Solutions for Practice Test I, Math 291 Fall 2012

October 16, 2012

1: (a) Find a right handed orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ such that $\mathbf{u}_1$ is a positive multiple of $(2,2,1)$ and $\mathbf{u}_2$ is orthogonal to $(1,1,0)$.

In the rest of this problem, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ refers to this orthonormal basis .

(b) Let $\mathbf{x}_1 = (1,2,3)$ and $\mathbf{x}_2 = (3,2,1)$. Consider the two lines $\mathbf{x}_1(s) = \mathbf{x}_1 + s\mathbf{u}_1$ and $\mathbf{x}_2(t) = \mathbf{x}_2 + t\mathbf{u}_2$. Compute numbers y_1, y_2 and y_3 such that

$$\mathbf{x}_1 - \mathbf{x}_2 = y_1\mathbf{u}_1 + y_2\mathbf{u}_2 + y_3\mathbf{u}_3 \ ,$$

and compute

$$\|\mathbf{x}_1(s) - \mathbf{x}_2(t)\|^2$$

as a function of s and t.

(c) Find the point on the line parameterized by $\mathbf{x}_1(s)$ that is closest to the line parameterized by $\mathbf{x}_2(t)$, and find point on the line parameterized by $\mathbf{x}_2(t)$ that is closest to the line parameterized by $\mathbf{x}_1(t)$.

SOLUTION: For **(a)** we normalize $(2,2,1)$ to get $\mathbf{u}_1 = \frac{1}{3}(2,2,1)$. We compute

$$(2,2,1) \times (1,1,0) = (-1,1,0) \ .$$

This vector is orthogonal to both $(2,2,1)$ and $(1,1,0)$, and so we take $\mathbf{u}_1 = \frac{1}{\sqrt{2}}(-1,1,0)$. (You could also use $\mathbf{u}_1 = -\frac{1}{\sqrt{2}}(-1,1,0)$ which you would get taking the cross product in the other order.)

Finally, continuing with the first choice for $\mathbf{u}_2$ made above, we take

$$\mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2 = \frac{1}{3\sqrt{2}}(-1,-1,4) \ .$$

For **(b)**, we compute $\mathbf{x}_1 - \mathbf{x}_2 = (-2,0,2)$, and then

$$\begin{aligned}
(\mathbf{x}_1 - \mathbf{x}_2)\cdot\mathbf{u}_1 &= -\frac{2}{3} \\
(\mathbf{x}_1 - \mathbf{x}_2)\cdot\mathbf{u}_2 &= \sqrt{2} \\
(\mathbf{x}_1 - \mathbf{x}_2)\cdot\mathbf{u}_3 &= \frac{5}{3}\sqrt{2} \ .
\end{aligned}$$

Therefore,

$$(y_1, y_2, y_3) = \left(-\frac{2}{3}\ ,\ \sqrt{2}\ ,\ \frac{5}{3}\sqrt{2}\right) \ .$$

It then follows that the coordinates of $\mathbf{x}_1(s) - \mathbf{x}_2(t)$ with respect to the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ are

$$(y_1 + s, y_2 - t, y_3) = \left(s - \frac{2}{3}\ ,\ \sqrt{2} - t\ ,\ \frac{5}{3}\sqrt{2}\right) .$$

Therefore,

$$\|\mathbf{x}_1(s) - \mathbf{x}_2(t)\|^2 = \left(s - \frac{2}{3}\right)^2 + \left(\sqrt{2} - t\right)^2 + \left(\frac{5}{3}\sqrt{2}\right)^2 .$$

(Do not expand this; completed squares are a good thing; leave them be.)

For **(c)** , it is evident from our computation above that $\|\mathbf{x}_1(s) - \mathbf{x}_2(t)\|$ is minimized by choosing $s = 2/3$ and $t = \sqrt{2}$. We then compute

$$\mathbf{x}_1(2/3) = (1, 2, 3) + \frac{2}{9}(2, 2, 1) = \frac{1}{9}(13, 22, 29) .$$

and

$$\mathbf{x}_2(\sqrt{2}) = (3, 2, 1) + (-1, 1, 0) = (2, 3, 1) .$$

These are the two closest points.

2: For $t > 0$, let $\mathbf{x}(t)$ be the curve given by

$$\mathbf{x}(t) = \left(2t^2 + 2t^3/3\ , -t^2 - 4t^3/3\ , 2t^2 - 4t^3/3\right)$$

(a) Compute the arc length along the curve between $\mathbf{x}(0)$ and $\mathbf{x}(t)$ as a function of t.

(b) Compute curvature $\kappa(t)$ as a function of t, and the binormal vector $\mathbf{B}(t)$. .

(c) Find the torsion $\tau(t)$ as a function of t, justifying your answer.

SOLUTION: For **(a)**, we compute

$$\mathbf{v}(t) = \mathbf{x}'(t) = (4t + 2t^2\ ,\ -2t - 4t^2\ ,\ 4t - 4t^2) ,$$

and hence

$$v(t) = \|\mathbf{v}(t)| = 6t\sqrt{1 + t^2} .$$

Hence the arc length is

$$\int_0^t 6r\sqrt{1 + r^2}\mathrm{d}r = 2[(1 + t^2)^{3/2} - 1] .$$

For **(b)**, we compute

$$\mathbf{a}(t) = (4 + 4t\ ,\ -2 - 8t\ ,\ 4 - 8t) ,$$

and then

$$\mathbf{v}(t) \times \mathbf{a}(t) = 12t^2(2, 2, -1) .$$

Therefore $\|\mathbf{v}(t) \times \mathbf{a}(t)\| = 36t^2$, and so

$$\kappa(t) = \frac{\|\mathbf{v}(t) \times \mathbf{a}(t)\|}{v^3(t)} = \frac{36t^2}{(6t(1 + t^2)^{1/2})^3} = \frac{1}{6t(1 + t^2)^{3/2}} .$$

Finally

$$\mathbf{B}(t) = \frac{1}{\|\mathbf{v}(t) \times \mathbf{a}(t)\|}\mathbf{v}(t) \times \mathbf{a}(t) = \frac{1}{3}(2, 2, -1) .$$

For **(c)**, since $\mathbf{B}(t)$ is constant, $\tau(t) = 0$.

Extra Credit: Let $\mathbf{x}(t)$ be the curve from Problem **2**. Find the distance from $\mathbf{x}(t)$ to to the plane $2x + 2y - z = 3$ as a function of t, as well as the volume of the tetrahedron with vertices $\mathbf{x}(0)$, $\mathbf{x}(1)$, $\mathbf{x}(3)$, and $\mathbf{x}(4)$. Justify your answers.

SOLUTION: Since the torsion is zero, the curve is planar, and the equation of the plane of motion is $\mathbf{B} \cdot (\mathbf{x} - \mathbf{x}(0)) = 0$, which is

$$\frac{1}{3}(2, 2, -1) \cdot (x, y, z) = 0 \ .$$

This plane is parallel to the plane $2x + 2y - z = 3$, and hence the distance to this plane is constant. Since $(0, 0, -3)$ lies in the plane $2x + 2y - z = 3$, the distance between the two planes, and hence between the curve and the plane, is

$$|\mathbf{B} \cdot ((0, 0, -2) - \mathbf{x}(0))| = \frac{1}{3}|(2, 2, -1) \cdot (0, 0, -3)| = 1 \ .$$

Next, since $\mathbf{x}(0)$, $\mathbf{x}(1)$, $\mathbf{x}(3)$, and $\mathbf{x}(4)$ lie in a plane, the tetrahedron spanned by them is degenerate and contains zero volume.

3: Let $f(x, y)$ and $g(x, y)$ be given by

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases} \qquad \text{and} \qquad g(x,y) = \begin{cases} \dfrac{x^2 y^2}{x^4 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases} .$$

(a) Is the function f continuous at $(0, 0)$? Is is bounded on the closed unit disc $\{(x, y) \ : \ x^2 + y^2 \leq 1\}$? Justify your answers.

(b) Is the function g continuous at $(0, 0)$? Is is bounded on the closed unit disc $\{(x, y) \ : \ x^2 + y^2 \leq 1\}$? Justify your answer.

SOLUTION: For **(a)**, we note that since $|x^2 y| \leq (x^4 + y^2)/2$, with equality if any only if $y = \pm x^2$, we have that for all $(x, y) \neq (0, 0)$,

$$-\frac{1}{2} \leq \frac{x^2 y}{x^4 + y^2} \leq \frac{1}{2} \ .$$

Hence this function is bounded on all of $\mathbb{R}^2$. However, we see that all along the curve $y = x^2$, $x \neq 0$, we have $f(x, y) = f(x, x^2) = 1/2 \neq 0$. Thus, f takes on the value $1/2$ at points that are arbitrariy close to $(0, 0)$ where it takes on the value 0. Thus f is not continuous. Alternately,

$$\frac{1}{2} = \lim_{n \to \infty} f(1/n, 1/n^2) \neq 0 = f(0, 0)$$

while

$$\lim_{n \to \infty} (1/n, 1/n^2) = (0, 0) \ .$$

Thus, f is not continuous.

For **(b)**, by the above, for $\mathbf{x} \neq (0,0)$,

$$|g(\mathbf{x})| \leq \frac{1}{2}|y| \leq \frac{1}{2}\|\mathbf{x}\| \ . \tag{1}$$

Then since

$$\lim_{t \to 0} \frac{1}{2} t = 0 \ ,$$

g is continuous by the squeeze principle. Then since continuous functions are always bounded on closed bounded sets, g is bounded on the unit disk. Alternately, from (1) we see that $|g(\mathbf{x})| \leq 1/2$ for $\|\mathbf{x}\| \leq 1$.

4: Let $f(x,y)$ be given by

$$f(x,y) = x^2y^2 - xy^2 + x \ .$$

(a) Compute the gradient of f, and find all of the critical points, if any, of f.

(b) Find the equation of the tangent plane to the graph of f at the point $(-1,1)$.

(c) Think of $f(x,y)$ as the altitude on a landscape at the point with horizontal coordinates x, y. Identify the direction of the positive y-axis with due North, and the direction of the positive x-axis with due East.

Find all, points, if any, at which the direction of steepest descent is due West.

SOLUTION: For **(a)**, we compute

$$\nabla f(x,y) = (2xy^2 - y^2 + 1 \ , \ 2xy(x-1)) \ .$$

Thus, (x,y) is a critical pony if and only if (x,y) satisfies

$$\begin{aligned} 2xy^2 - y^2 &= -1 \\ 2xy(x-1)) &= 0 \ . \end{aligned}$$

The second equation is satisfied if and only if $x = 0$, $y = 0$ or $x = 1$. Suppose $x = 0$. The first equation becomes $y^2 = 1$, so $y = \pm 1$. Thus, $(0,1)$ and $(0,-1)$ are critical points. Suppose $y = 0$. The first equation becomes $0 = -1$, and so there are no critical points with $y = 0$. Suppose $x = 1$. The first equation becomes $y^2 = -1$, so there are no critical points with $x = 1$. Thus, $(0,1)$ and $(0,-1)$ are the only critical points.

For **(b)**, we evaluate

$$\nabla f(-1,1) = (-2,4) \qquad \text{and} \qquad f(-,1) = 1 \ .$$

The equation of the tangent plane is

$$z = 1 + (-2,4)\cdot(x+1, y-1) = -2x + 4y - 5 \ .$$

For **(c)**, since the direction of steepest descent is opposite the direction of the gradient, we seek the set of points (x,y) at which $\nabla f(x,y) = (a,0)$ with $a > 0$. In particular,

$$\frac{\partial}{\partial y} f(x,y) = 0 \qquad \text{which is} \qquad 2xy(x-1)) = 0 \ .$$

This is the set of all points at which either $x = 0$, $y = 0$ or $x = 1$. But we also need

$$\frac{\partial}{\partial x} f(x, y) > 0 \qquad \text{which is} \qquad 2xy^2 - y^2 + 1 > 0 \ .$$

Thus, for $x = 0$, we require $-1 < y < 1$. For $y = 0$, there is no restriction on x, and for $x = 1$, there is no restriction on y. Therefore, the set consists of all points $(0, y)$ with $-1 < y < 1$, and all points $(x, 0)$ and all points $(y, 1)$.

5: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = x^2y - 4y + x^3 \ .$$

(a) Find the equation of the tangent plane to the graph of f at the point $(1, 2)$.

(b) Let $\mathbf{x}(t) := (t + t^2, t - t^3)$. Compute

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x}(t))\bigg|_{t=1} \ .$$

SOLUTION: For **(a)**, we compute

$$\nabla f(x, y) = (2xy + 3x^2, x^2 - 4) \ , \qquad \nabla f(1, 2) = (7, -3) \qquad \text{and} \qquad f(1, 2) = -5 \ .$$

Then the equation of the tangent plane is

$$z = -5 + (7, -3) \cdot (x - 1, y - 2) 7x - 3y - 6 \ .$$

For **(a)**, we compute

$$\mathbf{x}(1) = (2, 0) \ , \qquad \mathbf{x}'(1) = (3, -2) \qquad \text{and} \qquad \nabla f(2, 0) = (12, 0) \ .$$

By the Chain Rule, the answer is

$$\nabla f(\mathbf{x}(1)) \cdot \mathbf{x}'(2) = (12, 0) \cdot (3, -2) = 36 \ .$$