Solutions for Practice Test IB, Math 291 Spring 2011

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1: Let $\mathbf{a} = (1,1,1)$

(a) Find a vector $\mathbf{x}$ such that

$$\mathbf{x} \times \mathbf{a} = (-7,2,5) \qquad \text{and} \qquad \mathbf{x} \cdot \mathbf{a} = 0 \ .$$

(b) There is no vector $\mathbf{x}$ such that

$$\mathbf{x} \times \mathbf{a} = (1,0,0) \qquad \text{and} \qquad \mathbf{x} \cdot \mathbf{a} = 0 \ .$$

Show that no such vector exists.

SOLUTION: For any vector $\mathbf{x}$,

$$\mathbf{a} \times (\mathbf{a} \times \mathbf{x}) = -\|\mathbf{a}\|^2 \mathbf{x}_\perp \ ,$$

where $\mathbf{x}_\perp$ is the component of $\mathbf{x}$ orthogonal to $\mathbf{a}$. If $\mathbf{x} \cdot \mathbf{a} = 0$, this reduces to

$$\mathbf{x} = -\frac{1}{\|\mathbf{a}\|^2}(\mathbf{x} \times \mathbf{a}) \times \mathbf{a} = -\frac{1}{3}(-7,2,5) \times (1,1,1) = (-1,4,-3) \ .$$

For **(b)**, note that $(1,0,0)$ is not orthogonal to $\mathbf{a}$, and hence no vector can satisfy $\mathbf{x}\times\mathbf{a} = (1,0,0)$ since the cross product of rwo vectors is orthogonal to each of them.

2: Let P_1 denote the plane through the three points $\mathbf{a_1} = (1,2,1)$ $\mathbf{a_2} = (-1,2,-3)$ and $\mathbf{a_3} = (2,-3,-2)$. Let P_2 denote the plane through the three points $\mathbf{b_1} = (1,1,0)$ $\mathbf{b_2} = (1,0,1)$ and $\mathbf{b_3} = (0,1,1)$.

(a) Find equations for the planes P_1 and P_2.

(b) Parameterize the line given by $P_1 \cap P_2$, and find the distance between this line and the point $\mathbf{a_1}$.

(c) Consider the line through $\mathbf{b_1}$ and $\mathbf{b_2}$. Determine the point of intersection of this line with the plane P_1, and find a parametrization of the line given by reflecting this line off plane P_1. That is, the direction vector of the reflected line is what you get by reflecting the direction vector of the original line using the unit normal direction $\mathbf{u}$ to the plane.

SOLUTION: We compute

$$(\mathbf{a}_2 - \mathbf{a}_2) \times (\mathbf{a}_3 - \mathbf{a}_1) = 10(-2, -1, 1) \ .$$

Hence the equation is

$$(-2, -1, 1) \cdot (x - 1, y - 2, z - 1) = 0$$

which reduces to

$$2x + y - z = 3 \ .$$

In the same way, or by inspection using the symmetry, the equation of the other plane is

$$x + y + z = 2 \ .$$

For **(b)**, looking for a solution $\mathbf{x}_0$ of the system $2x + y - z = 3$ and $x + y + z = 2$ with $x = 0$, we find

$$\mathbf{x}_0 + \frac{1}{2}(0, 5, -1) \ .$$

The direction vector $\mathbf{v}$ is the cross product of the normal vectors to the planes: $\mathbf{v} = ((-2, -1, 1) \times (1, 1, 1) = (-2, 3, -1)$. Hence the parameterization is

$$\mathbf{x}(t) = \frac{1}{2}(0, 5, -1) + t(-2, 3, -1) \ .$$

The equation of the line then is

$$(-2, 3, -1) \times \left(\mathbf{x} - \frac{1}{2}(0, 5, -1)\right) = \mathbf{0} \ .$$

The distance in question therefore is

$$\frac{1}{\|(-2, 3, -1)\|} \left\| (-2, 3, -1) \times \left(\mathbf{a}_1 - \frac{1}{2}(0, 5, -1)\right) \right\| = \frac{\|4, 2, -2\|}{\|(-2, 3, -1)\|} = 2\sqrt{3/7} \ .$$

For **(c)**, the line through $\mathbf{b}_1$ and $\mathbf{b}_2$ has direction vector $\mathbf{w} := \mathbf{b}_2 - \mathbf{b}_1 = (0, -1, 1)$. Hence this line is parameterized by

$$\mathbf{x}(t) = (1, 1, 0) + t(0, -1, 1) = (1, 1 - t, t) \ .$$

Plugging this into the equation $2x + y - z = 3$ for P_1, we find $t = 0$; i.e., $(1, 1, 0)$ is on the line and on the plane P_1. So this is the reflection point.

The reflected line therefore has $(1, 1, 0)$ as its base point, and has $\mathbf{h}_\mathbf{u}(\mathbf{w})$ as its direction vector, when $\mathbf{u}$ is the unit normal to the plane P_1. Then since

$$\mathbf{u} = \frac{1}{\sqrt{6}}(2, 1, -1) \qquad \text{and} \qquad \mathbf{h}_\mathbf{u}(\mathbf{w}) = \mathbf{w} - 2(\mathbf{w} \cdot \mathbf{u})\mathbf{u} \ ,$$

we find

$$\mathbf{h}_\mathbf{u}(\mathbf{w}) = \frac{1}{3}(-4, 1, -1) \ .$$

Thus, the reflected line is given by

$$\mathbf{x}(t) = (1, 1, 0) + \frac{1}{3}(-4, 1, -1) \ .$$

3: Let $\mathbf{x}(t)$ be the curve given by

$$\mathbf{x}(t) = (e^t \cos(t), e^t \sin(t), e^t) \ .$$

(a) Compute the arc length $s(t)$ as a function of t, measured from the starting point $\mathbf{x}(0)$, and find an arc length parameterization of this curve

(b) Compute curvature $\kappa(t)$ as a function of t.

(c) Find an equation for the osculating plane at time $t = 0$

SOLUTION: We compute

$$\mathbf{v}(t) = \mathbf{x}'(t) = e^t(\cos t - \sin t, \cos t + \sin t, 1) \qquad \text{and hence} \qquad v(t) = \sqrt{3}e^t \ .$$

Thus,

$$s(t) = \int_0^t \sqrt{3}e^u \mathrm{d}u = \sqrt{3}(e^t - 1) \ ;$$

Solving $s = \sqrt{3}(e^t - 1)$, we find

$$t(s) = \ln\left(\frac{s}{\sqrt{3}} + 1\right) \ .$$

The arc length parameterization is then given by $\mathbf{x}(t(s))$, which is

$$\left(\frac{s}{\sqrt{3}} + 1\right)\left(\cos\left(\ln\left(\frac{s}{\sqrt{3}} + 1\right)\right), \sin\left(\ln\left(\frac{s}{\sqrt{3}} + 1\right)\right), 1\right) \ .$$

Next, to find the curvature, we compute

$$\mathbf{a}(t) = \mathbf{v}'(t) = e^t(-2\sin t, 2\cos t, 1)$$

and

$$\mathbf{v}(t) \times \mathbf{a}(t) = e^{2t}(\sin t - \cos t, -\sin t - \cos t, 2) \ .$$

Thus, $\|\mathbf{v}(t) \times \mathbf{a}(t)\| = 6e^{2t}$. The curvature is

$$\kappa(t) = \frac{\|\mathbf{v}(t) \times \mathbf{a}(t)\|}{v^3(t)} = \frac{2}{\sqrt{3}}e^{-t} \ .$$

Finally, the normal to the osculating plane at $t = 0$ is

$$\mathbf{v}(0) \times \mathbf{a}(0) = (-1, -1, 2) \ .$$

We take $\mathbf{x}(0) = (1, 0, 1))$ as the base point. The equation then is

$$-x - y + 2z = 1 \ .$$

4: **(a)** Let $f(x, y)$ be given by

$$f(x,y) = \begin{cases} \dfrac{xy}{|x| + |y|} & (x,y) \neq (0,0) \\ 0 & (x,y) \neq (0,0) \ . \end{cases}$$

Does

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

exist? If so, evaluate the limit. If not, explain why not.

(b) Let $g(x,y)$ be given by

$$g(x,y) = \begin{cases} \dfrac{x^2+y^2}{\sqrt{x^2+y^4+1}-1} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0)\ . \end{cases}$$

Does

$$\lim_{(x,y)\to(0,0)} g(x,y)$$

exist? If so, evaluate the limit. If not, explain why not.

SOLUTION: For **(a)**,

$$0 \le |f(x,y)| = \frac{|x||y|}{|x|+|y|} \le \frac{|x||y|}{|x|} = |y| \le \|\mathbf{x}\|\ .$$

Hence, by the squeeze principle,

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0\ .$$

For **(b)**, Note thas as (x,y) approaches $(0,0)$, both the numerator and the denominator tend to zero. Since y^4 goes to zero more quickly than y^2 as $\mathbf{x}$ tends to zero, let us look at what happens as $\mathbf{x}$ approaches $(0,0)$ along the y-axis:

$$g(0,y) = \frac{y^2}{\sqrt{1+y^4}-1}\ .$$

Two applications of l'Hospital's rule show that

$$\lim_{y\to 0} g(0,y) = \infty\ .$$

The function is not continuous at $(0,0)$.

5: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = x^2y + yx - xy^2\ .$$

(a) Compute the gradient of f, and find all critical points of f.

(b) Find the equation of the tangent plane to the graph f at the point $(1,1)$.

(c) Let $\mathbf{x}(t) = (1+t-t^3, 2-t+t^2)$. Compute $\left.\dfrac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x}(t))\right|_{t=0}$.

(d) Find all points (x,y) at which the tangent plane to the graph of f is orthogonal to the line parameterized by $t(1,0,1)$.

SOLUTION: For **(a)**, we compute

$$\nabla f(x,y) = ((2x-y+1)y\ , (x-2y+1)x)\ .$$

To find the critical points, we solve

$$\begin{aligned}(2x - y + 1)y &= 0\\ (x - 2y + 1)x &= 0\end{aligned}$$

We can solve the first equation by taking $y = 0$ or $y = 2x + 1$. If $y = 0$, the second equation becomes $(x + 1)x = 0$ which is solved by $x = -1$ and $x = 0$. Hence $(-1, 0)$ and $(0, 0)$ are critical points. If $y = 2x + 1$, the second equation becomes $(3x + 1)x = 0$, which is solved by $x = 0$ and $x = -1/2$. Thus $(0, 1)$ and $(-1/3, 1/3)$ are critical points. The four that we have listed above constitute the complete set of critical points.

For **(b)**, we compute $\nabla f(1, 1) = (2, 0)$ and $f(1, 1) = 1$ Hence

$$z = f(x, y) \approx 1 + (2, 0) \cdot (x - 1, y - 1) = 2x - 1 \ .$$

The equation is $z = 2x - 1$, or equivalently,

$$2x - z = 1 \ .$$

For **(c)** we compute

$$\mathbf{x}(0) = (1, 2) \qquad \mathbf{x}'(0) = (1, -1) \quad \text{and} \quad \nabla f(1, 2) = (2, -2) \ .$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x}(t))\bigg|_{t=0} = \nabla f(\mathbf{x}(0)) \cdot \mathbf{x}'(0) = (2, -2) \cdot (1, -1) = 4 \ .$$

For **(d)** we note that the normal vector to the tangent plane at $(x, y, f(x, y))$ is

$$((2x - y + 1)y \ , (x - 2y + 1)x) \ , \ -1) \ .$$

Setting this equal to $a(1, 0, 1)$, we see we must have $a = -1$, and hence

$$\begin{aligned}(2x - y + 1)y &= -1\\ (x - 2y + 1)x &= 0\end{aligned}$$

From the second equation we have that either $x = 0$ or $x = 2y - 1$. If $x = 0$, then the first equation becomes $(-y + 1)y = -1$ which has the roots $y = (1 \pm \sqrt{5})/2$. Hence we have the points

$$\left(0, \frac{1 - \sqrt{5}}{2}\right) \qquad \text{and} \qquad \left(0, \frac{1 + \sqrt{5}}{2}\right) \ .$$

If $x = 2y - 1$, the second equation becomes $3y^2 - y = 1$, which has no real roots. Hence there are only two such points, which are listed above.

Extra Credit: Let $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ be three vectors in $\mathbb{R}^3$ such that $\mathbf{w}_2 \times \mathbf{w}_3 \neq \mathbf{0}$. Consider the curve

$$\mathbf{x}(t) = t\mathbf{w}_1 + t^2\mathbf{w}_2 + t^3\mathbf{w}_3 \ .$$

Show that this curve is planar if $\mathbf{w}_1 \cdot \mathbf{w}_2 \times \mathbf{w}_3 = 0$, and for more extra credit, show that "if" can be upgraded to "if and only if".

SOLUTION: We compute

$$\mathbf{v}(t) = \mathbf{w}_1 + 2t\mathbf{w}_2 + 3t^2\mathbf{w}_3 \qquad \text{and} \qquad \mathbf{a}(t) = 2\mathbf{w}_2 + 6t\mathbf{w}_3 \ .$$

Now, $\mathbf{w}_1 \cdot \mathbf{w}_2 \times \mathbf{w}_3 = 0$ if and only if $\mathbf{w}_1$ lies in the plane spanned by $\mathbf{w}_2$ and $\mathbf{w}_3$. In this case, we see from the calculation obove that for all t, $\underline{(}t)$ and $\mathbf{a}(t)$ belong to the plane given by

$$(\mathbf{w}_2 \times \mathbf{w}_3) \cdot \mathbf{x} = 0 \ .$$

This means that

$$\mathbf{B}(t) = \pm \frac{\mathbf{w}_2 \times \mathbf{w}_3}{\|\mathbf{w}_2 \times \mathbf{w}_3\|} \ .$$

Hence the torsion is zero, and the curve is planar.

Conversely., if the curve is planar, for all t, $\mathbf{v}(t) \times \mathbf{a}(t)$ is a multiple of $\mathbf{v}(0) \times \mathbf{a}(0) = \mathbf{w}_1 \times \mathbf{w}_2$. But then $\mathbf{w}_2 \times \mathbf{w}_3$ and $\mathbf{w}_3 \times \mathbf{w}_1$ must also be multiples of $\mathbf{w}_1 \times \mathbf{w}_2$. Therefore,

$$(\mathbf{w}_2 \times \mathbf{w}_3) \cdot [(\mathbf{w}_3 \times \mathbf{w}_1) \times (\mathbf{w}_1 \times \mathbf{w}_2)] = 0 \ .$$

Now recall the identity proved in Exercise 1.37:

$$(\mathbf{b} \times \mathbf{c}) \cdot [(\mathbf{c} \times \mathbf{a}) \times (\mathbf{a} \times \mathbf{b})] = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|^2 \ .$$

Applying this inequality, we see that

$$|\mathbf{w}_1 \cdot (\mathbf{w}_2 \times \mathbf{w}_3)|^2 = 0.$$