Practice Test I, Math 291 Fall 2012

October 11, 2012

1: (a) Find a right handed orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ such that $\mathbf{u}_1$ is a positive multiple of $(2, 2, 1)$ and $\mathbf{u}_2$ is orthogonal to $(1, 1, 0)$.

In the rest of this problem, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ refers to this orthonormal basis .

(b) Let $\mathbf{x}_1 = (1, 2, 3)$ and $\mathbf{x}_2 = (3, 2, 1)$. Consider the two lines $\mathbf{x}_1(s) = \mathbf{x}_1 + s\mathbf{u}_1$ and $\mathbf{x}_2(t) = \mathbf{x}_2 + t\mathbf{u}_2$. Compute numbers y_1, y_2 and y_3 such that

$$\mathbf{x}_1 - \mathbf{x}_2 = y_1\mathbf{u}_1 + y_2\mathbf{u}_2 + y_3\mathbf{u}_3 \ ,$$

and compute

$$\|\mathbf{x}_1(s) - \mathbf{x}_2(t)\|^2$$

as a function of s and t.

(c) Find the point on the line parameterized by $\mathbf{x}_1(t)$ that is closest to the line parameterized by $\mathbf{x}_2(t)$, and find point on the line parameterized by $\mathbf{x}_2(t)$ that is closest to the line parameterized by $\mathbf{x}_1(t)$.

2: For $t > 0$, let $\mathbf{x}(t)$ be the curve given by

$$\mathbf{x}(t) = \left(2t^2 + 2t^3/3 \ , -t^2 - 4t^3/3 \ , 2t^2 - 4t^3/3\right)$$

(a) Compute the arc length along the curve between $\mathbf{x}(0)$ and $\mathbf{x}(t)$ as a function of t.

(b) Compute curvature $\kappa(t)$ as a function of t, and the binormal vector $\mathbf{B}(t)$. .

(c) Find the torsion $\tau(t)$ as a function of t, justifying your answer.

Extra Credit: Let $\mathbf{x}(t)$ be the curve from Problem **2**. Find the distance from $\mathbf{x}(t)$ to to the plane $2x + 2y - z = 3$ as a function of t, as well as the volume of the tetrahedron with vertices $\mathbf{x}(0)$, $\mathbf{x}(1)$, $\mathbf{x}(3)$, and $\mathbf{x}(4)$. Justify your answers.

3: Let $f(x,y)$ and $g(x,y)$ be given by

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases} \qquad \text{and} \qquad g(x,y) = \begin{cases} \dfrac{x^2y^2}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases} .$$

(a) Is the function f continuous at $(0,0)$? Is is bounded on the closed unit disc $\{(x,y) \ : \ x^2+y^2 \leq 1\}$? Justify your answers.

(b) Is the function g continuous at $(0,0)$? Is is bounded on the closed unit disc $\{(x,y) \ : \ x^2+y^2 \leq 1\}$? Justify your answer.

4: Let $f(x,y)$ be given by

$$f(x,y) = x^2y^2 - xy^2 + x \ .$$

(a) Compute the gradient of f, and find all of the critical points, if any, of f.

(b) Find the equation of the tangent plane to the graph of f at the point $(-1,1)$.

(c) Think of $f(x,y)$ as the altitude on a landscape at the point with horizontal coordinates x,y. Identify the direction of the positive y-axis with due North, and the direction of the positive x-axis with due East.

Find all, points, if any, at which the direction of steepest descent is due West.

5: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = x^2y - 4y + x^3 \ .$$

(a) Find the equation of the tangent plane to the graph of f at the point $(1,2)$.

(b) Let $\mathbf{x}(t) := (t+t^2, t-t^3)$. Compute

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x}(t))\bigg|_{t=1} .$$