5.1. (a) $\nabla f = (3x^2y - 3y^2, x^3 + 2 - 6xy)$.

The equations $\nabla f = 0$ may be written $3y(x^2 - y) = 0,\ x^3 - 6xy + 2 = 0$. The first equation gives $y = 0$ or $y = x^2$.

If $y = 0$ then $x = -2^{1/3}$; so $(0, -2^{1/3})$ is a critical point.

If $y = x^2$ then $x^3 = 2/5$; so $((2/5)^{1/3}, (2/5)^{2/3})$ is a critical point.

In particular there are exactly two critical points.

(b) No. The contour plot shows three critical points, one in the first quadrant, one near the negative x-axis, and one near the negative y-axis.

5.3. (a) $\nabla f = (3x^2 + 3y, 3y^2 + 3x)$.

Critical points occur where $y = -x^2$ and $x = -y^2$. These equations lead to $x = -x^4$, so $x = 0$ or -1, and then $y = -x^2$ determines y. The critical points are $(0, 0)$ and $(-1, -1)$.

(b) The tangent line is given by $\nabla f(1, 1) \bullet (\mathbf{x} - (1, 1)) = 0$, that is, $(6, 6) \bullet (x - 1, y - 1) = 0$, or $x + y = 2$.

(c) Yes; at least it seems to show critical points at the right places and has the right tangent line at $(1, 1)$ (slope -1).

5.5. $\nabla f = (y^2 - y, 2xy - x)$.

(a) $y^2 = y$ gives $y = 0$ or 1; for either of these, $2xy - x = 0$ gives $x = 0$. So there are two critical points, at $(0, 0)$ and $(0, 1)$.

(b) $\nabla f(3/2, 1/3) \bullet (\mathbf{x} - (3/2, 1/3)) = 0$, that is, $(3/4, -1/2) \bullet (x - 3/2, y - 1/3) = 0$, that is, $3x - 2y = 23/6$.

(c) No, the contour plot shows a critical point that's not at $(0, 0)$ or $(0, 1)$.

5.7. The domain D is a closed elliptical disk. The critical points of f occur at $\nabla f = (y, x) = \mathbf{0}$, i.e., $(0, 0)$ is the only critical point. On the boundary, $\nabla g = (2x, 8y)$ is never $\mathbf{0}$. So the candidate points on the boundary are those where

$$\det \begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = 0, \text{ i.e., } \det \begin{bmatrix} y & x \\ 2x & 8y \end{bmatrix} = 8y^2 - 2x^2 = 0.$$

Solving this together with the boundary equation $x^2 + 4y^2 = 6$ gives $16y^2 = 12$, $y = \pm\sqrt{3}/2$, $x^2 = 3$, $x = \pm\sqrt{3}$. We have 5 candidates:

$$(0, 0),\ \pm(\sqrt{3}/2, \sqrt{3}),\ \pm(\sqrt{3}/2, -\sqrt{3}).$$

These give $f(x, y) = 0$, $3/2$, and $-3/2$, respectively.

Therefore the maximum value is $3/2$, achieved at the maximizers $\pm(\sqrt{3}/2, \sqrt{3})$, and the minimum value is $-3/2$, achieved at the minimizers $\pm(\sqrt{3}/2, -\sqrt{3})$.

5.9. Again $f(x, y) = xy$ has a unique critical point at $(0, 0)$. The boundary is given by $g(x, y) = x^4 + 2x^2y^2 + y^4 - x^2 + y^2 = 0$.

The gradient of g is

$$\nabla g = (4x^3 + 4xy^2 - 2x, 4x^2y + 4y^3 + 2y).$$

Setting $\nabla g = 0$ gives $(4x^2 + 4y^2 - 2)x = 0 = (4x^2 + 4y^2 + 2)y = 0$, so $y = 0$, as $4x^2 + 4y^2 + 2$ never equals 0. Then $x = 0$ or $4x^2 - 2 = 0$. But on the boundary $g = 0$, $x^4 = x^2$, so $x = 0$ or $x^2 = 1$. The upshot is that on the boundary there is a unique point where $\nabla g = 0$, and that point is $(0, 0)$, which is already on our list of possibilities.

The remaining possibilities therefore must come from the boundary points where

$$\det \begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = 0, \text{ i.e., } \det \begin{bmatrix} y & x \\ 4x^3 + 4xy^2 - 2x & 4x^2y + 4y^3 + 2y \end{bmatrix} = -4x^4 + 4y^4 + 2(x^2 + y^2) = 0.$$

Completing the square for the y-terms and x-terms separately, we add and subtract $1/4$ and get

$$(2y^2 + 0.5)^2 - (2x^2 - 0.5)^2 = 0.$$

Therefore $2y^2 + 0.5 = \pm(2x^2 - 0.5)$. The minus sign gives $2y^2 = -2x^2$, so $(x, y) = (0, 0)$, a point we already have. So consider the $+$ sign:

$$2x^2 - 2y^2 = 1, \quad x^2 - y^2 = 1/2.$$

Therefore $(x^2 + y^2)^2 = 1/2$, $x^2 + y^2 = \sqrt{2}/2$. Combining this with $x^2 - y^2 = 1/2$, $2x^2 = (1 + \sqrt{2})/2$,

$$x = \pm\frac{1}{2}\sqrt{\sqrt{2} + 1}, \quad y = \pm\frac{1}{2}\sqrt{\sqrt{2} - 1}.$$

These give four points, one in each quadrant. The maximum value of xy is

$$\frac{1}{2}\sqrt{\sqrt{2} + 1} \cdot \frac{1}{2}\sqrt{\sqrt{2} - 1} = \frac{1}{4}$$

at the points in the first and third quadrant, and the minimum value is $-1/4$, at the other two points.

5.11. $\nabla f = (2x - y, 2y - x)$ equals $\mathbf{0}$ only at $(0, 0)$.

Here $g(x, y) = x^2 + y^2 - 1$, and $\nabla g = (2x, 2y)$ is never 0 on the boundary of D. So other than $(0, 0)$ our candidate points are the solutions of

$$\det \begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = 0, \text{ i.e., } \det \begin{bmatrix} 2x - y & 2y - x \\ y & x \end{bmatrix} = 2(x^2 - y^2) = 0, \text{ i.e., } y = \pm x.$$

On the boundary of D these are the points $(\pm\sqrt{2}/2, \pm\sqrt{2}/2)$, which together with $(0,0)$ are our candidates. We compute $f(0,0) = 0$, $f(\pm(\sqrt{2}/2, \sqrt{2}/2)) = 1/2$, and $f(\pm(\sqrt{2}/2, -\sqrt{2}/2)) = 3/2$. So the minimum value of f on D is 0, at the minimizer $(0,0)$. The maximum value of f is $3/2$, at the maximizers $(\sqrt{2}/2, -\sqrt{2}/2))$ and $(-\sqrt{2}/2, \sqrt{2}/2))$.

5.13. We have to check (1) the interior of D for critical points of f; (2) the parabola $y = 1 - x^2$ for $-1 < x < 1$, using Lagrange's method; (3) the segment $y = 0$, $-1 < x < 1$; and (4) the vertices $(\pm 1, 0)$.

(1) The critical points of f are at $\nabla f = \mathbf{0}$, i.e., $y + 2 = x - 2 = 0$, i.e., the point $(2, -2)$. However, this point is not in D so it is to be ignored.

(2) On this curve $g(x,y) = x^2 + y - 1 = 0$ and $\nabla g = (2x, -1)$ is never $\mathbf{0}$. The Lagrange method gives

$$\det \begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = 0, \text{ i.e., } \det \begin{bmatrix} y+2 & x-2 \\ 2x & - \end{bmatrix} 1 = -y - 2 - 2x^2 + 4x = 0.$$

Solving together with $y = 1 - x^2$ gives $-2x^2 + 4x - 2 = y = 1 - x^2$, $x^2 - 4x + 3 = 0$, $x = 1, 3$. Neither of these is in the interval $(-1, 1)$, however, so we ignore them in this step.

(3) On the interval $y = 0$, $-1 < x < 1$, $f(x,y) = 2x$ has no critical point.

(4) All our candidates therefore have to come from this step: they are $(0,0)$ and $(1,0)$. The minimum value is $f(0,0) = 0$ and the maximum is $f(1,0) = 2$.

5.15. (This problem is easily solved geometrically by drawing the level lines of $f(x,y) = (x-y)^2$. These lines are easy to track as they cross the region D.)

Analytically, as in 5.13, we examine (1) the interior of D, (2) the lower boundary $y = x^2/2$, $-2 < x < 2$; (3) the upper boundary $y = 2$, $-2 < x < 2$, and (4) the vertices $(\pm 2, 2)$.

(1) $\nabla f = (2(x-y), 2(y-x)) = 0$ everywhere on the line $y = x$, where $f(x,y) = (x-y)^2 = 0$. These are obviously minimizers, since $f(x,y)$ is a square.

(2) Lagrange method, with $g(x,y) = (x^2/2) - y$, points to

$$\det \begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = 0, \text{ i.e., } \det \begin{bmatrix} 2(x-y) & 2(y-x) \\ x & -1 \end{bmatrix} = 2(y-x)(1-x) = 0,$$

so $y = x$ (already considered) or $x = 1$. This gives us the point $(1, 1/2)$, where $f(1, 1/2) = 1/4$.

(3) On $y = 2$, $-2 < x < 2$, $f(x,y) = f(x,2) = (x-2)^2$ has no critical point.

(4) In addition to the points (x,x) and $(1,1/2)$ above we have the vertices $(-2,2)$ and $(2,2)$, giving $f(x,y) = 16$ and 0, respectively.

The maximizer is $(-2, 2)$, where $f(-2, 2) = 16$. The minimizers, where $f(x, y) = 0$, are all the points on the line $y = x$, $0 \le x \le 2$.

5.17. The set C is the intersection of the vertical cylinder $x^2 + y^2 = 1$ with the nonvertical plane $x + y + z = 3$, so it's a closed curve. Notice that $\nabla f = (3, 1, -1)$ is never zero. We look at the points on C where

$$\det \begin{bmatrix} \nabla f \\ \nabla g \\ \nabla h \end{bmatrix} = 0, \text{ i.e., } \det \begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 2x & 2y & 0 \end{bmatrix} = 4x - 8y = 0$$

and satisfying the two constraints. So $x = 2y$. Then $x^2 + y^2 = 1$ gives $(x, y) = (\pm 2/\sqrt{5}, \pm 1/\sqrt{5})$. Since $x + y + z = 3$ we get the points

$$\begin{aligned}(x, y, z) = &(1/\sqrt{5})(2, 1, 3(\sqrt{5} - 1)),\ (1/\sqrt{5})(-2, 1, (1 + 3\sqrt{5})), \\ &(1/\sqrt{5})(2, -1, (-1 + 3\sqrt{5})),\ (1/\sqrt{5})(-2, -1, 3(1 + \sqrt{5})),\end{aligned}$$

so $f(x, y, z) = (1/\sqrt{5})(7 + 3(1 - \sqrt{5})), (1/\sqrt{5})(-5 + (-1 - 3\sqrt{5})), (1/\sqrt{5})(5 + (1 + 3\sqrt{5})), (1/\sqrt{5})(-7 + 3(-1 + \sqrt{5}))$, respectively. The largest and smallest of these are the third and second, resp.

5.19. We want to maximize $f(x, y, z) = d^2 = (x - 1)^2 + (y - 3)^2 + (z - 4)^2$ subject to the constraint $g(x, y, z) = 0$, where $g(x, y, z) = x^2 + y^2 - z$. This can only occur at a point where $\nabla f = \mathbf{0}$, or $\nabla g = \mathbf{0}$, or the tangent planes to the level surface of f and to the surface $g(x, y, z) = 0$ are parallel, i.e., ∇f and ∇g are parallel.

$\nabla f = 2(x - 1, y - 3, z - 4)$ and $\nabla g = (2x, 2y, -1)$. Clearly ∇g is never $\mathbf{0}$. We can check the parallel condition in more than one way. One way is to set $\nabla f \times \nabla g = \mathbf{0}$, giving three equations in addition to the equation $g(x, y, z) = 0$. Another way (sometimes called the method of Lagrange Multipliers) is to set $\nabla f = \lambda \nabla g$ where λ is an unknown scalar, giving 3 equations in addition to $g(x, y, z) = 0$.

Let's use the cross product. The z-component of $\nabla f \times \nabla g$ is $4[(x - 1)y - (y - 3)x]$ so

$$4[(x - 1)y - (y - 3)x] = 0, \quad y = 3x.$$

The y-component of $\nabla f \times \nabla g$ is $4x(z - 4) + 2(x - 1)$, so

$$4x(z - 4) + 2(x - 1) = 0, \quad z = \frac{6 - 2x}{4x} = \frac{3}{2x} - \frac{1}{2}$$

With y and z expressed in terms of x, the equation $z = x^2 + y^2$ then becomes

$$\frac{3}{2x} - \frac{1}{2} = x^2 + (3x)^2 = 10x^2.$$

Equivalently, $3 = 20x^3 + x$. By inspection, luck, or Newton's Method, $x = 1/2$, so $y = 3/2$ and $z = x^2 + y^2 = 5/2$. The point is $(0.5, 1.5, 2.5)$.