NOTE: In these solutions (for example in #6) we sometimes use the subscript notation for partial derivatives. In that notation,

$$f_x = \frac{\partial f}{\partial x}, \quad f_y = \frac{\partial f}{\partial y}, \quad f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{xy} = \frac{\partial^2 f}{\partial y \partial x}, \quad f_{yx} = \frac{\partial^2 f}{\partial x \partial y}, \quad \text{etc.}$$

6.1. $[Hess_f(x,y)] = \begin{bmatrix} 2y & 2x+2y-1 \\ 2x+2y-1 & 2x \end{bmatrix}$ so $[Hess_f(\mathbf{x}_0)] = [Hess_f(1,1)] = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$.

For $\mathbf{v} = (1,2)$,

$$\frac{d^2}{dt^2} f(\mathbf{x}_0 + t\mathbf{v})\Big|_{t=0} = \mathbf{v}[Hess_f(\mathbf{x}_0)] \bullet \mathbf{v} = (1\,2) \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \bullet (1,2) = 22.$$

6.3. $[Hess_f(x,y,z)] = \begin{bmatrix} 2yz & 2xz+2y & 2xy-1 \\ 2xz+2y & 2x & x^2 \\ 2xy-1 & x^2 & 0 \end{bmatrix}$ so $[Hess_f(\mathbf{x}_0)] = [Hess_f(1,1,1)] = \begin{bmatrix} 2 & 4 & 1 \\ 4 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

For $\mathbf{v} = (1,0,1)$,

$$\frac{d^2}{dt^2} f(\mathbf{x}_0 + t\mathbf{v})\Big|_{t=0} = \mathbf{v}[Hess_f(\mathbf{x}_0)] \bullet \mathbf{v} = (1,0,1) \begin{bmatrix} 2 & 4 & 1 \\ 4 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \bullet (1,0,1) = 4.$$

6.5. $\det \begin{bmatrix} 1-t & 2 \\ 2 & 4-t \end{bmatrix} = t^2 - 5t = t(t-5)$. The eigenvalues are 0 and 5. Since $A - 0I = A$, an eigenvector for the eigenvalue 0 is $(1,2)^\perp = (-2,1)$, and an eigenvector for the eigenvalue 5 is $(-2,1)^\perp = (-1,-2)$. An orthonormal basis of $\mathbf{R}^2$ consisting of eigenvectors is

$$\left\{ \frac{1}{\sqrt{5}}(-2,1), \frac{1}{\sqrt{5}}(1,2) \right\}$$

6.6. The formulas for f_x, f_y, f_{xy} away from $(0,0)$ are routine consequences of the quotient rule. That $f_{xy} = f_{yx}$ away from $(0,0)$ is a consequence of Clairaut's Theorem.

(a) $f_x(0,0) = \lim\limits_{x\to 0} \dfrac{f(x,0) - f(0,0)}{x} = \lim\limits_{x \to 0} \dfrac{0-0}{x} = \lim\limits_{x\to 0} 0 = 0.$

Similarly $f_y(0,0) = \lim_{y\to 0}(f(0,y) - f(0,0))/y = 0$.

Continuity of f_x and f_y away from $(0,0)$ is obvious. To prove continuity of f_x at $(0,0)$ we must prove that

$$\lim_{(x,y)\to(0,0)} \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2+y^2)^2} = 0.$$

This follows from the Squeeze Principle, because

$$0 \le \left| \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2} \right| = \frac{|y(x^4+4x^2y^2-y^4)|}{(x^2+y^2)^2} \le |y| \frac{|x^4|+4|x^2y^2|+|y^4|}{(x^2+y^2)^2} \le |y| \frac{2x^4+4x^2y^2+2y^4}{(x^2+y^2)^2} = 2|y|$$

and $\lim_{(x,y)\to(0,0)} 2|y| = 0$. Continuity of f_y at 0 is similarly proved.

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}$ does not exist because along the x-axis ($y=0$) the function equals $x^6/x^6 = 1$ while along the y-axis ($x=0$) the function equals -1.

(c) Because of (b), f_{xy} and f_{yx} can't possibly be continuous at $(0,0)$. So Clairaut's Theorem does not apply.

Note that $f_y(x,0) = x^5/x^4 = x$ for $x \ne 0$, and $f_x(0,y) = -y^5/y^4 = -y$ for $y \ne 0$. Then

$$\begin{aligned} f_{yx}(0,0) &= \lim_{x\to 0} \frac{f_y(x,0)-f_y(0,0)}{x} = \lim_{x\to 0} \frac{x}{x} = 1, \\ f_{xy}(0,0) &= \lim_{y\to 0} \frac{f_x(0,y)-f_x(0,0)}{y} = \lim_{y\to 0} \frac{-y}{y} = -1, \end{aligned}$$

so the conclusion of Clairaut's Theorem doesn't hold. This is fine since the theorem doesn't apply. This example shows that the continuity assumption in the theorem cannot be removed.

6.7. Similar to 6.5; eigenvalues are 2 and 6.

$A - 2I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ so $(2,2)^\perp = (-2,2)$ is an eigenvector for the eigenvalue 2, and $(-2,2)^\perp = (-2,-2)$ is an eigenvector for the eigenvalue 6. An orthonormal basis of $\mathbf{R}^2$ consisting of eigenvectors is

$$\left\{ \frac{1}{\sqrt{2}}(-1,1), \frac{1}{\sqrt{2}}(1,1) \right\}$$

6.11. $\partial f/\partial x = 3y^3 + 2 + x^3 = 0$, $\partial f/\partial y = 9xy^2 + 9y = 9y(xy+1) = 0$.

The Hessian is $H(x,y) = \begin{bmatrix} 3x^2 & 9y^2 \\ 9y^2 & 18xy+9 \end{bmatrix}$

If $y = 0$ we get the critical point $y = 0$, $x = -2^{1/3}$; the Hessian there is $H = \begin{bmatrix} 3\cdot 2^{2/3} & 0 \\ 0 & 9 \end{bmatrix}$. Since $a = 3 \cdot 2^{2/3} > 0$ and $\det H > 0$, there is a local minimum at $(-2^{1/3}, 0)$.

If $y \ne 0$ then $xy+1 = 0$, $y = -1/x$, $-3/x^3 + 2 + x^3 = 0$, $x^6 + 2x^3 - 3 = 0$, a quadratic equation for x^3. So $x^3 = 1, -3$, and $x = 1$ or $x = -3^{1/3}$. As $y = -1/x$, this gives critical points $(1,-1)$ and $(-3^{1/3}, 3^{-1/3})$.

$H(1,-1) = \begin{bmatrix} 3 & 9 \\ 9 & -9 \end{bmatrix}$. Since $\det H(1,-1) = -108 < 0$, f has a saddle point at $(1,-1)$. The signs of the entries of H at $(-3^{1/3}, 3^{-1/3})$ are $\begin{bmatrix} + & + \\ + & - \end{bmatrix}$, so the determinant is negative and f has another saddle point at $(-3^{1/3}, 3^{-1/3})$.

(b) $\mathbf{x}_0 = (-2^{1/3}, 0)$ is a good choice since H is already diagonal. An orthonormal basis of eigenvectors consists of $\mathbf{e}_1$ (eigenvalue $3 \cdot 2^{2/3}$) and $\mathbf{e}_2$ (eigenvalue 9). The level curves near $\mathbf{x}_0$ are approximately ellipses with major axes parallel to $\mathbf{e}_1$ and minor axes parallel to $\mathbf{e}_2$.

6.13. (a) $\partial f/\partial x = 2x(1-2y) = 0$, $\partial f/\partial y = 2(y - x^2) = 0$. Also, $H(x,y) = \begin{bmatrix} 2(1-2y) & -4x \\ -4x & 2 \end{bmatrix}$

If $x = 0$ then $y = 0$; $(0,0)$ is a critical point. $H(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ so f has a local minimum at $(0,0)$.

If $x \neq 0$ then $1 - 2y = 0$, $y = 1/2$, $x = \pm 1/\sqrt{2}$. $H(\pm 1/\sqrt{2}, 1/2) = \begin{bmatrix} 0 & \mp 2\sqrt{2} \\ \mp 2\sqrt{2} & 2 \end{bmatrix}$ has negative determinant so f has saddle points at $(\pm 1/\sqrt{2}, 1/2)$.

(b) Since $H(0,0) = 2I$ we can use $\mathbf{e}_1$ and $\mathbf{e}_2$ as eigenvectors. Since the eigenvalues are equal, the level curves near $(0,0)$ are approximated by circles $u^2 + v^2 = K$, where $u = x - 0 = x$ and $v = y$.

At $(1/\sqrt{2}, 1/2)$, $H(1/\sqrt{2}, 1/2) = \begin{bmatrix} 0 & \pm 2\sqrt{2} \\ \pm 2\sqrt{2} & 2 \end{bmatrix}$ has characteristic polynomial $\lambda^2 - 2\lambda - 8 = 0$, so the eigenvalues are $\lambda = 4$ and $\lambda = -2$. Further,

$$H - 4I = \begin{bmatrix} -4 & \pm 2\sqrt{2} \\ \pm 2\sqrt{2} & -2 \end{bmatrix}$$

so an eigenvector for the eigenvalue 4 is $(1, \sqrt{2})$. Then an eigenvector for the eigenvalue -2 is $(1, \sqrt{2})^{\perp} = (-\sqrt{2}, 1)$.

The level curves of f near $(1/\sqrt{2}, 1/2)$ are approximately the hyperbolas $4u^2 - 2v^2 = K$, where the u and v-axes intersect at $(1/\sqrt{2}, 1/2)$, and the u- and v-axes are parallel to $(1, \sqrt{2})$ and $(-\sqrt{2}, 1)$, respectively.

The analysis near $(-1/\sqrt{2}, 1/2)$ is similar; the eigenvalues of $H(-1/\sqrt{2}, 1/2)$ are also 4 and -2, but the eigenvectors are $(1, -\sqrt{2})$ and $(\sqrt{2}, 1)$.

6.15. $f(x,y) = x^4 + y^4 - 2x^2 y$. We seek a critical point (x_0, y_0) with $x_0 > 0$ and $y_0 > 0$.

$\partial f/\partial x = 4x^3 - 4xy = 0$, $\partial f/\partial y = 4y^3 - 2x^2 = 0$. The first equation gives $y = x^2$ (since we are assuming that $x > 0$), so the second equation gives $4y^3 - 2y = 0$, $y = 1/\sqrt{2}$ (since we are assuming that $y > 0$). Then $x = \sqrt{y} = 2^{-1/4}$.

(a) $H(x,y) = \begin{bmatrix} 12x^2 - 4y & -4x \\ -4x & 12y^2 \end{bmatrix}$ so $H(2^{-1/4}, 2^{-1/2}) = \begin{bmatrix} 4\sqrt{2} & -4 \cdot 2^{-1/4} \\ -4 \cdot 2^{-1/4} & 6 \end{bmatrix}$ so

$$\det H(2^{-1/4}, 2^{-1/2}) = 16\sqrt{2} > 0.$$

As the upper left entry of H is positive, f has a local minimum at $(2^{-1/4}, 2^{-1/2})$.

(b) The largest (resp. smallest) directional second derivative has value equal to the largest (resp. smallest) eigenvalue of $H(2^{-1/4}, 2^{-1/2})$. These occur in the directions of the respective unit eigenvectors.

The eigenvalues are roots of $\lambda^2 - (6 + 4\sqrt{2})\lambda + 16\sqrt{2} = 0$, namely,

$$\lambda_{\pm} = 3 + 2\sqrt{2} \pm \sqrt{17 - 4\sqrt{2}}.$$

$H - \lambda_+ I = \begin{bmatrix} -3 + 2\sqrt{2} - \sqrt{17 - 4\sqrt{2}} & -4 \cdot 2^{-1/4} \\ * & * \end{bmatrix}$ so an eigenvector corresponding to λ_+ is $(-4 \cdot 2^{-1/4}, -3 + 2\sqrt{2} - \sqrt{17 - 4\sqrt{2}})$. An eigenvector for λ_- is similarly found.