Solutions to Odd-Numbered Problems 17-31, Chapter 1

1.17. (a) Use any point on the plane for $\mathbf{x}_0$, and any two nonparallel vectors in the plane for $\mathbf{v}_1$ and $\mathbf{v}_2$. For instance, $\mathbf{x}_0 = \mathbf{p}_1 = (-2, 0, 2)$, $\mathbf{v}_1 = \mathbf{p}_2 - \mathbf{p}_1 = (3, -2, 0)$, $\mathbf{v}_2 = \mathbf{p}_3 - \mathbf{p}_1 = (5, -1, -4)$ gives

$$\mathbf{x}(s, t) = (-2, 0, 2) + s(3, -2, 0) + t(5, -1, -4)$$

or equivalently

$$\begin{aligned} x &= -2 + 3s + 5t \\ y &= -2s - t \\ z &= 2 - 4t. \end{aligned}$$

There are many correct parametrizations (as in (b), too).

(b) Most convenient is to use $\mathbf{w} = \mathbf{z}_1 - \mathbf{z}_0 = (-1, -7, 3)$. This gives

$$\mathbf{z}(u) = (1, 4, -2) + u(-1, -7, 3)$$

or equivalently

$$\begin{aligned} x &= 1 - u \\ y &= 4 - 7u \\ z &= -2 + 3u. \end{aligned}$$

(c) Using $\mathbf{v}_1$ and $\mathbf{v}_2$ from (a) we compute a normal vector

$$\mathbf{N} = \mathbf{v}_1 \times \mathbf{v}_2 = (8, 12, 7)$$

and the equation $8x + 12y + 7z = -2$ (the right side obtained as $8x + 12y + 7z\Big|_{\mathbf{x}=\mathbf{p}_1}$).

(d) Many answers are possible here. One approach is to equate the three expressions for u from (b) to get

$$1 - x = (4 - y)/7 = (z + 2)/3.$$

Another approach is to use the form $\mathbf{w} \times (\mathbf{z} - \mathbf{z}_0) = \mathbf{0}$, with $\mathbf{w}$ the direction vector in (b) and $\mathbf{z} = (x, y, z)$. This yields

$$(-1, -7, 3) \times (x - 1, y - 4, z + 2) = \mathbf{0}$$

or equivalently

$$\begin{aligned} -3y - 7z &= 2 \\ 3x + z &= 1 \\ 7x - y &= 3. \end{aligned}$$

(e) Putting (b) into (c) gives $8(1-u)+12(4-7u)+7(-2+3u)=-2$, $u=44/71$. The point of intersection is

$$\mathbf{z}(44/71)=(1/71)(27,-24,-10).$$

(f) By (1.64) the distance is $D=\|(\mathbf{p}_1-\mathbf{z}_0)_\perp\|$ where $\perp$ means $\perp$ $\mathbf{w}$ and $\mathbf{w}$ is a direction vector, as in (b). Here any point on the line could be used in place of $\mathbf{z}_0$. Now $\mathbf{p}_1-\mathbf{z}_0=(-3,-4,4)$ and $\mathbf{w}=(-1,-7,3)$. So D is the length of

$$(-3,-4,4)-\frac{(-3,-4,4)\bullet(-1,-7,3)}{(-1,-7,-3)\bullet(-1,-7,3)}(-1,-7,3)=(-3,-4,4)-\frac{43}{59}(-1,-7,3).$$

(g) This distance D is the length of $(\mathbf{z}_0-\mathbf{x}_0)_\|$, where $\|$ means $\|\mathbf{N}$, $\mathbf{N}$ being a normal vector. Now $\mathbf{z}_0-\mathbf{x}_0=(3,4,-4)$ and we can use $\mathbf{N}=(8,12,7)$:

$$D=\frac{|(\mathbf{z}_0-\mathbf{x}_0)\bullet\mathbf{N}|}{\|\mathbf{N}\|}=\frac{44}{\sqrt{257}}.$$

1.19. (a) Equating the expressions for $\mathbf{x}$ in the two parametrizations gives

$$\begin{aligned}1+t&=x=2-s\\ 1-t&=y=s\\ 2t&=z=2.\end{aligned}$$

Solving for s and t gives $t=1$, $s=0$, and we check, as we must, that these values solve *all three* equations. The point of intersection is then $(2,0,2)+0(-1,1,0)=(2,0,2)$ (which equals $(1,1,0)+1(1,-1,2)$).

(b) As a normal vector for P we can use the cross product of the two direction vectors: $\mathbf{N}=(1,-1,2)\times(-1,1,0)=(-2,-2,0)$. Using the point $(1,1,0)$ leads to $-2x-2y=-4$, or $x+y=2$.

1.21. Choose any point $\mathbf{q}$ in the plane, say $\mathbf{q}=(2,0,0)$. Then the distance from $\mathbf{p}$ to the plane is

$$\frac{|(\mathbf{p}-\mathbf{q})\bullet\mathbf{N}|}{\|\mathbf{N}\|}=\frac{|(-4,-5,1)\bullet(1,-3,1)|}{\sqrt{11}}=\frac{12}{\sqrt{11}}.$$

1.23. Let the reflected line be ℓ_1, and let P be the plane $x+2y-z=1$. First, to get one point $\mathbf{p}_1$ on ℓ_1, note that the point of intersection of ℓ and P lies on ℓ_1. This point satisfies the equations for ℓ and P, i.e., the system

$$\begin{aligned}x-3y+z&=2\\ 2y+z&=3\\ x+2y-z&=1.\end{aligned}$$

This gives $x = 16/9$, $y = 5/9$, $z = 17/9$, so $\mathbf{p}_1 = (1/9)(16, 5, 17)$.

Next we get a direction vector $\mathbf{a}$ for ℓ, and reflect it in P to get a direction vector $\mathbf{a}_1$ for ℓ_1. The vector $\mathbf{a}$ lies in the two planes corresponding to the first two equations above, so it's orthogonal to the normal vectors for both those planes. We can therefore use

$$\mathbf{a} = (1, -3, 1) \times (0, 2, 1) = (-5, -1, 2)$$

We want to reflect this in the plane given by the third equation above, for which a unit normal vector is $\mathbf{u} = (1/\sqrt{6})(1, 2, -1)$. Our desired direction vector is therefore

$$\mathbf{a}_1 = \mathbf{h}_{\mathbf{u}}(\mathbf{a}) = \mathbf{a} - 2(\mathbf{a} \bullet \mathbf{u})\mathbf{u} = (2, 5, -1).$$

We have all the ingredients; the line ℓ_1 is parametrized by $\mathbf{x} = \mathbf{p}_1 + t\mathbf{a}_1$, or

$$\mathbf{x} = (1/9)(16, 5, 17) + t(2, 5, -1)$$

1.25. See Example 21 and Theorem 13. A unit vector $\mathbf{u}$ orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$ can be obtained first:

$$\mathbf{v}_1 \times \mathbf{v}_2 = (1, 0, -1) \times (2, 1, 1) = (1, -3, 1), \text{ so } \mathbf{u} = \frac{1}{\sqrt{11}}(1, -3, 1)$$

and then the distance between the lines is

$$|(\mathbf{x}_1 - \mathbf{x}_2) \bullet \mathbf{u}| = \frac{1}{\sqrt{11}}|(0, 3, 1) \bullet (1, -3, 1)| = \frac{8}{\sqrt{11}}.$$

1.27. As on pp. 47-48, the key is to obtain an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ of $\mathbf{R}^3$ such that

$$\mathbf{u}_1 \perp \mathbf{v}_1, \ \mathbf{u}_1 \perp \mathbf{v}_2, \text{ and } \mathbf{u}_2 \perp \mathbf{v}_1.$$

Then we can use formulae (1.68) and (1.69) (where $\mathbf{b} = \mathbf{x}_1 - \mathbf{x}_2$) to find the values of the parameters s and t giving the "nearest neighbors" on the two lines. Notice that since $\mathbf{v}_1$ will be perpendicular to $\mathbf{u}_1$ and $\mathbf{u}_2$, it will be parallel to $\mathbf{u}_3$, so we can use

$$\mathbf{u}_3 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \frac{1}{\sqrt{35}}(3, -5, -1).$$

Next, to get $\mathbf{u}_1$,

$$\mathbf{v}_1 \times \mathbf{v}_2 = (3, -5, -1) \times (-1, 3, 3) = (-12, -8, 4), \text{ so}$$
$$\mathbf{u}_1 = \frac{1}{\sqrt{224}}(-12, -8, 4) = \frac{1}{\sqrt{14}}(-3, -2, 1).$$

Finally

$$\mathbf{u}_2 = \mathbf{u}_3 \times \mathbf{u}_1 = \frac{1}{\sqrt{10}}(-1, 0, -3).$$

Next, $\mathbf{b} = \mathbf{x}_1 - \mathbf{x}_2 = (2, 1, 2)$, and we use (1.68) and (1.69) to find the critical values s_0 and t_0 of s and t, respectively.

$\mathbf{b} \bullet \mathbf{u}_2 = (2, 1, 2) \bullet (1/\sqrt{10})(-1, 0, -3) = -8/\sqrt{10}$ and $\mathbf{v}_2 \bullet \mathbf{u}_2 = (-1, 3, 3) \bullet (1/\sqrt{10})(-1, 0, -3) = -8/\sqrt{10}$, so by (1.68)

$$t_0 = \frac{\mathbf{b} \bullet \mathbf{u}_2}{\mathbf{v}_2 \bullet \mathbf{u}_2} = 1 \qquad !!$$

Also we calculate $\mathbf{b} \bullet \mathbf{u}_3 = -1/\sqrt{35}$, $\mathbf{v}_1 \bullet \mathbf{u}_3 = \sqrt{35}$, and $\mathbf{v}_2 \bullet \mathbf{u}_3 = -21/\sqrt{35}$. So by (1.69)

$$s_0 = \frac{1(\mathbf{v}_2 \bullet \mathbf{u}_3) - \mathbf{b} \bullet \mathbf{u}_3}{\mathbf{v}_1 \bullet \mathbf{u}_3} = -\frac{4}{7}.$$

The point on the first line closest to the second line is

$$\mathbf{x}_1 + s_0 \mathbf{v}_1 = \mathbf{x}_1 - \frac{4}{7}\mathbf{v}_1 = \left(\frac{9}{7}, \frac{34}{7}, \frac{11}{7}\right).$$

The point on the second line closest to the first is

$$\mathbf{x}_2 + t_0 \mathbf{v}_2 = \mathbf{x}_2 + \mathbf{v}_2 = (0, 4, 2).$$

The distance between the two lines is the distance between these two points,

$$\|(1/7)(9, 34, 11) - (0, 4, 2)\| = \frac{1}{7}\|(9, 6, -3)\| = \frac{3}{7}\sqrt{14}.$$

This distance is, as a check, equal to $|\mathbf{b} \bullet \mathbf{u}_1|$, as it must be.

1.29. (a) One checks that $\mathbf{u}_1 \bullet \mathbf{u}_1 = \mathbf{u}_2 \bullet \mathbf{u}_2 = \mathbf{u}_3 \bullet \mathbf{u}_3 = 1$ and $\mathbf{u}_1 \bullet \mathbf{u}_2 = \mathbf{u}_1 \bullet \mathbf{u}_3 = \mathbf{u}_2 \bullet \mathbf{u}_3 = 0$, so it is an orthonormal basis. (Actually it is right-handed, because $\mathbf{u}_1 \times \mathbf{u}_2 = \mathbf{u}_3$.)

(b) Use Lemma 3 and in particular Equation (1.34). Thus, such a vector $\mathbf{u}$ is given by

$$\mathbf{u} = \frac{1}{\|\mathbf{u}_1 - \mathbf{e}_1\|}(\mathbf{u}_1 - \mathbf{e}_1) = \frac{1}{\sqrt{3}}(-1, 1, -1).$$

(c) $\mathbf{h}_{\mathbf{u}}(\mathbf{u}_2) = \mathbf{u}_2 - 2(\mathbf{u}_2 \bullet \mathbf{u})\mathbf{u} = (0, 1, 0) = \mathbf{e}_2$.

Since Householder reflections preserve dot products, hence lengths and angles, we know *in advance* that $\mathbf{h}_{\mathbf{u}}(\mathbf{u}_1), \mathbf{h}_{\mathbf{u}}(\mathbf{u}_2), \mathbf{h}_{\mathbf{u}}(\mathbf{u}_3)$ form an orthonormal basis. At this point, we know that the first two are $\mathbf{e}_1$ and $\mathbf{e}_2$, so the third one must be $\pm\mathbf{e}_3$. A calculation like the one for $\mathbf{u}_2$ shows that $\mathbf{h}_{\mathbf{u}}(\mathbf{u}_3) = -\mathbf{e}_3$.

(Remark: Thus $\mathbf{h}_{\mathbf{u}}$, like all Householder reflections, takes a right-handed ON basis to a left-handed one.)

1.31. (a) It's not an orthonormal basis; $\mathbf{u}_1 \bullet \mathbf{u}_3 \neq 0$.

(b) Same as 1.29(b).

(c) $\mathbf{h}_{\mathbf{u}}(\mathbf{u}_2) = \mathbf{u}_2$ because $\mathbf{u}_2 \perp \mathbf{u}$. Also $\mathbf{h}_{\mathbf{u}}(\mathbf{u}_3) = \mathbf{u}_3 - 2(\mathbf{u}_3 \bullet \mathbf{u})\mathbf{u} = \frac{1}{9\sqrt{2}}(8, 7, -7)$.