7.2. (a) $\displaystyle\int_{y=-\sqrt{3}}^{\sqrt{3}} \int_{x=1-\sqrt{4-y^2}}^{-1+\sqrt{4-y^2}} x^2y^2\,dx\,dy$

(b) The region D is symmetric with respect to the y-axis and the integrand is unchanged when x is replaced by $-x$. So let D^+ be the half of D in the first and 4th quadrants. Then

$$\int_D x^2y^2\,dA = 2\int_{D^+} x^2y^2\,dA = 2\int_{x=0}^{1} \int_{y=-\sqrt{4-(x+1)^2}}^{y=\sqrt{4-(x+1)^2}} x^2y^2\,dy\,dx$$

(c) $\displaystyle 2\int_{x=0}^{1} x^2 \frac{y^3}{3}\Bigg|_{y=-\sqrt{4-(x+1)^2}}^{y=\sqrt{4-(x+1)^2}} dx = 2\int_{x=0}^{1} 2x^2\sqrt{4-(x+1)^2}^{\,3}\,dx$

Let $x+1 = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$, $\sqrt{4-(x+1)^2} = 2\cos\theta$. The integral equals

$$4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2\sin\theta-1)^2\cdot 8\cos^3\theta\cdot 2\cos\theta\,d\theta = 64\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (4\sin^2\theta - 4\sin\theta+1)\cos^4\theta\,d\theta \text{ etc.}$$

7.4. Draw the region D. It is a triangle with vertices $(0,0)$, $(1.5,1.5)$, and $(3,9)$.

(a) The line $y=1.5$ separates D into two triangular regions E_1 and E_2, each with a horizontal side. The lower triangle E_1 is bounded on the left by $y=3x$ and on the right by $y=x$; the triangle E_2 is bounded on the left by $y=3x$ and on the right by $y=5x-6$. Thus,

$$\int_D xy\,dA = \int_{E_1} xy\,dA + \int_{E_2} xy\,dA = \int_{y=0}^{1.5}\int_{x=y/3}^{y} xy\,dx\,dy + \int_{y=1.5}^{9}\int_{x=y/3}^{(y+6)/5} xy\,dx\,dy$$

(b) The line $x=1.5$ separates D into two triangular regions D_1 and D_2, each with a vertical side. The left triangle D_1 is bounded below by $y=x$ and above by $y=3x$; the triangle D_2 is bounded below by $y=5x-6$ and above by $y=3x$. Thus,

$$\int_D xy\,dA = \int_{D_1} xy\,dA + \int_{D_2} xy\,dA = \int_{x=0}^{1.5}\int_{y=x}^{3x} xy\,dy\,dx + \int_{x=1.5}^{3}\int_{y=5x-6}^{3x} xy\,dy\,dx$$

(c) Using (b),

$$\begin{aligned}\int_D xy\,dA &= \int_{x=0}^{1.5}\left(x\frac{(3x)^2}{2} - x\frac{x^2}{2}\right)dx + \int_{x=1.5}^{3}\left(x\frac{(3x)^2}{2} - x\frac{(5x-6)^2}{2}\right)dx \\ &= \int_0^{1.5} 4x^3\,dx + \int_{1.5}^{3} -8x^3+30x^2-18x\,dx \\ &= x^4\Big|_0^{3/2} + (-2x^4+10x^3-9x^2)\Big|_{3/2}^{3} = \frac{81}{16} - 162 + 270 - 81 + \frac{162}{16} - \frac{270}{8} + \frac{81}{4} = \frac{459}{16}\end{aligned}$$

7.8. Let the change of variable $\mathbf{u} = \mathbf{u}(\mathbf{x})$ be defined by $u = xy$ and $v = y/x$. Let $\bar{D}$ be the region in the u, v-plane defined by $1 \le u \le 2$ and $1 \le v \le 2$. Then $\bar{D}$ corresponds to D under the correspondence $\mathbf{u}(\mathbf{x})$.

Solving $u = xy$ and $v = y/x$ for x and y gives the inverse correspondence $\mathbf{x}(\mathbf{u})$, namely $uv = y^2$, $y = \sqrt{uv}$, $x = u/y = \sqrt{u/v}$.

The Jacobian of the inverse correspondence and its determinant are

$$D_{\mathbf{x}}(\mathbf{u}) = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}\begin{bmatrix} \frac{1}{\sqrt{uv}} & -\frac{\sqrt{u}}{\sqrt{v^3}} \\ \frac{\sqrt{v}}{\sqrt{u}} & \frac{\sqrt{u}}{\sqrt{v}} \end{bmatrix}, \quad \det D_{\mathbf{x}}(\mathbf{u}) = \frac{1}{4}\frac{2}{v} = \frac{1}{2v}$$

Finally

$$\begin{aligned} \int_D xy \, d^2\mathbf{x} &= \int_{\bar{D}} u \cdot |\det D_{\mathbf{x}}(\mathbf{u})| \, d^2\mathbf{u} = \int_{u=1}^{2}\int_{v=1}^{2} \frac{u}{2v} \, dv \, du \\ &= \int_{u=1}^{2} \frac{u \ln 2}{2} \, du = \frac{3\ln 2}{4}. \end{aligned}$$

7.10. The curve is $r^2 = (r^2 - r\cos\theta)^2 = r^2(r - \cos\theta)^2$. Apart from the point at the origin, this is equivalent to $(r - \cos\theta)^2 = 1$ or $r - \cos\theta = \pm 1$. Since r is always positive except at the origin, and since $\cos\theta \le 1$ for all θ, this is equivalent to $r - \cos\theta = 1$, or $r = 1 + \cos\theta$.

The curve looks like a heart resting on its left side. The enclosed area D is described in polar coordinates by $0 \le \theta \le 2\pi$, $0 \le r \le 1 + \cos\theta$. The area of D is

$$\int_D d^2\mathbf{x} = \int_{\theta=0}^{2\pi}\int_{r=0}^{1+\cos\theta} r \, dr \, d\theta = \int_{\theta=0}^{2\pi} \frac{1}{2}(1+\cos\theta)^2 \, d\theta = \frac{3}{2}\pi$$

7.12. As θ goes from 0 to 2π, this equation traces out a figure eight, going from NorthEast (the point $(\sqrt{2}, \sqrt{2})$ at $\theta = \pi/4$) to SouthWest (the point $(-\sqrt{2}, -\sqrt{2})$ at $\theta = 5\pi/4$). The area of the enclosed region D is

$$\int_D d^2\mathbf{x} = \int_{\theta=0}^{2\pi}\int_{r=0}^{1+\sin 2\theta} r \, dr \, d\theta = \int_{\theta=0}^{2\pi} \frac{1}{2}(1+\sin 2\theta)^2 \, d\theta = \frac{3}{2}\pi.$$

7.14. Let $\mathbf{u}(\mathbf{x})$ be defined by $u = xy$ and $v = x^2 y$. Let $\bar{D}$ be the region in the u, v-plane defined by $1 \le u \le 2$ and $3 \le v \le 4$. Then $\bar{D}$ corresponds to D under the correspondence $\mathbf{u}(\mathbf{x})$.

Solving $u = xy$ and $v = x^2 y$ for x and y gives the inverse correspondence $\mathbf{x}(\mathbf{u})$, namely $x = v/u$, $y = u^2/v$.

The Jacobian of the inverse correspondence and its determinant are

$$D_{\mathbf{x}}(\mathbf{u}) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} -v/u^2 & 1/u \\ 2u/v & -u^2/v^2 \end{bmatrix}, \quad \det D_{\mathbf{x}}(\mathbf{u}) = -\frac{1}{v}.$$

Finally

$$\int_D xy\, d^2\mathbf{x} = \int_{\bar{D}} u \cdot |\det D_{\mathbf{x}}(\mathbf{u})|\, d^2\mathbf{u} = \int_{u=1}^{2} \int_{v=3}^{4} \frac{u}{v}\, dv\, du = \frac{3}{2} \ln\left(\frac{4}{3}\right).$$

7.16. The region D has two symmetric pieces, D_1 in the first quadrant and D_2 in the third quadrant. The integrand is unchanged by the double sign change $x \to -x$ and $y \to -y$, so

$$\int_D (x^2 + y^2)\, d^2\mathbf{x} = 2 \int_{D_1} x^2 + y^2 d^2\mathbf{x}$$

Define $\mathbf{u}(\mathbf{x})$ by $u = x^2 - y^2$ and $v = xy$ and let $\bar{D}_1$ be the region $0 \leq u \leq 4$, $1 \leq v \leq 2$ in the u, v-plane.

The inverse transformation is messy but $D_{\mathbf{u}}(\mathbf{x})$ is easy to compute; it's

$$D_{\mathbf{u}}(\mathbf{x}) = \begin{bmatrix} 2x & -2y \\ y & x \end{bmatrix}, \qquad \det D_{\mathbf{u}}(\mathbf{x}) = 2(x^2 + y^2)$$

Therefore by the change of variable formula in reverse (tricky!)

$$\int_D (x^2 + y^2)\, d^2\mathbf{x} = 2 \int_{D_1} (x^2 + y^2)\, d^2\mathbf{x} = \int_{D_1} |\det D_{\mathbf{u}}(\mathbf{x})|\, d^2\mathbf{x} = \int_{\bar{D}_1} 1\, d^2\mathbf{u} = \text{ area of } \bar{D}_1 = 4 \cdot 1 = 4.$$