Solutions for Practice Test IIA, Math 291 Fall 2013

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1: Let $f(x,y) = x^2 + y^2 - 2yx^2$.

(a) Find all of the critical points of f. Evaluate the Hessian matrix of f at each of these critical points, and determine where each is a local maximum, a local minimum, a saddle, or undecidable from the Hessian.

SOLUTION We compute $\nabla f(x,y) = 2(x(1-2y), y - x^2)$. Therefore, a critical point (x,y) must satisfy

$$\begin{aligned} x(1-2y) &= 0 \\ y - x^2 &= 0 \end{aligned}$$

From the first equation, either $x = 0$ or $y = 1/2$. From the second equation, if $x = 0$, then $y = 0$, and if $y = 1/2$, then $x = \pm 1/\sqrt{2}$. Hence there are three critical points:

$$(0,0) \qquad (1/\sqrt{2}, 1/2) \quad \text{and} \quad (-1/\sqrt{2}, 1/2)\ .$$

We next compute

$$\mathrm{Hess}_f(x,y) = \begin{bmatrix} 2-4y & -4x \\ -4x & 2 \end{bmatrix}\ .$$

Evaluating this at $(0,0)$, we find $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. The two principle curvatures are both 2. Since both are positive, the surface curves upward from this critical point, which is a local minimum. Since the two principle curvatures are equal, the contour curves will look circular near this critical point.

Evaluating the Hessian at $(\pm 1/\sqrt{2}, 1/2)$, we find $\mp \begin{bmatrix} 0 & 2\sqrt{2} \\ 2\sqrt{2} & 2 \end{bmatrix}$. In either case, the principle curvatures are the roots of

$$t(t-2) - 8 = 0$$

which are 4 and -2. Hence theses two critical points are saddle points: The surface curves upwards in some directions, and downwards in others.

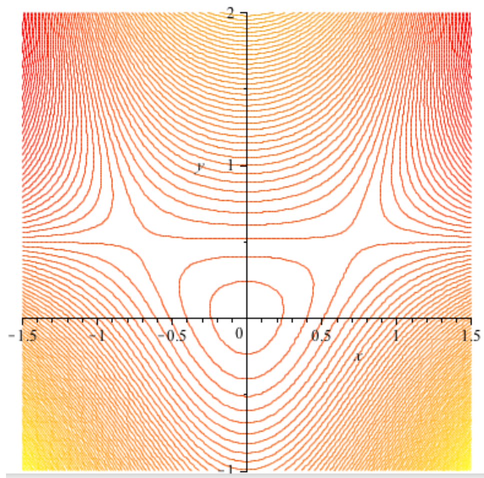
(b) Sketch a contour plot of f in the vicinity of each of the critical points. Show the computations that lead to the plots to get credit.

SOLUTION For the critical point $(0,0)$, the principle curvatures are 2 and 2, so in an orthogonal system of coordinates centered on $(0,0)$, the quadratic approximation takes the from $f(0,0)+2\tilde{x}^2+2\tilde{y}^2$. So the curves are of the quadratic approxmation are of the from $2\tilde{x}^2+2\tilde{y}^2 = \text{constant}$. These are circles and we do not need the axes to draw them.

For the critical point $(1/\sqrt{2}, 1/2)$, the principle curvatures are 4 and -2, and hence in an orthogonal system of coordinates centered on $(0,0)$, the quadratic approximation takes the from $f(0,0)+2\tilde{x}^2-2\tilde{y}^2$. To get the direction of the $\tilde{x}$ axis, we form

$$\mathrm{Hess}_f(1/\sqrt{2}, 1/2) - \begin{bmatrix} -2 & 0 \\ 0 & -22 \end{bmatrix} = \begin{bmatrix} 2 & -2\sqrt{2} \\ -2\sqrt{2} & 0 \end{bmatrix}$$

and take $\mathbf{v}_1 = (2, -2\sqrt{2})$. That is, we subtract the *other* principle curvature form the diagonal of the Hessian, and that the tope row of what is left. This points along the $\tilde{x}$ axis, and the tilde y axis is orthogonal to that. The remaining critical point is handled the same way. Here is the resulting sketch:



2: Let $f(x,y) = xy$. Let D denote the region in the plane consisting of all of the points (x,y) such that

$$x^2 + 4y^2 \leq 6 \ .$$

Find the minimum and maximum values of f in D. Also, find all of the minimizers and maximizers in D.

SOLUTION We compute $\nabla f(x,y) = (y,x)$, and defining $g(x,y) = x^2+4y^2 = 6$, the constraint is $g(x,y) = 0$. We compute $\nabla g(x,y) = 2(x,4y)$, and so Lagrange's equation yields

$$x^2 = 4y^2 \ ,$$

so that $x = \pm 2y$. Elimonating x from the constraint eqaution we have $8y^2 = 6$, so that $y = \pm\sqrt{3}/2$ and $x = \sqrt{3}$. The maximum value is $3/2$, which is achieved at $(\sqrt{3}, \sqrt{3}/2)$ and $(-\sqrt{3}, -\sqrt{3}/2)$. The minimum value is $-3/2$, which is achieved at $(-\sqrt{3}, \sqrt{3}/2)$ and $(\sqrt{3}, -\sqrt{3}/2)$.

3: Let $\mathbf{f}(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}))$ where $f(x,y) = xy - x^3 - 1/4$, and $g(x,y) = 1 - 4y^2 - x^2$.

(a) How many solutions to the system $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ are there? Draw a plot showing their approximate location.

(b) In the previous part, you should have found that there is one solution not too far from

$$\mathbf{x}_0 = (-1, 1/2) \ .$$

Compute $[D_{\mathbf{f}}(\mathbf{x})]$, and then use $\mathbf{x}_0$ as a starting point for Newton's method, and compute the next approximate solution $\mathbf{x}_1$.

(c) Evaluate $\mathbf{f}(\mathbf{x}_1)$, and compare this with $\mathbf{f}(\mathbf{x}_0)$.

SOLUTION The equation $g(x,y) = 0$ describes an ellipse centered at the origin whose major axis has length 2 and runs along the x-axis, and whose minor axis has length 1 and runs along the y axis. This is easily sketched. The solution of $f(x,y) = 0$ is the graph of

$$y = x^2 + \frac{1}{4x} \ .$$

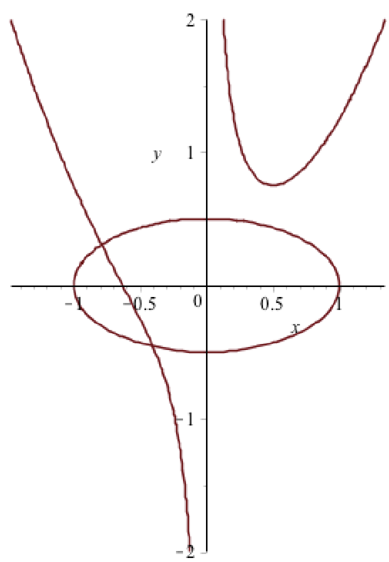
To understand the graph, we solve

$$y(x) = 0 \ , \quad y'(x) = 0 \quad \text{and} \quad y''(x) = 0 \ .$$

Each has a unique solution: $y(x) = 0$ only at $x = -2^{-2/3}$, $y'(x) = 0$ only at $x = 1/2$, and $y''(x) = 0$ only at $x = -2^{-2/3}$.

There is evidently a vertical asymptote at $x = 0$. For $x > 0$, $y''(x) > 0$, and so so the curve is convex, with a minimum at $x = 1/2$. Since $y(1/2) = 3/4$, the branch of the curve $f(x,y) = 0$ in the right half plane lies on or above the ;line $y = 3/4$, and therefore strictly above the ellipse. There is no intersection with the ellipse in the right half plane.

The branch of the curve $f(x,y) = 0$ in the left half plane crosses the x axis at $x = -2^{-2/3}$, which is inside the ellipse, and has an inflection point there, and the slope is strictly negative for all $x < 0$. Hence the curve $f(x,y) = 0$ crosses the ellipse exactly twice: In the upper left quadrant it does so a bit to the left of $x = -2^{2/3}$, and in the lower left quadrant it does so a bit to the right of $x = -2^{2/3}$. Here is a plot:



As you see, there is one solution not to far from $(-1, 12/)$. It would be better to take, say $(-3/4, 1/3)$, but to make the number come out nicely we will take $\mathbf{x}_0 = (-1, 1/2)$.

(b) We compute

$$[D_{\mathbf{f}}(\mathbf{x})] = \begin{bmatrix} y - 3x^2 & x \\ -2x & -8y \end{bmatrix} ,$$

and then

$$[D_{\mathbf{f}}(\mathbf{x}_0)] = -\frac{1}{2}\begin{bmatrix} 5 & 2 \\ -4 & 8 \end{bmatrix} , \text{and} \qquad \mathbf{f})\mathbf{x}_0) = (1/4, -1) \ .$$

Then

$$\mathbf{x}_1 = \mathbf{x}_0 - [D_{\mathbf{f}}(\mathbf{x}_0)]^{-1}\,\mathbf{f})\mathbf{x}_0) = (-5/6, 1/3) \ .$$

(c) We now compute

$$\mathbf{f}(\mathbf{x}_1) = \left(\frac{11}{216}, -\frac{5}{36}\right) \approx (0.051, -0.139) \ .$$

This is already not bad.

4: Let D be the set in $\mathbb{R}^2$ given by

$$2x^2 + 2xy + 2y^2 \le 1 \ .$$

Let $f(x,y) = x^2 + y^2$. Compute $\int_D f(x,y)\mathrm{d}A$. (**Hint**: Find a change of coordinate $(u(x,y), v(x,y)$ under which $2x^2 + 2xy + 2y^2 = 1$ becomes $u^2 + v^2 = 1$.)

SOLUTION Let us write $g(x,y) = 2x^2 + 2xy + 2y^2$. This is a quadratic, and we know how to choose new coordinates to simplify it: We write

$$g(x,y) = \mathbf{x} \cdot A\mathbf{x}$$

where $\mathbf{x} = (x,y)$ and

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} .$$

This is a doubly symmetric 2×2 matrix, and so eigenvectors are

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}(1,1) \qquad \text{and} \qquad \mathbf{u}_2 = \frac{1}{\sqrt{2}}(1,-1) \ ,$$

and the corresponding eigenvalues are 3 and 1. Then with

$$u(x,y) = \mathbf{u}_1 \cdot \mathbf{x} \qquad \text{and} \qquad v(x,y) = \mathbf{u}_2 \cdot \mathbf{x} \ , \tag{0.1}$$

the equation $g(x,y) = 1$ becomes

$$3u^2 + v^2 = 1 \ .$$

Finally, if we introduce $w := \sqrt{3}u$, the equation becomes

$$w^2 + v^2 = 1 \ ,$$

which is the equation for the unit circle in the w, v plane. Combining (0.1) with $w := \sqrt{3}u$, we have

$$(w,v) = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 1 & -1 \end{bmatrix}(x,y) \ .$$

Inverting,

$$x = \frac{1}{\sqrt{6}}w + \frac{1}{\sqrt{2}}v \qquad \text{and} \qquad y = \frac{1}{\sqrt{6}}w - \frac{1}{\sqrt{2}}v \ ,$$

and hence

$$f(x,y) = \frac{1}{6}w^2 + \frac{1}{2}v^2 \ .$$

Also,

$$\left|\frac{\partial(x,y)}{\partial(w,v)}\right| = \frac{1}{\sqrt{3}} \ .$$

Thus, with $\widehat{D}$ denoting the unit disk in the w,v plane,

$$\int_D f(x,y)\mathrm{d}A = \frac{1}{\sqrt{3}}\int_{\widehat{D}} \left(\frac{1}{6}w^2 + \frac{1}{2}v^2\right)\mathrm{d}A \ .$$

The last integral is easily done in polar coordinates: Let $r^2 = w^2 + v^2$, $w = r\cos\theta$ and $v = r\sin\theta$. Then

$$\begin{aligned}\frac{1}{\sqrt{3}}\int_{\widehat{D}} \left(\frac{1}{6}w^2 + \frac{1}{2}v^2\right)\mathrm{d}A &= \frac{1}{\sqrt{3}}\int_0^{2\pi}\left(\frac{1}{6}\cos^2\theta + \frac{1}{2}\sin^2\theta\right)\mathrm{d}\theta\int_0^1 r^3\mathrm{d}r \\ &= \frac{1}{\sqrt{3}}\frac{2\pi}{3}\frac{1}{4} = \frac{\pi}{6\sqrt{3}} \ .\end{aligned}$$

5: (a) Let $\mathcal{V}$ be the region in $\mathbb{R}^3$ that lies below the graph of $z = 1 - x^2$, and above the graph of $z = y^2$. Compute the volume of $\mathcal{V}$.

(b) Let $\mathcal{S}$ be the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the plane $x + z = 1$. Compute $\int_{\mathcal{S}} f(x,y,z)\mathrm{d}S$ where $f(x,y,z) = y/\sqrt{x^2+y^2}$. To get full credit, carry the computations through to the point that only an integral over a single variable remains to be evaluated.

SOLUTION The two surfaces meet where

$$1 - x^2 = z = y^2$$

which is at $x^2 + y^2 = 1$. It is easy to express the limits of integration in Cylindrical coordinates:

$$\begin{aligned} r^2\sin^2\theta &< z < 1 - r^2\cos^2\theta \\ 0 &< \theta < 2\pi \\ 0 &< r < 1 \ . \end{aligned}$$

Thus,

$$\begin{aligned}\mathrm{vol}(\mathcal{V}) = \int_{\mathcal{V}} 1\mathrm{d}V &= \int_0^{2\pi}\left(\int_0^1\left(\int_{r^2\sin^2\theta}^{1-r^2\cos^2\theta}\mathrm{d}z\right)r\mathrm{d}r\right)\mathrm{d}\theta \\ &= \int_0^{2\pi}\left(\int_0^1(1-r^2)r\mathrm{d}r\right)\mathrm{d}\theta = \frac{\pi}{2} \ .\end{aligned}$$

(b)

The key to solving both parts is coming up with a good parameterization. To find the intersection of the plane and paraboloid, we equate their z values and find

$$1 - x^2 - y^2 = 1 - x$$

which is the same as

$$x^2 + y^2 = x \qquad \text{or} \qquad (x - 1/2)^2 + y^2 = 1/4 \ .$$

This is the circle bounding the disk in the x, y plane centered on $(1/2, 0)$ with radius $1/2$. This is what we would see in a top view diagram. Our surface $\mathcal{S}$ is the part of the paraboloid that lies above this disk. Let us use cylindrical coordinates:

$$(x, y, z) = (r\cos\theta, r\sin\theta, z) \ .$$

The equation for the paraboloid is $x = 1 - r^2$, and so we have our parameterization

$$\mathbf{x}(r, \theta) = (r\cos\theta, r\sin\theta, 1 - r^2) \ .$$

The equation $x^2 + y^2 = x$ translates to $r^2 = r\cos\theta$, so the limits on our parameters are

$$0 \le r \le \cos\theta \qquad \text{and} \qquad -\pi/2 \le \theta \le \pi/2 \ .$$

We next compute

$$\frac{\partial \mathbf{x}}{\partial r} \times \frac{\partial \mathbf{x}}{\partial \theta} = (2r^2\cos\theta, 2r^2\sin\theta, r) \ . \tag{0.2}$$

Now we are ready to do part (**b**). Translating $f(x, y, z)$ into cylindrical coordinates, we find $f(x, y, z) = \sin\theta$. From (refsurel), we find

$$\mathrm{d}S = r\sqrt{4r^2 + 1}\mathrm{d}r\mathrm{d}\theta \ .$$

Hence

$$\int_{\mathcal{S}} f(x, y, z)\mathrm{d}S = \int_{-\pi/2}^{\pi/2} \left(\int_0^{\cos\theta} r\sqrt{4r^2 + 1}\mathrm{d}r \right) \sin\theta \mathrm{d}\theta \ .$$

The inner integral is easily done by substitution:

$$\int_0^{\cos\theta} r\sqrt{4r^2 + 1}\mathrm{d}r = \frac{1}{8}\frac{2}{3}u^{3/2}\bigg|_1^{4\cos^2\theta + 1} = \frac{1}{12}((4\cos^2\theta + 1)^{3/2} - 1) \ .$$

We finally have

$$\int_{\mathcal{S}} f(x, y, z)\mathrm{d}S = \frac{1}{12}\int_{-\pi/2}^{\pi/2} ((4\cos^2\theta + 1)^{3/2} - 1)\sin\theta \mathrm{d}\theta \ ,$$

which is easily evaluated, but this is all we are asked for.

Extra Credit: $\mathcal{S}$ be upper hemisphere of the unit sphere in $\mathbb{R}^3$. Let $f(x, y, z) = xyz$. Find the minimum and maximum values of f on $\mathcal{S}$, and all of the points at which f takes on these values. Explain how you are taking into account both of the constraints $x^2 + y^2 + z^2 = 1$ and $z \ge 0$.

SOLUTION The functions whose maximum and minimum we seek is pretty symmetric, so let us find the maximum and minimum on the whole sphere, and see if they occur in the upper hemisphere.

To do this we write the constrain in the form $g(x,y,z) = 1$ with

$$g(x,y,z) = x^2 + y^2 + z^2 \ .$$

The Lagrange condition then gives us

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) \ .$$

This means

$$\nabla f(x,y,z) \times \nabla g(x,y,z) = 0 \ .$$

Computing the gradients and cross product, we get the equations

$$\begin{aligned} x(y^2 - z^2) &= 0 \\ y(x^2 - z^2) &= 0 \\ z(y^2 - x^2) &= 0 \end{aligned}$$

We also have the equation

$$x^2 + y^2 + z^2 = 1 \ .$$

We see that if any two coordinates are zero, the third must be ± 1. This give the points

$$(\pm 1, 0, 0) \qquad (0, \pm 1, 0) \qquad (0, 0, , \pm 1) \ .$$

Now suppose that only one coordinate, say x is zero. Since y and z are non zero, we can divide by them and conclude

$$\begin{aligned} x^2 - z^2 &= 0 \\ y^2 - x^2 &= 0 \end{aligned}$$

Thus we have $x^2 = y^2 = z^2$, and all coordinates would be zero ,which is impossible. So the only other choice is all coordinates non-zero. Then we still have $x^2 = y^2 = z^2$. From the constraint equation we have that the remaining candidates are

$$(\pm 1/\sqrt{3}, \pm 1/\sqrt{3}, \pm 1/\sqrt{3}) \ .$$

Since the sphere is compact, and f is continuous, there will be a minimum and a maximum. The maximum is $3^{3/2}$, and is attained at the points $(\pm 1/\sqrt{3}, \pm 1/\sqrt{3}, \pm 1/\sqrt{3})$ with an even number of minus signs. There are such points, namely $(-1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$ and $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ in the upper hemisphere. Since these are maximizers on the whole sphere, they are certainly maximizers on the hemisphere. The minimum $3^{-3/2}$, and is attained at the points $(\pm 1/\sqrt{3}, \pm 1/\sqrt{3}, \pm 1/\sqrt{3})$ with an odd number of minus signs. There are such points, namely $(-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ and $(1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$ in the upper hemisphere. Since these are minimizers on the whole sphere, they are certainly maximizers on the hemisphere.