1.1. The equation $\mathbf{a} = s\mathbf{b} + t\mathbf{c}$ is equivalent to the system

$$\begin{aligned} 2s + t &= 3 \\ s + 3t &= -1 \end{aligned}$$

whose solution is $s = 2$, $t = -1$. So $\mathbf{a} = 2\mathbf{b} - \mathbf{c}$.

1.3. $\|\mathbf{x}\| = \sqrt{81} = 9$, $\|\mathbf{y}\| = \sqrt{9} = 3$. The angle θ between $\mathbf{x}$ and $\mathbf{y}$ (in the range $0 \le \theta \le \pi$) is

$$\theta = \arccos \frac{\mathbf{x} \bullet \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} = \arccos \frac{-7}{9 \cdot 3} = \arccos \frac{-7}{27}.$$

1.5. $\|\mathbf{x}\| = 9$, $\|\mathbf{y}\| = 3$, $\theta = \arccos \frac{23}{27}$.

1.7. (a) Since $1^2 + 4^2 + 8^2 = 4^2 + 7^2 + 4^2 = 81 = 9^2$, $\mathbf{u}_1 \bullet \mathbf{u}_1 = \frac{1}{9} \cdot \frac{1}{9}(81) = 1$ and similarly $\mathbf{u}_2 \bullet \mathbf{u}_2 = 1 = \mathbf{u}_3 \bullet \mathbf{u}_3$. Also

$$\begin{aligned} \mathbf{u}_1 \bullet \mathbf{u}_2 &= \frac{1}{81}(1, -4, -8) \bullet (8, 4, -1) = \frac{8 - 16 + 8}{81} = 0, \\ \mathbf{u}_1 \bullet \mathbf{u}_3 &= \frac{1}{81}(1, -4, -8) \bullet (4, -7, 4) = 0, \\ \mathbf{u}_2 \bullet \mathbf{u}_3 &= \frac{1}{81}(8, 4, -1) \bullet (4, -7, 4) = 0, \end{aligned}$$

so $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis of $\mathbf{R}^3$. Therefore $\mathbf{u}_1 \times \mathbf{u}_2 = \pm\mathbf{u}_3$. Moreover if $\mathbf{u}_1 \times \mathbf{u}_2 = +\mathbf{u}_3$, then it is right-handed, while if $\mathbf{u}_1 \times \mathbf{u}_2 = -\mathbf{u}_3$, it is left-handed (Theorem 9). In fact

$$\mathbf{u}_1 \times \mathbf{u}_2 = \frac{1}{81}\big[(1, -4, -8) \times (8, 4, -1)\big] = \left(\frac{36}{81}, *, *\right) = \frac{1}{9}(4, *, *),$$

so the + sign must be correct. Thus, $\mathbf{u}_1 \times \mathbf{u}_2 = +\mathbf{u}_3$ and the basis is right-handed.

(b) Let $\mathbf{v} = (10, 11, -11)$. Then since $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis of $\mathbf{R}^3$,

$$\mathbf{v} = \sum_{i=1}^{3} (\mathbf{v} \bullet \mathbf{u}_i)\mathbf{u}_i.$$

Thus, $y_1 = \mathbf{v} \bullet \mathbf{u}_1 = \frac{1}{9}(10 - 44 + 88) = 6$, $y_2 = \mathbf{v} \bullet \mathbf{u}_2 = 15$, $y_3 = \mathbf{v} \bullet \mathbf{u}_3 = -9$. As for the lengths, $\|(10, 11, -11)\| = \sqrt{342} = \|(6, 15, -9)\|$. One knows that the two lengths are equal *in advance* because $\|(10, 11, -11)\|^2 = y_1^2 + y_2^2 + y_3^2 = \|(y_1, y_2, y_3)\|^2$ by by the second equation in Theorem 5.

1.9. (a) To solve $\mathbf{x} \times \mathbf{a} = (-7, 2, 5)$, first check that $\mathbf{a} \perp (-7, 2, 5)$: namely, $\mathbf{a} \bullet (-7, 2, 5) = (1, 1, 1) \bullet (-7, 2, 5) = 0$. Rewriting the equation as $\mathbf{a} \times \mathbf{x} = -(-7, 2, 5) = (7, -2, -5)$, we have a particular solution

$$\mathbf{x} = -\frac{1}{\|\mathbf{a}\|^2}(\mathbf{a} \times (7, -2, -5)) = -\frac{1}{3}\big[(1, 1, 1) \times (7, -2, -5)\big] = -\frac{1}{3}(-3, 12, -9) = (1, -4, 3)$$

See the bottom half of page 37. Since $\mathbf{x} = \mathbf{a} \times$ something, it follows that $\mathbf{x} \perp \mathbf{a}$, that is, $\mathbf{x} \bullet \mathbf{a} = 0$. So $\mathbf{x} = (1, -4, 3)$ satisfies the requirements.

(b) Since $\mathbf{a} \bullet (1, 0, 0) = 1 \neq 0$, $\mathbf{a}$ is not perpendicular to $(1, 0, 0)$, so the equation $\mathbf{x} \times \mathbf{a} = (1, 0, 0)$ has no solution.

1.11. (a) The correct assertion to be proved is

$$\mathbf{b}_{2m} = (-1)^m \|\mathbf{a}\|^{2m} \mathbf{b} \text{ for all positive integers } m.$$

Proof. The proof is by induction.

Proof for $m = 1$: $\mathbf{b}_2 = \mathbf{a} \times \mathbf{b}_1 = \mathbf{a} \times (\mathbf{a} \times \mathbf{b}_0) = \mathbf{a} \times (\mathbf{a} \times \mathbf{b}) = -\|\mathbf{a}\|^2 \mathbf{b}$. (The last equation is valid only because $\mathbf{a} \perp \mathbf{b}$ by assumption.) Thus the statement holds for $m = 1$.

Assume the statement is true for some $m > 0$, and prove it for $m + 1$:

$$\begin{aligned} \mathbf{b}_{2(m+1)} &= \mathbf{b}_{2m+2} = \mathbf{a} \times \mathbf{b}_{2m+1} = \mathbf{a} \times (\mathbf{a} \times \mathbf{b}_{2m}) \\ &= -\|\mathbf{a}\|^2 \mathbf{b}_{2m} \end{aligned}.$$

The last equation is true because $\mathbf{a} \perp \mathbf{b}_{2m}$, which in turn is true because $\mathbf{b}_{2m} = \mathbf{a} \times \mathbf{b}_{2m-1}$. Continuing,

$$\begin{aligned} \mathbf{b}_{2(m+1)} &= -\|\mathbf{a}\|^2 \mathbf{b}_{2m} \\ &= -\|\mathbf{a}\|^2 (-1)^m \|\mathbf{a}\|^{2m} \mathbf{b} \text{ because the statement is assumed true for } m \\ &= (-1)^{m+1} \|\mathbf{a}\|^{2m+2} \mathbf{b} = (-1)^{m+1} \|\mathbf{a}\|^{2(m+1)} \mathbf{b}, \text{ Q.E.D.} \end{aligned}$$

(b) One way to generalize this to the case in which $\mathbf{a}$ is not assumed to be orthogonal to $\mathbf{b}$ is to use the fact that $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}_\perp$ (because $\mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{b}_\perp = \mathbf{a} \times (\mathbf{b} - \mathbf{b}_\perp) = \mathbf{a} \times \mathbf{b}_\| = \mathbf{0}$). Since $\mathbf{a}$ is orthogonal to $\mathbf{b}_\perp$, the result of (a) applies to $\mathbf{b}_\perp$ instead of $\mathbf{b}$ and so

$$\mathbf{b}_{2m} = (-1)^m \|\mathbf{a}\|^{2m} \mathbf{b}_\perp.$$

1.13. If $\mathbf{b} \times \mathbf{c} = \mathbf{0}$ then the left side is 0, while the right side is positive as $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are assumed nonzero. So the statement holds in this case. Assume then that $\mathbf{b} \times \mathbf{c} \neq \mathbf{0}$. Then $\mathbf{b}$ and $\mathbf{c}$ are not parallel. Let

θ be the angle between $\mathbf{b}$ and $\mathbf{c}$, and let ϕ be the angle between $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$. Therefore $0 < \theta < \pi$ and $0 \leq \phi \leq \pi$. Now

$$\begin{aligned} |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| &= \|\mathbf{a}\| \|\mathbf{b} \times \mathbf{c}\| |\cos\phi| = \|\mathbf{a}\| (\|\mathbf{b}\| \|\mathbf{c}\| \sin\theta) |\cos\phi| \\ &= \|\mathbf{a}\| \|\mathbf{b}\| \|\mathbf{c}\| |\cos\phi| \sin\theta \\ &\leq \|\mathbf{a}\| \|\mathbf{b}\| \|\mathbf{c}\|. \end{aligned}$$

Clearly equality holds if and only if $|\cos\phi| = \sin\theta = 1$, that is, if and only if $\phi = 0$ or π and $\theta = \pi/2$. This is true if and only if $\mathbf{a} \| \mathbf{b} \times \mathbf{c}$ and $\mathbf{b} \perp \mathbf{c}$, which is true if and only if $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are mutually orthogonal, which finally is true if and only if $\{(1/\|\mathbf{a}\|)\mathbf{a}, (1/\|\mathbf{b}\|)\mathbf{b}, (1/\|\mathbf{c}\|)\mathbf{c}\}$ is an orthonormal basis of $\mathbf{R}^3$.

1.15. For our orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ we start with

$$\mathbf{u}_3 = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{26}}(1, 4, 3).$$

Then for $\mathbf{u}_1$ we first take any nonzero vector $\mathbf{w}$ that is orthogonal to $\mathbf{v}$ – many choices are possible here – such as $(4, -1, 0)$. We take

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{w}\|}\mathbf{w} = \frac{1}{\sqrt{17}}(4, -1, 0).$$

Finally take

$$\mathbf{u}_2 = \mathbf{u}_3 \times \mathbf{u}_1 = \frac{1}{\sqrt{442}}(3, 12, -17)$$

(Note: because we took $\mathbf{u}_2 = \mathbf{u}_3 \times \mathbf{u}_1$ and not $\mathbf{u}_1 \times \mathbf{u}_3$, the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is right-handed.)