3.3. (a) Two (or more) proofs are possible. The case $x_1 = y_1$ is obvious, so assume that $x_1 \neq y_1$. In fact, assume that $y_1 < x_1$; if the reverse inequality holds, then a similar argument can be made. Also, we only prove the first inequality; the second is similarly proved.

(a1)

$$|\sin(x_1) - \sin(y_1)| = \left|\int_{y_1}^{x_1} \cos t\, dt\right| \leq \int_{y_1}^{x_1} |\cos t|\, dt \leq \int_{y_1}^{x_1} 1\, dt = x_1 - y_1 = |x_1 - y_1|.$$

(a2) By the Mean Value Theorem, there is c such that $y_1 < c < x_1$ and

$$\frac{\sin(x_1) - \sin y_1}{x_1 - y_1} = \cos c.$$

Therefore the absolute value of the left side is at most 1, which implies the desired ineequality.

(b) Let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$. The key ingredients are the Triangle inequality, (3.21), and at the end, Cauchy-Schwarz.

$$\begin{aligned}
|f(\mathbf{x}) - f(\mathbf{y})| &= |(\sin(x_1) - \sin(y_1))\cos(x_2) + \sin(y_1)(\cos(x_2) - \cos(y_2))| \\
&\leq |\sin(x_1) - \sin(y_1)||\cos(x_2)| + |\sin(y_1)||\cos(x_2) - \cos(y_2)| \quad \text{(Triangle Inequality)} \\
&\leq |\sin(x_1) - \sin(y_1)| + |(\cos(x_2) - \cos(y_2))| \\
&\leq |x_1 - y_1| + |x_2 - y_2| \qquad (3.21) \\
&= (|x_1 - y_1|, |x_2 - y_2|) \bullet (1, 1) \\
&\leq \|(|x_1 - y_1|, |x_2 - y_2|)\|\|(1, 1)\| \qquad \text{Cauchy-Schwarz} \\
&= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}\sqrt{2} = \sqrt{2}\|\mathbf{x} - \mathbf{y}\|, \text{ Q.E.D.}
\end{aligned}$$

3.5. Yes, f is continuous. The justification uses the fact that

$$\lim_{t \to 0^+} t \ln t = 0,$$

a fact from elementary calculus that can be derived, for example, by using L'Hôpital's Rule for ∞/∞.

Using Theorem 29 and the continuity of the ln function, one sees that f is certainly continuous at any $\mathbf{x} \neq \mathbf{0}$. The sticky point is to prove continuity at $\mathbf{x} = \mathbf{0}$, that is, to prove that

(1) $$\lim_{\mathbf{x} \to \mathbf{0}} (x + y) \ln(x^2 + y^2) = 0.$$

We use the Squeeze Principle, first establishing

(2) $$|(x + y) \ln(x^2 + y^2)| \leq 4\|\mathbf{x}\|| \ln(\|\mathbf{x}\|)|$$

Namely,

$$\begin{aligned}|(x+y)\ln(x^2+y^2)| &= |x+y||\ln(\|\mathbf{x}\|^2)| \\ &\le (|x|+|y|)|2\ln\|\mathbf{x}\|| \\ &\le (\|\mathbf{x}\|+\|\mathbf{x}\|)|2\ln\|\mathbf{x}\|| \\ &= 4\|\mathbf{x}\||\ln\|\mathbf{x}\||.\end{aligned}$$

This proves (2). By the limiting fact above, the right side of (2) approaches 0 as $\mathbf{x} \to \mathbf{0}$ and hence by the Squeeze Principle, (1) holds. Q.E.D.

3.7. f is continuous for any given $r > 0$. As usual, Theorem 29 implies that f is continuous at any $\mathbf{x} = (x, y)$ as long as $x \neq 0$. Note that continuity is in question here at each point of the y-axis, not just at a single point. We take an arbitrary point $(0, y_0)$ on the y-axis.

Suppose that $r > 0$. Then for all (x, y),

$$|f(x,y)| = \begin{cases} |x|^r|\sin x| \le |x|^r & \text{if } x \neq 0 \\ 0 = |x|^r & \text{if } x = 0 \end{cases}$$

Hence $|f(x, y)| \le |x|^r$ for all (x, y). Since $r > 0$, $\lim_{x\to 0}|x|^r = 0$ and so by the Squeeze Principle,

$$\lim_{\mathbf{x}\to(0,y_0)} f(x,y) = 0 = f(0,y_0)$$

so f is continuous at $(0, y_0)$. As $(0, y_0)$ was arbitrary, f is continuous at every point of the y-axis and hence at every point of $\mathbf{R}^2$.

3.9. The closed unit ball $\overline{B}$ is closed and bounded so it's compact. The function f is continuous, so f has a maximizer and minimizer on $\overline{B}$ by Theorem 34.

More specifically, observe that f is a sum of squares, so $f(\mathbf{x}) \ge 0$ for all $\mathbf{x}$ in R^n. But $f(\mathbf{0}) = \sum(\mathbf{a}_i \bullet \mathbf{0})^2 = 0$, so $\mathbf{x} = \mathbf{0}$ is a minimizer of f on $\mathbf{R}^n$, hence on both B and $\overline{B}$.

Also, f is *homogeneous of degree* 2, that is, for any scalar λ,

$$f(\lambda\mathbf{x}) = \lambda^2 f(\mathbf{x}).$$

This is because

$$f(\lambda\mathbf{x}) = \sum\left[\mathbf{a}_i \bullet (\lambda\mathbf{x})\right]^2 = \sum\left[\lambda(\mathbf{a}_i \bullet \mathbf{x})\right]^2 = \sum\lambda^2(\mathbf{a}_i \bullet \mathbf{x})^2 = \lambda^2\sum(\mathbf{a}_i \bullet \mathbf{x})^2 = \lambda^2 f(\mathbf{x}).$$

Now we can show there's no maximizer of f on B. Suppose that there were such a maximizer $\mathbf{M}$. Clearly $\mathbf{M} \neq \mathbf{0}$. Let $c = \|\mathbf{M}\| > 0$. Since $\mathbf{M} \in B$, $c < 1$ (strict inequality!). Choose any number b such that $c < b < 1$ (such as $b = (1 + c)/2$) and let $\mathbf{z} = (b/c)\mathbf{M}$. Then $\|\mathbf{z}\| = |b/c|\|\mathbf{M}\| = b < 1$ so $\mathbf{z} \in B$. Moreover

$$f(\mathbf{z}) = f((b/c)\mathbf{M}) = (b/c)^2 f(\mathbf{M}) > f(\mathbf{M}).$$

Therefore $\mathbf{M}$ is not really a maximizer, a contradiction. So no maximizer exists on B.