Problem Set 5 for Math 292, April 8, 2013

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April 9, 2013

The problems concern the use of symmetry for finding a change of variables that permits the solution of a differential equation. In these problems, the symmetry transformations will all be a general scaling transformation given by

$$g^u(x,y) = \begin{bmatrix} e^{u\alpha} & 0 \\ 0 & e^{u\beta} \end{bmatrix} (x,y) = (e^{u\alpha}x, e^{u\beta}y) \ .$$

Recall that slope field $(1, f(x,y)$ is invariant under the transformations g^u in case

$$\begin{bmatrix} e^{u\alpha} & 0 \\ 0 & e^{u\beta} \end{bmatrix} (1, f(e^{-u\alpha}x, e^{-u\beta}y) = a(x,y)(1, f(x,y)) \ .$$

for some non-zero function $a(x,y)$, so that the transformed slope field had the same slope as the original slope field.

We have seen that in this case the change of variables

$$h(x,y) = \left(\frac{1}{\alpha} \ln x \ \ x^{-\beta/\alpha} y \right) \tag{$*$}$$

will transform $(1, f(x,y)$ into a slope field of the form $(1, w(v))$, corresponding to the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}u} v = w(v) \ ,$$

which we can solve, at least implicitly by integration.

1. Consider the equation

$$y' = -3\frac{y}{x} - y^{3/2}x^{1/2} \ , \tag{1}$$

corresponding to the slope field

$$(a, f(x,y) = \left(1, -3\frac{y}{x} - y^{3/2}x^{1/2}\right) \ .$$

(a) Show that this slope field is invariant under the scaling transformation with $\alpha = -1$ and $\beta = 3$.

(b) Show that under the change of variables $(*)$, the slope field $(1, f(x,y))$ transforms into (a multiple of) the slope field

$$(1, v^{3/2}) \ ,$$

corresponding to the equation

$$\frac{\mathrm{d}}{\mathrm{d}u}v = v^{3/2} \ . \tag{2}$$

(c) Find the general solution of (2), and then change this back into x, y terms to find the general solution of (1). You should find

$$y(x) = x^{-3}\left(C + \frac{\ln(x)}{2}\right)^{-2} \ .$$

Check that this is indeed a solution.

2. Consider the equations

$$y' = 4x^2y^2 + x^5y^3 \tag{3}$$

and

$$y' = 4x^4y^2 + x^5y^3 \tag{4}$$

(a) One of these equations is scale invariant for some α and β and one is not. Determine which one is, and find values of α and β for which it is invariant.

(b) Find an equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}u}v = w(v) \tag{5}$$

that is equivalent to the invariant equation under a change of variables of the form $(*)$.

(c) Apply our usual method of solving such an equation to find a function $g(v)$ so that the solutions of (5) satisfy

$$g(v) + u = C \ ,$$

C an arbitrary constant.

(d) Use the change of variables to express this as a function $F(x, y)$ that is constant on the solution of the original equation in the x, y variables. That is, $F(x, y) = C$ gives the solution curves of the differential equation in implicit form.