Practice Final Exam, Math 291 Fall 2013

Eric A. Carlen[1]
Rutgers University

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Material from Part 1 of the Course:

1: (a) Let $\mathbf{a} = (2,1,2)$ and $\mathbf{b} = (1,2,-2)$. The set of points $\mathbf{x} \in \mathbb{R}^3$ satisfying

$$\mathbf{a} \times \mathbf{x} = \mathbf{b}$$

is a line.

(a) Find a parameterization of this line.

(b) Find the distance between this line and the origin $(0,0,0)$.

(c) Find the distance between this line and the z-axis.

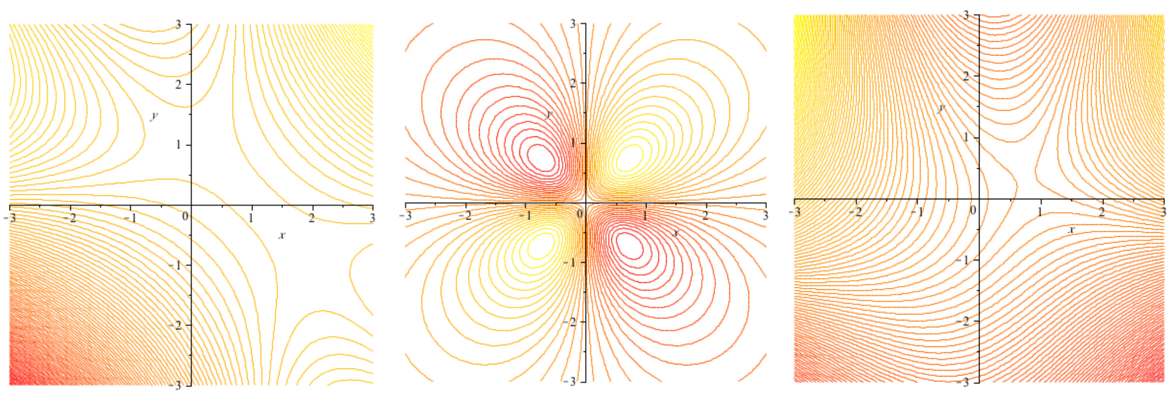
2: Let $f(x,y) = \dfrac{xy}{(1+x^2+y^2)^2}$.

(a) Find all of the critical points of f, and find the value of f at each of the critical points.

(b) Does f have a maximum value? Explain why or why not. If it does, find all points at which the value of f is maximal; i.e, find all maximizers.

(c) Does f have a minimum value? Explain why or why not. If it does, find all points at which the value of f is minimal; i.e, find all minimizers.

(d) One of the following is a contour plot for f. Which one is it? Explain your answer to receive credit.



3: Let $\mathbf{x}(t)$ be the curve given by $\mathbf{x}(t) = (\ln(t), 2t, t^2/2)$ for $t > 0$.

(a) Compute the arc length $s(t)$ as a function of t, measured from the starting point $\mathbf{x}(1)$, and find an arc-length parameterization of this curve

(b) Compute curvature $\kappa(t)$ and torsion $\tau(t)$ as a function of t.

(c) Find an equation for the osculating plane at time $t = 1$, and compute the distance from the origin to the osculating plane at $t = 1$.

4: Let $f(x, y) = x^2y + (x - 1)(y - 1)^2$.

(a) Find the equations for the tangent planes to the graph of $z = f(x, y)$ at $\mathbf{x}_1 = (2, -1)$ and at $\mathbf{x}_2 = (-1, 1)$.

(b) Parameterize the line that is the intersection of the two planes found in **(a)**.

(c) Parameterize the line that is the tangent to the contour curve of f passing through $(1, 1)$. (Note that this is a line in the plane, $\mathbb{R}^2$.)

curve

Material from Part 2 of the Course:

5: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = 4xy - x^4 - y^4$.

(a) Let $\mathbf{x}(t)$ b given by $\mathbf{x}(t) = (t + t^2, t^2 + t^3)$ Compute $\left.\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x}(t))\right|_{t=1}$.

(b) Find all of the critical points of f, and find the value of f at each of the critical points.

(c) Compute the Hessian of f at each critical point and determine whether each critical point is a local minimum, a local maximum, a saddle point, or if it cannot be classified through a computation of the Hessian.

(d) Sketch a contour plot of f near each critical point.

6: Let $f(x, y) = \dfrac{xy}{(1 + x^2 + y^2)^2}$, as in problem **2.** Find the minimum and maximum values of f in the set where

$$|x| + |y| \leq 1 .$$

Also, find all of the minimizers and maximizers.

7: **(a)** Let D be the set in the positive quadrant of $\mathbb{R}^2$ that bounded by

$$\begin{aligned} y &= x \\ y &= \sqrt{3}x \\ y &= x^2 + y^2 \end{aligned}$$

Let $f(x, y) = \sqrt{1 + x^2 + y^2}$. Compute $\int_D f(x, y)\mathrm{d}A$.

(b) Let D be the set in $\mathbb{R}^2$ that is given by

$$x^2 \leq y \leq 2x^2 \qquad \text{and} \qquad x^3 \leq y \leq 2x^3 .$$

Let $f(x, y) = \dfrac{x}{y}$. Compute $\int_D f(x, y)\mathrm{d}A$.

8: Let $\mathcal{V}$ be the region in $\mathbb{R}^3$ that lies inside the sphere $x^2 + y^2 + z^2 = 4$, and above the graph of $z = 1/\sqrt{x^2 + y^2}$. Compute the volume of $\mathcal{V}$ **and** the total surface area of its boundary. (There are two pieces to the boundary.)

Material from Part 3 of the Course:

9: Consider the two vector fields

$$\mathbf{F} = (y + z^2, x + z^2, 2zx + 2zy) \qquad \text{and} \qquad \mathbf{G} = (y + z^2, x + z^2, 2x + 2y)\ .$$

(a) Compute the divergence and curl of $\mathbf{F}$ and $\mathbf{G}$.

(b) Let $\mathcal{S}$ be the unit sphere, and $\mathbf{N}$ its outward normal. Compute **either**

$$\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{N} \mathrm{d}S \qquad \textbf{or} \qquad \int_{\mathcal{S}} \mathbf{G} \cdot \mathbf{N} \mathrm{d}S\ .$$

The choice is yours. Do whichever one you find easier, and justify your answer to receive credit.

(c) One of the vector fields $\mathbf{F}$ and $\mathbf{G}$ is equal to $\nabla\varphi$ for some potential function φ. Which one is it? Find such a potential function.

(d) Let C be the curve that is given by

$$x^2 + y^2 + y^2 = 4 \qquad \text{and} \qquad x + y + z = 1\ .$$

Orient C so that it is traversed in the counter-clockwise direction when viewed from above. Compute **either**

$$\int_C \mathbf{F} \cdot \mathbf{T} \mathrm{d}s \qquad \textbf{or} \qquad \int_C \mathbf{G} \cdot \mathbf{T} \mathrm{d}s\ .$$

The choice is yours. Do whichever one you find easier, and justify your answer to receive credit.

10: Let $\mathcal{V}$ be the region in $\mathbb{R}^3$ that lies inside the sphere $x^2 + y^2 + z^2 = 4$, and above the graph of $z = 1/\sqrt{x^2 + y^2}$, as in problem **8**. Let $\mathbf{F} = (xz, x^2y + z^2, y^3 - zx^2)$ and let $\mathbf{N}$ be the outward normal to $\mathcal{S}$, the boundary of $\mathcal{V}$. Compute the total flux

$$\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{N} \mathrm{d}S\ .$$

11: Let C be the (open) curve that runs along the line segment from $(1, 0, 0)$ to $(0, 1, 0)$, and then along the line segment from there to $(0, 0, 1)$. Let $\mathbf{G} = (y + z^2, x + z^2, 2x + 2y)$. Compute

$$\int_C \mathbf{G} \cdot \mathbf{T} \mathrm{d}s\ .$$