4.1. First express $\mathbf{v} = s\mathbf{v}_1 + t\mathbf{v}_2$, which gives $s = 2$, $t = -5$. Then

$$\mathbf{v} \bullet \nabla f(\mathbf{x}_0) = (2\mathbf{v}_1 - 5\mathbf{v}_2) \bullet \nabla f(\mathbf{x}_0) = 2\mathbf{v}_1 \bullet \nabla f(\mathbf{x}_0) - 5\mathbf{v}_2 \bullet \nabla f(\mathbf{x}_0) = 2(2) - 5(-2) = 14.$$

4.3. First express $\mathbf{v} = s\mathbf{v}_1 + t\mathbf{v}_2 + u\mathbf{v}_3$, which gives $s = t = u = 1$. Then

$$\mathbf{v} \bullet \nabla f(\mathbf{x}_0) = (\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) \bullet \nabla f(\mathbf{x}_0) = \mathbf{v}_1 \bullet \nabla f(\mathbf{x}_0) + \mathbf{v}_2 \bullet \nabla f(\mathbf{x}_0) + \mathbf{v}_3 \bullet \nabla f(\mathbf{x}_0) = 5 + 3 + 2 = 10.$$

4.5. (a) First,

$$\frac{\partial f}{\partial x} = \frac{y(1+x^2+y^2)^2 - xy \cdot 2(1+x^2+y^2) \cdot 2x}{(1+x^2+y^2)^4} = \frac{y(1+x^2+y^2-4x^2)}{(1+x^2+y^2)^3} = \frac{y(1-3x^2+y^2)}{(1+x^2+y^2)^3},$$

and by symmetry, $\dfrac{\partial f}{\partial y} = \dfrac{x(1+x^2-3y^2)}{(1+x^2+y^2)^3}$.

Then $\dfrac{\partial f}{\partial x} = 0$ if and only if $y = 0$ or $3x^2 - y^2 = 1$; and $\dfrac{\partial f}{\partial y} = 0$ if and only if $x = 0$ or $3y^2 - x^2 = 1$. There are four combinations:

$x = y = 0$: critical point $(0,0)$, $f(0,0) = 0$

$3y^2 - x^2 = 1,\ y = 0$: no solution

$3x^2 - y^2 = 1, x = 0$: no solution

$3x^2 - y^2 = 1 = 3y^2 - x^2$: $x^2 = y^2 = 1/2$, four critical points $(\pm\sqrt{1/2}, \pm\sqrt{1/2})$.

$f(\sqrt{1/2}, \sqrt{1/2}) = 1/8 = f(-\sqrt{1/2}, -\sqrt{1/2})$ and $f(-\sqrt{1/2}, \sqrt{1/2}) = -1/8 = f(\sqrt{1/2}, -\sqrt{1/2})$.

(b) The 5 critical points in (a) are the only possible maximizers and minimizers. $(0,0)$ is obviously out as a maximizer or minimizer, so the only possible maximizers are $\pm(\sqrt{1/2}, \sqrt{1/2})$, and the maximum value of f, if it exists, is $1/8$.

Here is an argument why the points $P = (\sqrt{1/2}, \sqrt{1/2})$ and $Q = (-\sqrt{1/2}, -\sqrt{1/2})$ are both actually maximizers. Consider first the region A defined by $x^2 + y^2 \geq 9$, the closed exterior of a circle of radius 3. Using the inequality $|xy| \leq 2|xy| \leq x^2 + y^2$, we find that on A,

$$|f(x,y)| = \frac{|xy|}{(1+x^2+y^2)^2} \leq \frac{(x^2+y^2)}{(x^2+y^2)^2} = \frac{1}{x^2+y^2} \leq \frac{1}{9} < \frac{1}{8}$$

So $f(P)$ and $f(Q)$ “beat” $f(x,y)$ for any $(x,y) \in A$.

On the set B defined by $x^2 + y^2 \leq 9$, f has maximizers because f is continuous and B is compact. The maximizers can’t be on the boundary of B, because we have already seen that P and Q beat such boundary points. Therefore the maximizers must occur at critical points. Therefore P and Q are maximizers of f with respect to B. Since they also beat every point outside B, they are genuine maximizers.

(c) A simlar argument shows that $(-\sqrt{1/2}, \sqrt{1/2})$ and $(\sqrt{1/2}, -\sqrt{1/2})$ are minimizers, and the minimum value of f is $-1/8$.

4.7. A normal vector to the tangent plane of f at $\mathbf{x}_0$ is given by $\mathbf{n}(\mathbf{x}_0) = \left(\frac{\partial f}{\partial x}(\mathbf{x}_0), \frac{\partial f}{\partial y}(\mathbf{x}_0), -1\right)$. At an arbitrary point $\mathbf{x}_0 = (x, y)$,

$$\mathbf{n} = (3y - 3x^2, 3x - 3y^2, -1).$$

For this normal vector to be parallel to $(3, 3, 1)$, $\mathbf{n}$ must equal $(-3, -3, -1)$. This leads to the equations

$$3y - 3x^2 = -3$$
$$3x - 3y^2 = -3$$

Thus $y = x^2 - 1$, $x = y^2 - 1 = (x^2 - 1)^2 - 1 = x^4 - 2x^2$, $x^4 - 2x^2 - x = 0$. One solution is $x = 0$, corresponding to $y = -1$; another is $x = -1$, corresponding to $y = 0$. Thus $x(x+1)$ is a factor of $x^4 - 2x^2 - x$, and we find

$$x^4 - 2x^2 - x = x(x+1)(x^2 - x - 1), \text{ with roots } x = 0, -1, \frac{1 \pm \sqrt{5}}{2}.$$

The corresponding values of y are given by $y = x^2 - 1$, and we have four points in all:

$$(0, -1),\ (-1, 0),\ \left(\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right),\ \left(\frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right).$$

4.9. Let $\nabla f(\mathbf{x}_0) = (a, b, c)$. We first find a, b, and c. The three given equations are

$$a + b + c = 5,\ b + c = 3, c = 2, \text{ so } (a, b, c) = (2, 1, 2).$$

We take $\mathbf{u}_1 = (1/3)(2, 1, 2)$. There are many choices for $\mathbf{u}_2$, which need only be a unit vector orthogonal to $\mathbf{u}_1$; then $\mathbf{u}_3$ must be $\mathbf{u}_1 \times \mathbf{u}_2$. For example, $\mathbf{u}_2 = (1/3)(1, 2, -2)$ and $\mathbf{u}_3 = (1/3)(-2, 2, 1)$.