Solutions to odd problems 1-13, Chapter 2

**2.1.** (a) $\mathbf{v}(t) = \mathbf{x}'(t) = (1, 2t)$ and $\mathbf{a}(t) = \mathbf{v}'(t) = (0, 2)$.

(b) $v(t) = \|\mathbf{v}(t)\| = \sqrt{1+4t^2}$ and $\mathbf{T}(t) = (1/v(t))\mathbf{v}(t) = (1/\sqrt{1+4t^2})(1, 2t)$.

(c) At $t = 1$, $\mathbf{v}(1) = (1, 2)$ is a direction vector for the tangent line, which passes through $\mathbf{x}(1) = (2, 1)$. The tangent line is $\mathbf{x} = (2 + t, 1 + 2t)$. Other parametrizations are possible, as usual.

**2.3.** We can write $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))$ and $\mathbf{y}(t) = (y_1(t), \ldots, y_n(t))$. Then

$$\mathbf{x}(t) \bullet \mathbf{y}(t) = x_1(t)y_1(t) + \cdots + x_n(t)y_n(t).$$

Since $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are assumed continuous, each of the real-valued functions $x_i(t)$ and $y_i(t)$ is continuous, by the theorem on page 63, line 4. By elementary calculus the product and sum of real-valued continuous functions of one variable are continuous. Therefore $\mathbf{x}(t) \bullet \mathbf{y}(t)$ is continuous.

Suppose that $n = 3$. Write $\mathbf{x}(t) \times \mathbf{y}(t) = (z_1(t), z_2(t), z_3(t))$. By the same theorem on page 63 it suffices to show that $z_1, z_2$, and $z_3$ are continuous. But $z_1(t) = x_2(t)y_3(t) - x_3(t)y_2(t)$. As above, this product-sum combination of real-valued functions of one variable is continuous, by elementary calculus. Similarly $z_2(t)$ and $z_3(t)$ are continuous, so $\mathbf{x}(t) \times \mathbf{y}(t)$ is continuous. Q.E.D.

**2.5.** (a) $s'(t) = v(t) = \|\mathbf{v}(t)\| = \|\mathbf{x}'(t)\| = \|(e^t(\cos t - \sin t), e^t(\sin t + \cos t), e^t)\| = \sqrt{3}e^t$. Therefore taking the reference point at $t = 0$,

$$s = s(t) - s(0) = \int_0^t \sqrt{3}e^t \, dt = \sqrt{3}(e^t - 1).$$

Consequently $e^t = 1 + (s/\sqrt{3})$ and $t = t(s) = \ln(1 + (s/\sqrt{3}))$. So

$$\mathbf{x}(s) = \left(\left(1 + \frac{s}{\sqrt{3}}\right) \cos\ln\left(1 + \frac{s}{\sqrt{3}}\right), \left(1 + \frac{s}{\sqrt{3}}\right) \sin\ln\left(1 + \frac{s}{\sqrt{3}}\right), 1 + \frac{s}{\sqrt{3}}\right).$$

(b) $\mathbf{x}''(t) = \mathbf{a}(t) = (-2e^t \sin t, 2e^t \cos t, e^t)$. We compute

$$\begin{aligned}
\mathbf{v} \times \mathbf{a} &= e^{2t}(\cos t - \sin t, \sin t + \cos t, 1) \times (-2\sin t, 2\cos t, 1) \\
&= e^{2t}(\sin t - \cos t, -\cos t - \sin t, 2), \\
\|\mathbf{v} \times \mathbf{a}\| &= \sqrt{6}e^{2t}, \\
\kappa = \kappa(t) &= \frac{\|\mathbf{v} \times \mathbf{a}\|}{v^3} = \frac{\sqrt{6}e^{2t}}{(\sqrt{3}e^t)^3} = \frac{\sqrt{2}}{3}e^{-t}, \\
\mathbf{a}' \bullet (\mathbf{v} \times \mathbf{a}) &= (-2e^t(\sin t + \cos t), 2e^t(\cos t - \sin t), e^t) \bullet e^{2t}(\sin t - \cos t, -\cos t - \sin t, 2) = 2e^{3t}, \\
\tau &= \frac{\mathbf{a}' \bullet (\mathbf{v} \times \mathbf{a})}{v^6\kappa^2} = \frac{2e^{3t}}{(\sqrt{3}e^t)^6(2/9)e^{-2t}} = \frac{1}{3}e^{-t}
\end{aligned}$$

(c) The vector $\mathbf{v} \times \mathbf{a}$ is normal to the osculating plane. At $t = 0$, $\mathbf{v} \times \mathbf{a} = (-1, -1, 2)$ and the plane passes through $\mathbf{x}(0) = (1, 0, 1)$, giving the equation $-x - y + 2z = 1$.

**2.7.** (a) $\mathbf{v} = \mathbf{x}'(t) = (1, t, t^2)$ and $\mathbf{a} = \mathbf{v}'(t) = (0, 1, 2t)$. At $t = 1$, $\mathbf{v} = (1, 1, 1)$ and $\mathbf{a} = (0, 1, 2)$, so a normal to the osculating plane is $\mathbf{v} \times \mathbf{a} = (1, -2, 1)$. The osc. plane passes through $\mathbf{x}(0) = (1, 1/2, 1/3)$ and so the osc. plane is

$$x - 2y + z = 1/3.$$

(b) Using $\mathbf{x}_1 = (0, 0, 0)$ and and $\mathbf{x}_2 = \mathbf{x}(0) = (1, 1/2, 1/3)$, and $\mathbf{n} = (1, -2, 1)$ as normal to the plane, the distance is

$$\frac{|(\mathbf{x}_1 - \mathbf{x}_2) \bullet \mathbf{n}|}{\|\mathbf{n}\|} = \frac{1}{3\sqrt{6}}.$$

**2.9.** (a) First, for any $t$, if $x = a\cos t$ and $y = b\sin t$, then $(x/a)^2 + (y/b)^2 = \cos^2 t + \sin^2 t = 1$. Therefore the given parametrization lands on the given ellipse.

Conversely, for any $(x, y)$ on the ellipse, that is, for any $x$ and $y$ such that $(x/a)^2 + (y/b)^2 = 1$, the point $(x/a, y/b)$ is on the unit circle, so there exists $t$ such that $x/a = \cos t$ and $y/b = \sin t$. Therefore for that $t$, $(x, y) = (a\cos t, b\sin t)$. This shows that the alleged parametrization indeed passes through each point of the ellipse, so it is a parametrization of the *entire* ellipse.

(b) For this calculation $\mathbf{x} = (a\cos t, b\sin t, 0)$, $\mathbf{x}' = (-a\sin t, b\cos t, 0)$, $\mathbf{x}'' = (-a\cos t, -b\sin t, 0)$, $v = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$, $\mathbf{x}' \times \mathbf{x}'' = (0, 0, ab)$, so

$$\kappa = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}.$$

So $\kappa$ is minimized (resp. maximized) when $A := a^2\sin^2 t + b^2\cos^2 t$ is maximized (resp. minimized).

Suppose first that $a < b$. Then

$$a^2 = a^2\cos^2 t + a^2\sin^2 t \le A = a^2\cos^2 t + b^2\sin^2 t \le b^2\cos^2 t + b^2\sin^2 t = b^2.$$

So the maximum $\kappa$ occurs for $\sin^2 t = 0$, $\cos^2 t = 1$, i.e., at $t = 0$ and $\pi$, i.e., $(x, y) = (\pm a, 0)$; the maximum value is $\kappa = b/a^2$. Similarly the minimum $\kappa$ occurs at $(0, \pm b)$ and is $a/b^2$.

If $a > b$, these are still the critical points for $\kappa$ but the maximum and minimum points are reversed.

Finally if $a = b$ then the ellipse is a circle and $\kappa = 1/a = 1/b$ is constant.

**2.11.** Any parametrization may be used; we use $x = t + 1$, $y = t^2$, and $-1 \le t \le 0$ to go from $(0, 1)$ to $(1, 0)$. Then $v = \sqrt{1 + 4t^2}$ and as $s' = v$, the desired arc length is

$$s(0) - s(-1) = \int_{-1}^{0} \sqrt{1 + 4t^2}\, dt = \frac{1}{2}\int_{\arctan(-2)}^{0} \sec^3 u\, du = \frac{1}{2}\int_{0}^{\arctan 2} \sec^3 u\, du$$

$$= \frac{1}{4}\Big[\sec u \tan u + \ln|\sec u + \tan u|\Big]\Big|_0^{\arctan 2} = \frac{1}{4}\big(2\sqrt{5} + \ln(\sqrt{5} + 2)\big)$$

**2.13.** See Example 38; but it is not necessary to make a detailed calculation of $\mathbf{x}(t)$. The curve $\mathbf{x}(t)$ is a circle, in a plane orthogonal to $\mathbf{b}$, and traversed with constant angular speed $\|\mathbf{b}\| = 3\, rad/unit\ time$. As $t$ goes from 0 to $\pi$, therefore, $3\pi$ radians are traversed, so the arc length is 1.5 times the circumference, and $\mathbf{x}(\pi)$ is the point on the circle opposite the initial point. The vector from the center of the circle to the initial point $\mathbf{x}(0)$ is $\mathbf{r} = \mathbf{x}(0)_{\perp\mathbf{b}}$, which we calculate to be $\mathbf{r} = (1/9)(-1, 4, -1)$. Thus the radius of the circle is

$$\|\mathbf{r}\| = \frac{\sqrt{2}}{3}.$$

(a) $\mathbf{x}(\pi) = \mathbf{x}(0) - 2\mathbf{r} = (1/9)(11, 1, 11)$, and the arc length is $3\pi\|\mathbf{r}\| = \pi\sqrt{2}$.

(b) Since the path is a circle, the curvature is constant, $\kappa = 1/\|\mathbf{r}\| = 3/\sqrt{2}$, and since the path is planar, the torsion is $\tau = 0$.