ell$,

$$\mathbf{a}_\| = -\frac{\|\mathbf{v}\|^2}{\ell}\mathbf{n} = -\frac{\|\mathbf{v}\|^2}{\ell^2}\mathbf{x} \ . \tag{1.1}$$

To see this, note that since the curve remains on the sphere, $\|\mathbf{x}\|^2 = \ell^2$ for all $t$. Differentiating, we see that

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{x}\|^2 = 2\mathbf{x}\cdot\mathbf{v} \ . \tag{1.2}$$

Differentiating once more, we see that

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}\cdot\mathbf{v} = \|\mathbf{v}\|^2 + \mathbf{x}\cdot\mathbf{a}\ .$$

Remembering that $\mathbf{x} = \ell\mathbf{n}$, we se that (1.1) is true.

For the spherical pendulum, the tangential component of the acceleration is given by $1/m$ times the tangential component of the gravitational force, $\mathbf{F} = (0,0,-mg)$ where $g$ is the gravitational acceleration, which is approximately 9.8 $m/s^2$.

The tangential component of $\mathbf{F}$ is the component orthogonal to $\mathbf{n}$; i.e., $\mathbf{F}_\perp = \mathbf{F} - (\mathbf{F}\cdot\mathbf{n})\mathbf{n}$. We will consider small oscillations so that, in particular, the mass stays in the lower half of the sphere. The unit normal vector $\mathbf{n}$ to the surface of the sphere at any such point $(x,y,z)$ is

$$\mathbf{n} = \frac{1}{\ell}(x,y,z)\ .$$

Since on the lower half of the sphere, $z = -\sqrt{\ell^2 - x^2 - y^2}$, we can write $\mathbf{n}$ as a function of $x$ and $y$ alone:

$$\mathbf{n} = \frac{1}{\ell}(x,y,-\sqrt{\ell^2 - x^2 - y^2})\ .$$

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**Exercise 1** Do the computations to find that

$$\begin{aligned}\mathbf{a}_\perp = \frac{1}{m}\mathbf{F}_\perp = \ &= (0,0,-g) - \frac{g\sqrt{\ell^2 - x^2 - y^2}}{R^2}(x,y,-\sqrt{\ell^2 - x^2 - y^2}) \\ &= \frac{g}{\ell^2}(-x\sqrt{\ell^2 - x^2 - y^2},\ -y\sqrt{R - x^2 - y^2},\ -x^2 - y^2) \qquad (1.3)\end{aligned}$$

Then combine this with (1.1) and (1.3) to show that

$$\mathbf{a} = \mathbf{a}_\parallel + \mathbf{a}_\perp = -\frac{1}{\ell^2}\left(\|\mathbf{v}\|^2\mathbf{x} + g(x\sqrt{\ell^2 - x^2 - y^2},\ y\sqrt{R - x^2 - y^2},\ x^2 + y^2)\right)\ ,$$

which is

$$\begin{aligned}\ddot{x} &= -\frac{1}{\ell^2}\left(xg\sqrt{\ell^2 - x^2 - y^2} + x(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\right) \\ \ddot{y} &= -\frac{1}{\ell^2}\left(yg\sqrt{\ell^2 - x^2 - y^2} + y(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\right) \\ \ddot{z} &= -\frac{1}{\ell^2}\left(g(x^2 + y^2) + z(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\right)\ .\end{aligned}$$

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Assuming the oscillations are small enough that the bob stays below the equator, we only need the $x$ and $y$ coordinates to describe the position of the bob since then $z = -\sqrt{\ell^2 - x^2 - z^2}$. Thus the equation for $z$ is redundant.

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**Exercise 2** Use (1.2) to show that as long as $\mathbf{x}$ stays below the equator,

$$\dot{z} = \frac{x\dot{x} + y\dot{y}}{\sqrt{\ell^2 - x^2 - y^2}}\ .$$

Use this to show that as long as $\mathbf{x}$ stays below the equator,

$$\begin{aligned} \ddot{x} &= -\frac{1}{\ell^2}\left(xg\sqrt{\ell^2-x^2-y^2} + x\left(\frac{(\ell^2-y^2)\dot{x}^2+(\ell^2-x^2)\dot{y}^2+2xy\dot{x}\dot{y}}{\ell^2-x^2-y^2}\right)\right) \\ \ddot{y} &= -\frac{1}{\ell^2}\left(yg\sqrt{\ell^2-x^2-y^2} + y\left(\frac{(\ell^2-y^2)\dot{x}^2+(\ell^2-x^2)\dot{y}^2+2xy\dot{x}\dot{y}}{\ell^2-x^2-y^2}\right)\right) \end{aligned} \tag{1.4}$$

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We now make the small oscillations approximation, which is that not only does $\mathbf{x}$ stay below the equator, it stays near the "south pole". This means that $x$ and $y$ are small compared to $\ell$. Then we can expect that $\dot{x}$ and $\dot{y}$ will also be small, and we make the small angle approximation by discarding all terms that are quadratic or higher order in $x$, $y$, $\dots x$ and $\dot{y}$. Later we will investigate the validity of this approximation, but not let us proceed with it. The equations simplify dramatically. They become

$$\begin{aligned} \ddot{x} &= -\frac{g}{\ell}x \\ \ddot{y} &= -\frac{g}{\ell}y \end{aligned}$$

To further simplify this, let us introduce use units of distance and time in which $\ell = 1$ and $g = 1$. Then the equations of motion become

If $x$ and $y$ are small, then $\sqrt{1-x^2-y^2} \approx 1$, and these simplify to

$$\begin{aligned} \ddot{x} &= -x \\ \ddot{y} &= -y\ . \end{aligned} \tag{1.5}$$

Notice that while in the full system (1.4), $\ddot{x}$ and $\ddot{y}$ both depend on all of $x$, $y$, $\dot{x}$ and $\dot{y}$, things are much simpler in (1.5): Here $\ddot{x}$ depends on $x$ alone, and $\ddot{y}$ depends on $y$ alone. The two equations in the system "decouple".

As we have seen, the general solutions of $\ddot{x} = -x$ and $\ddot{y} = -y$ are given by

$$x(t) = x(0)\cos t + \dot{x}(0)\sin t \qquad \text{and} \qquad y(t) = y(0)\cos t + \dot{y}(0)\sin t\ . \tag{1.6}$$

Thus, it is easy to write down the explicit parametric form of the curve $(x(t), y(t))$ given the initial data $(x(0), y(0))$ together with $(\dot{x}(0), \dot{y}(0))$.

The curve in $\mathbb{R}^2$ parameterized by $(x(t), y(t))$ is always an ellipse, though possibly degenerate. Our first goal is to prove this.

**Exercise 3:** Given numbers $a_0$, $b_0$, $c_0$ and $d_0$, let

$$x(t) = a_0\cos t + b_0\sin t \qquad \text{and} \qquad y(t) = c_0\cos t + d_0\sin t\ .$$

Show that

$$\begin{aligned} 2x^2(t) &= M_{1,1} + M_{1.2}\cos(2t) + M_{1,3}\sin(2t) \\ 2y^2(t) &= M_{2,1} + M_{2.2}\cos(2t) + M_{2,3}\sin(2t) \\ 2x(t)y(t) &= M_{3,1} + M_{3.2}\cos(2t) + M_{3,3}\sin(2t) \end{aligned}$$

where the $M_{i,j}$ are the entries of the $3 \times 3$ matrix

$$M := \begin{bmatrix} a_0^2 + b_0^2 & a_0^2 - b_0^2 & 2a_0b_0 \\ c_0^2 + d_0^2 & c_0^2 - d_0^2 & 2c_0d_0 \\ a_0c_0 + b_0d_0 & a_0c_0 - b_0d_0 & a_0d_0 + b_0c_0 \end{bmatrix} .$$

Then show that for any numbers $A$, $B$ and $C$,

$$\begin{aligned} Ax^2(t) + By^2(t) + Cx(t)y(t) &= (A, B, C) \cdot (M_{1,1}, M_{2,1}, M_{3,1}) \\ &+ (A, B, C) \cdot (M_{1,2}, M_{2,2}, M_{3,2}) \cos(2t) \\ &+ (A, B, C) \cdot (M_{1,3}, M_{2,3}, M_{3,3}) \sin(2t) . \end{aligned}$$

Finally, noting that for any two vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$, we can find a *non-zero* vector $(A, B, C)$ such that $(A, B, C) \cdot \mathbf{v} = 0$ and $(A, B, C) \cdot \mathbf{w} = 0$, show that there is a non zero vector $(A, B, C)$ and a number $D$ such that

$$Ax^2(t) + By^2(t) + Cx(t)y(t) = D$$

holds for all $t$. Conclude that the curve traced out by $(x(t), y(t))$ with $x(t)$ and $y(t)$ given by (1.6), must be a conic section, and then, since the only bounded conic sections are ellipses, that the curve traced out by $(x(t), y(t))$ is an ellipse.

**Exercise 4:** For the particular initial data

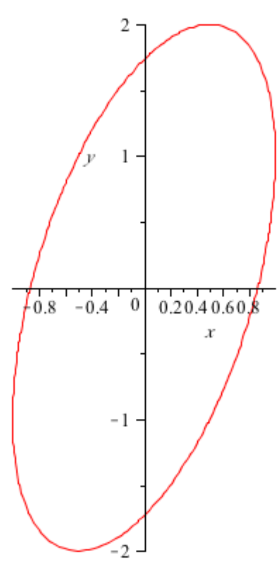
$$a_0 = 1 \qquad b_0 = 0 \qquad c_0 = 1 \quad \text{and} \quad d_0 = \sqrt{3} .$$

Show that the solution curve $(x(t), y(t))$ is given by

$$(x(t), y(t)) = (\cos t, \cos t + \sqrt{3} \sin t) .$$

Also, find an equation for this ellipse in the form $Ax^2 + By^2 + Cxy = D$.

Here is a plot of the ellipse:



## 1.1 The geometry of the phase curves

Now we shall study the geometry of the *phase curves* for the small oscillation spherical pendulum. As we have seen in the previous section, once one knows the vector $(x(0), \dot{x}(0), y(0), \dot{y}(0))$, one knows (after doing some computations) $(x(t), y(t))$ for all $t$, and hence one even knows $(x(t), \dot{x}(t), y(t), \dot{y}(t)$. Hence the phase space for the spherical pendulum is four dimensional. Let us now study the geometry of the phase curves in this four dimensional phase space.

The first order of business is to recast the system (1.5) of second order equations as a system of first order equations. To do this we introduce new variables $a$, $b$, $c$ and $d$:

$$a := x \qquad b := \dot{x} \qquad c := y \quad \text{and} \quad d := \dot{y} \ .$$

Then the system (1.5) is equivalently expressed by

$$\dot{a} = b \qquad \dot{b} = -a \qquad \dot{c} = d \quad \text{and} \quad \dot{d} = -c \ . \tag{1.7}$$

That is:

$$\frac{\mathrm{d}}{\mathrm{d}t}(a, b, c, d) = (b, -a, d, -c) \ . \tag{1.8}$$

This is a first order differential equation in a four dimensional phase space.

In the exercises that follow, Let $(a(t), b(t), c(t), d(t))$ be any solution of (1.7) with $(a(0), b(0), c(0), d(0)) = (a_0, b_0, c_0, d_0)$. It turns out that there are many simple constants of the motion; i.e., functions of $(a, b, c, d)$ that are constant along the phase curves. This will enable to identify the phase curves.

**Exercise 5:** Show that for all $t$,

$$\begin{aligned} a^2(t) + b^2(t) &= a_0^2 + b_0^2 \\ c^2(t) + d^2(t) &= c_0^2 + d_0^2 \\ a(t)c(t) + b(t)d(t) &= a_0 c_0 + b_0 d_0 \ . \end{aligned} \tag{1.9}$$

In particular, for all $t$, $(a(t), b(t), c(t), d(t))$ lies on the surface of the three dimensional sphere of radius $r$ in $\mathbb{R}^4$ given by

$$a^2 + b^2 + c^2 + d^2 = r^2$$

where

$$r := \sqrt{a_0^2 + b_0^2 + c_0^2 + d_0^2} \ . \tag{1.10}$$

**Exercise 6:** Going forward, let us suppose that

$$r_1 := \sqrt{a_0^2 + b_0^2} > 0 \qquad \text{and} \qquad r_2 := \sqrt{c_0^2 + d_0^2} > 0 \tag{1.11}$$

since if either of these quantities vanish, the motion takes place in a plane, and we already understand the small oscillations of a planar pendulum.

Define two time dependent vectors in $\mathbb{R}^2$ by

$$\mathbf{v}(t) = (a(t), b(t)) \qquad \text{and} \qquad \mathbf{w}(t) = (c(t), d(t)) \ .$$

Use the results of Exercise 3 to show that the angle between $\mathbf{v}(t)$ and $\mathbf{w}(t)$ does not depend on $t$. Geometrically, this is because the motion in the $a, b$ plane and the motion in the $c, d$ plane are both circular motion with the *same* frequency; the two motions "stay in phase".

**Exercise 7:** Show that there is some fixed angle $\theta$ so that for all $t$,

$$r_2\mathbf{v}(t) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} r_1\mathbf{w}(t) \ ,$$

where $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is the matrix representing counter-clockwise rotation through the angle $\theta$ in $\mathbb{R}^2$. Then use this to show that for all $t$,

$$r_2a(t) - [r_1\cos\theta]\, c(t) + [r_1\sin\theta]\, d(t) = 0$$

and

$$r_2b(t) - [r_1\sin\theta]\, c(t) - [r_1\cos\theta]\, d(t) = 0$$

**Exercise 8:** Define the vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ in $\mathbb{R}^4$ by

$$\begin{aligned} \mathbf{u}_1 &= \frac{1}{r}(r_2, 0, -r_1\cos\theta, +r_1\sin\theta) \\ \mathbf{u}_2 &= \frac{1}{r}(0, r_2, -r_1\sin\theta, -r_1\cos\theta) \\ \mathbf{u}_3 &= \frac{1}{r}(-r_1, 0, -r_2\cos\theta, -r_2\sin\theta) \\ \mathbf{u}_4 &= \frac{1}{r}(0, -r_1, -r_2\sin\theta, -r_2\cos\theta) \ . \end{aligned} \tag{1.12}$$

Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is orthonormal. Show also that for all $t$,

$$\mathbf{u}_1 \cdot (a(t), b(t), c(t), d(t)) = 0 \qquad \text{and} \qquad \mathbf{u}_2 \cdot (a(t), b(t), c(t), d(t)) = 0 \ .$$

Use this to show that

$$(a(t), b(t), c(t), d(t)) = \alpha(t)\mathbf{u}_3 + \beta(t)\mathbf{u}_4$$

where

$$\alpha(t) := \mathbf{u}_3 \cdot (a(t), b(t), c(t), d(t)) \qquad \text{and} \qquad \beta(t) := \mathbf{u}_4 \cdot (a(t), b(t), c(t), d(t)) \ , \tag{1.13}$$

and that $\alpha^2(t) + \beta^2(t) = r^2$ for all $t$.

This shows in particular that the phase curve is a circle produced by slicing the sphere of radius $r$ in $\mathbb{R}^4$ by a two dimensional plane through the origin; this is a so-called "great circle".

**Exercise 9:** Show that

$$\dot\alpha = \beta \qquad \text{and} \qquad \dot\beta = -\alpha \ ,$$

and consequently that

$$\begin{aligned} \alpha(t) &= \alpha(0)\cos t + \beta(0)\sin t \\ \beta(t) &= -\alpha(0)\sin t + \beta(0)\cos t \ . \end{aligned} \tag{1.14}$$

**Exercise 10:** Combining (1.12), (1.13) and (1.14), we can explicitly solve for the phase curve $(a(t), b(t), c(t), d(t))$ for given initial data $(a_0, b_0, c_0, d_0)$. Do this for

$$a_0 = 1 \qquad b_0 = 0 \qquad c_0 = 1 \quad \text{and} \quad d_0 = \sqrt{3}\ .$$

Check that $(a(t), c(t)) = (x(t), y(t))$ agrees wtih the answer found in Exercise 4.

**Exercise 11:** Prove the following theorem, which is a purely geometric summary of the analysis made in Exercises 3 through 8:

**1.1 THEOREM.** *Given any two non-negative numbers $r_1$ and $r_2$ with $r_1^2 + r_2^2 = 1$, and any number $s$ with $-r_1 r_2 \le s \le r_1 r_2$, the set of points $(a, b, c, d) \in \mathbb{R}^4$ such that*

$$\begin{aligned} a^2 + b^2 &= r_1 \\ c^2 + d^2 &= r_2 \\ ac + bd &= s \end{aligned}$$

*is a great circle on $S^3$, the unit sphere in $\mathbb{R}^4$.*

We can now draw some interesting geometric conclusions. First of all Given any point $(a_0, b_0, c_0, d_0)$ on the unit sphere in $\mathbb{R}^4$, there is a unique phase curve through this point, and as we have seen, that phase curve is a great circle. Now, two phase curves cannot intersect: Through each point in the phase space there is exactly one phase curve. Hence $S^3$ is the *disjoint union* of these great circles.

This is certainly not the case for $S^2$, the unit sphere in $\mathbb{R}^3$: Take out any great circle – the equation, for example –and what remains is two disconected pieces, neither of which contains any great circles at all.

The great circles that arise a phase curves for the small-oscillation spherical pendulum are rather special; not every great circle on $S^3$ is such a phase curve. Indeed, there are infinitely many great circles passing through any point on $\mathbf{p} \in S^3$. Let $\mathbf{q}$ be any unit vector orthogonal to $\mathbf{p}$, and consider the two dimensional plane spanned by $\mathbf{p}$ and $\mathbf{q}$. This slices $S^3$ in a great circle that passes though $\mathbf{p}$. In fact, it is not hard to see that all great circles passing through $\mathbf{p}$ arise this way. However, there is only one phase curve passing though any point in phase space, and so exactly one of these will be a phase curve.

The next section continues the investigation of the geometry of $S^3$ via differential equations.

## 2 The Hopf Fibration

The key to our analysis of the phase curves for spherical pendulum in the small oscillations approximation was provided by the constants of the motion we found in Exercise 3. Let us rephrase our conclusions form Exercise 3 as follows: Define the functions

$$\begin{aligned} f_1(a, b, c, d) &= a^2 + b^2 \\ f_2(a, b, c, d) &= c^2 + c^2 \\ f_3(a, b, c, d) &= ac + bd\ . \end{aligned}$$

Then we have see that if $a(t)$, $b(t)$, $c(t)$ and $d(t)$ satisfy (1.7), or equivalently (1.8),

$$\frac{\mathrm{d}}{\mathrm{d}t}f(a(t),b(t),c(t),d(t)) = 0 \qquad \text{for} \quad j = 1,2,3 \ .$$

This means that the phase curves through $(a_0, b_0, c_0, d_0)$ lie on the intersection of the three hypersurfaces given by

$$\begin{aligned} f_1(a,b,c,d) &= f_1(a_0,b_0,c_0,d_0) \\ f_2(a,b,c,d) &= f_2(a_0,b_0,c_0,d_0) \\ f_3(a,b,c,d) &= f_2(a_0,b_0,c_0,d_0) \ . \end{aligned}$$

Put differently, we have three constrains on the phase motion, which leave only one degree of freedom left.

There is, however, yet another constant of the motion. It comes from the fact that the $z$-component of the angular momentum is conserved for the spherical pendulum.

Define

$$f_4(a,b,c,d) = ad - bc \ .$$

**Exercise 12:** Show that if $a(t)$, $b(t)$, $c(t)$ and $d(t)$ satisfy (1.7), then

$$\frac{\mathrm{d}}{\mathrm{d}t}f_4(a,b,c,d) = 0 \ . \tag{2.1}$$

Thus, we have *four* constraints on the motion in a four dimensional phase space. This certainly suggests that there must be some relation among them. There is, and it is not hard to find.

**Exercise 13:** Show that

$$(2f_3)^2 + (2f_4)^2 + (f_1 - f_2)^2 = (f_1 + f_2)^2 \ . \tag{2.2}$$

The right hand side is of course simply $(a^2 + b^2 + c^2 + d^2)^2$, so if $(a,b,c,d) \in S^3$, (2.2) reduces to

$$(2f_3)^2 + (2f_4)^2 + (f_1 - f_2)^2 = 1 \ .$$

This means that for all $(a,b,c,d) \in S^3$, the vector $(2f_3, 2f_4, f_1 - f_2)$ is a unit vector in $\mathbb{R}^3$; i.e., $(2f_3, 2f_4, f_1 - f_2) \in S^2$. This bring us to the following definition:

**2.1 DEFINITION** (Hopf map)**.** Define the *Hopf map* to be function $h$ on $S^3$ with values in $S^2$ by

$$h(a,b,c,d) = (2(ad+bc), 2(ad-bc), a^2 + b^2 - c^2 - d^2) \ .$$

**Exercise 14:** Show that the Hopf map is surjective; i.e., for every $\mathbf{q} \in S^2$, there is a $\mathbf{p} \in S^3$ so that $h(\mathbf{p}) = \mathbf{q}$. Hint: Show first that for any $0 \le r \le 1$, and any two angles $\varphi$ and $\psi$

$$(r\cos\varphi, r\sin\varphi, \sqrt{1-r^2}\cos\psi, \sqrt{1-r^2}\sin\psi) \in S^3 \ ,$$

and that if $z$ is defined by $z := 2 - r^2$,

$$h(r\cos\varphi, r\sin\varphi, \sqrt{1-r^2}\cos\psi, \sqrt{1-r^2}\sin\psi) = (\sqrt{1-z^2}\cos(\varphi-\psi), \sqrt{1-z^2}\sin(\varphi-\psi), z) \ .$$

**Exercise 15:** Show that for every $\mathbf{q} \in S^2$, the set

$$C_{\mathbf{q}} = \{ \ (a,b,c,d) \in S^3 \ : \ h(a,b,c,d) = \mathbf{q} \ \}$$

is one of the phase curve great circles on $S^3$.

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Our next goal is to show that every pair of our phase curve great circles is linked exactly once. Here is a picture showing two closed curves in $\mathbb{R}^3$, essentially circles, that link once:

The number of times one closed curve in $\mathbb{R}^3$ "wraps around another" is the linking number of the two closed curves. It is a topological invariant; that is unchanged under continuous deformations of the curves.

Our curves are in $S^3$, not $\mathbb{R}^3$, but we can take care of that by identifying $S^3$ (except for the "north pole") with $\mathbb{R}^3$ using *stereographic projection*. We will follow up on this later.