Practice Final Exam, Math 292 Spring 2012

April 29, 2012

1: Consider the Riccati equation
$$y' = y^2 - y - 2 \ .$$

(a) Find a solution of the form $y(x) = Cx^m$.

(b) Find the general solution.

(c) Find the particular solution with $y(0) = -1$.

2: Find the general solution of

$$\begin{aligned} x' &= 4x + 4y \\ y' &= y + z \\ z' &= -4x - 5y - z \end{aligned}$$

as well as the particuar solution satisfying $x(0) = -1$, $y(0) = -1$ and $z(0) = 0$.

3: (a) Let $U = \{(x, y) \ : \ x > |y|\}$ and define $h(x, y) = (u(x, y), v(x, y))$ on U

$$u(x, y) = \frac{1}{2} \ln \left(\frac{y + x}{y - x} \right) \qquad \text{and} \qquad v = \sqrt{x^2 - y^2} \ .$$

Let
$$\mathbf{v}(x, y) = (x, y) - \sqrt{x^2 - y^2}(y, x) \ .$$

Compute
$$h_* \mathbf{v}(u, v) \ .$$

(b) Find the general solution of $\mathbf{x}(t) = \mathbf{v}(\mathbf{x}(t))$ for $\mathbf{x}_0 \in U$.

(c) Find an explicit formula for the associated phase flow on U.

4: Consider the vector fields
$$\mathbf{v}_1(x, y) = (x^3 + y, y^3 + x)$$
and
$$\mathbf{v}_2(x, y) = (-x - y^3, y + x^3) \ .$$

(a) Find all of the equilibrium points of $\mathbf{v}_1(x, y)$, and determine which (if any) are Lyapunov stable but not asymptotically stable, which (if any) are asymptotically stable and which (if any) are unstable

(b) Find all of the equilibrium points of $\mathbf{v}_2(x, y)$, and determine which (if any) are Lyapunov stable but not asymptotically stable, which (if any) are asymptotically stable and which (if any) are unstable

(c) Let $\mathbf{w}(y) = (xy + x + y, x^3 + y - 2x)$. Note that $(0, 0)$ is an equilibrium point of $\mathbf{w}(x, y)$. Let A be the matrix such that

$$\mathbf{x}'(t) = A\mathbf{x}(t)$$

is the linearization of

$$\mathbf{x}'(t) = \mathbf{w}(\mathbf{x}(t))$$

near $\mathbf{x} = 0$. Compute e^{tA}.

5: Let

$$H(x, y) := x^2 + 2y^2 + 2xy + x - 5y \ .$$

(a) Consider the solution $\mathbf{x}(t) = (x(t), y(t))$ of the differential equation

$$\mathbf{x}'(t) = \nabla^{\perp} H(\mathbf{x}(t) \qquad \text{with} \quad \mathbf{x}(0) = (1, 1) \ .$$

(Recall that $\nabla^{\perp} H(x, y) = (-\partial H(x, y)/\partial y, \partial H(x, y)/\partial x)$.)

Show that

$$\frac{\mathrm{d}}{\mathrm{d}t} H(x(t)), y(t)) = 0$$

along the solution curve, and sketch a plot of the solution curve. Also, determine

$$\min\{y(t) \ : t \in \mathbb{R}\} \ .$$

Justify your answer to be credited.

(b) Consider the solution $\mathbf{x}(t)$ of the differential equation

$$\mathbf{x}'(t) = -\nabla H(\mathbf{x}(t) \qquad \text{with} \quad \mathbf{x}(0) = (1, 1) \ .$$

Show that

$$\frac{\mathrm{d}}{\mathrm{d}t} H(x(t)), y(t)) = 0$$

and determine

$$\lim_{t \to \infty} \mathbf{x}(t) \ .$$

Justify your answer to be credited.

6: **(a)** Consider the second order equation

$$t^2 x''(t) - 2tx'(t) + 2x(t) = t \ln t$$

for $t > 0$. Find the general solution of this equation.

(b) Consider the equation

$$x''(t) + \left(\frac{5}{6} + \frac{1}{6}\cos(t)\right) x(t) = 0$$

such that $x(0) = 0$ and $x'(t) = 1$. Let T be such that $x(T) = 0$, and $x(t) > 0$ for $0 < t < T$. Find constants C_1 and C_2 such that

$$C_1 < T < C_2$$

and justify your answers.

7: Find the continuously differentiable curve $y(x)$ such that $y(0) = 1$ and $y(1) = 0$ that minimizes the functional

$$I[y] = \int_0^1 (y'(x))^2 + (y(x))^2)\mathrm{d}x \ .$$

Justify you answer.