MULTIVARIABLE CALCULUS, LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS

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Chapter 1

GEOMETRY, ALGEBRA AND Analysis IN SEVERAL VARIABLES

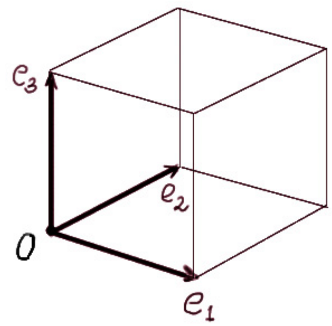
1.1 Algebra and geometry in $\mathbb{R}^n$

1.1.1 Vector variables and Cartesian coordinates

Our subject, multivariable calculus, is concerned with the analysis of functions taking several variables as input, and returning several variables of output. Many problems in science and engineering lead to the consideration of such functions. For instance, one such function might give the current temperature, barometric pressure, and relative humidity at a given point on the earth, as specified by latitude and longitude. In this case, there are two input variables, and three output variables.

You will also recognize the input variables in this example as *coordinates*. In fact, in most examples the variables we consider will arise either as coordinates or parameters of some kind. Our subject has its real beginning with a fundamental idea of Rene Descartes, for whom *Cartesisan coordinates* are named.

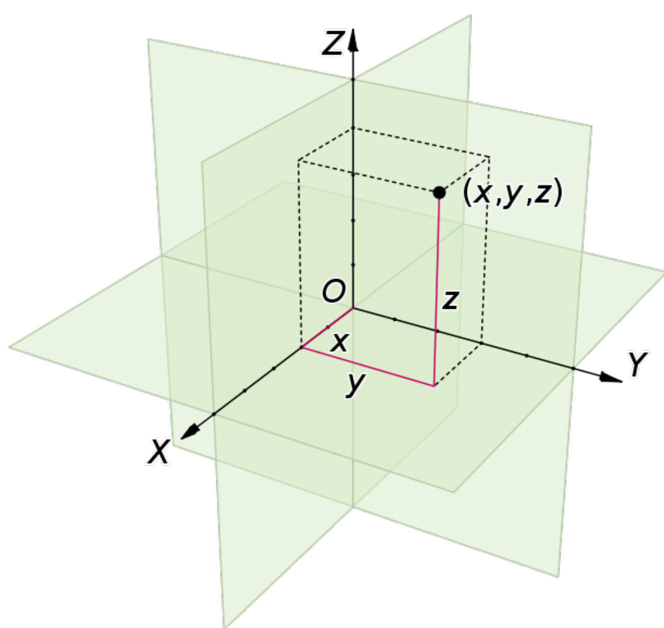
Descarte's idea was to specify points in three dimensional Euclidean space using lists of three numbers (x, y, z), now known as *vectors*. To do this one first fixes a *reference system*, by specifying a "base point" or "origin" that we shall denote by $\mathbf{0}$, and also a set of *three orthogonal directions*. For instance, if you are standing somewhere on the surface of the Earth, you might take the point at which you stand as the origin $\mathbf{0}$, and you might take East to be the first direction, North to be the second, and "straight up" to be the third. All of these directions are orthogonal to one another. Let us use the symbols $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ to denote these three directions.



The brilliant idea of Rene Descartes is this:

• *We can describe the exact position of any point in physical space by telling how to reach it by moving in the directions* $\mathbf{e}_1$, $\mathbf{e}_2$ *and* $\mathbf{e}_3$. *This is simply a matter of "giving directions": Start at the origin* $\mathbf{0}$, *and go out* x *units of distance in the* $\mathbf{e}_1$ *direction, then go out* y *units of distance in the* $\mathbf{e}_2$ *direction, and finally go out* z *units of distance in the* $\mathbf{e}_3$ *direction. The numbers* x, y *and* z *may be positive of negative (or zero). If, say,* x *is negative, this means that you should go* $|x|$ *units of distance in the direction* opposite *to* $\mathbf{e}_1$.

Thus, following Descartes' idea, we can specify the exact position of any point in physical space by giving the ordered list of numbers (x, y, z) that describes how to reach it from the origin of our reference system. The three numbers x, y and z are called the *coordinates* of the point with respect to the given reference system. (It is important to note that the reference system is part of the description too: Knowing how far to go in each direction is not much use if you do not know the directions, or the starting point.)



This representation of points in space as ordered triples of numbers, such as (x, y, z) allows one to use algebra and calculus to solve problems in geometry, and it literally revolutionized mathematics.

We now define a *three dimensional vector* to be an ordered triple (x, y, z) of real numbers. The geometric interpretation is that we regard x, y and z as the *coordinates* of a unique point in physical space – the one you get to by starting from the origin and moving x units of distance in the $\mathbf{e}_1$ direction, y units of distance in the $\mathbf{e}_2$ direction, and z units of distance in the $\mathbf{e}_3$ direction. We may identify the vector (x, y, z) with this point in physical space, once again keeping in mind that this identification depends on the reference system, and that what the vector really represents is not the point itself, but the translation that carries the origin to that point.

As we have said, this way of identifying three dimensional vectors with points in physical space is extremely useful because it brings algebra to bear on geometric problems. For instance, referring to the previous diagram, you see from (two application of) the Pythagorean Theorem that the distance from the origin $\mathbf{0}$ to the point represented by the vector (x, y, z) is

$$\sqrt{x^2 + y^2 + z^2} \ .$$

This distance is called the *length* or *magnitude* of the vector $\mathbf{x} = (x, y, z)$. The vector $\mathbf{x}$ also has a *direction*; namely the direction of the displacement that would carry one directly from $\mathbf{0}$ to $\mathbf{x}$. It is useful to associate to each direction the vector corresponding to a unit displacement in that direction. This provides a one-to-one correspondence between directions and *unit vectors*, i.e., vectors of unit length.

The unit sphere is defined to be the set of all unit vcctors; i.e., all points a unit distance from the origin. Thus, a point represented by the vector $\mathbf{x} = (x, y, z)$ lies on the unit sphere if and only if

$$x^2 + y^2 + z^2 = 1 \ . \tag{1.1}$$

You are probably familiar with this as the equation for the unit sphere. But before Descartes, geometry and algebra were very different subjects, and the idea of describing a geometric object in terms of an algebraic equation was unknown. It revolutionized mathematics.

1.1.2 Parameterization

Writing down the equation for the unit sphere is only a first step towards solving many problems involving spheres, such as, for example, computing the surface area of the unit sphere. Often the second step is to *solve the equation*. Now, for an equation like $x^2 = 1$, we can specify the set of all solutions by writing it out: $\{-1, 1\}$. But for $x^2 + y^2 + z^2 = 1$, there are clearly infinitely many solutions, and we cannot possibly write them all down.

What we can do, however, is to *parameterize* the solution set. Let us go through an example before formalizing this fundamental notion. Better yet, let us start with something even simpler: Consider the equation

$$x^2 + y^2 = 1 \tag{1.2}$$

in the x,y plane. (The x,y plane is the set of points (x, y, z) with $z = 0$.) You recognize (1.2) as the equation for the unit circle in the x,y plane. Recall the trigonometric identity

$$\cos^2\theta + \sin^2\theta = 1 \ . \tag{1.3}$$

Thus for all θ, the points

$$(x,y) = (\cos\theta, \sin\theta)$$

solve the equation (1.2).

Conversely, consider any solution (x,y) of (1.2). From the equation, $-1 \le x \le 1$, and hence we may define

$$\theta := \begin{cases} \arccos(x) & y \ge 0 \\ -\arccos(x) & y < 0 \end{cases} \tag{1.4}$$

By the definition of the arccos function, $-\pi < \theta \le \pi$. Since $\cos\theta$ is an even function of θ, it follows that $x = \cos\theta$, and then one easily sees that $y = \sin\theta$.

Thus, we have a *one-to-one* correspondence between the points in the interval $(-\pi.\pi]$ and the set of solutions of (1.2). The correspondence is given by the function

$$\theta \mapsto (\cos\theta, \sin\theta) \tag{1.5}$$

from $(-\pi.\pi]$ onto the unit circle. This is an example of a *parameterization*: As the *parameter* θ varies over $(-\pi.\pi]$, $(\cos\theta, \sin\theta)$ varies over the unit circle, covering each point for exactly one value of the parameter θ.

Since the function in (1.4) is one-to-one and onto, it is invertible. The inverse is simply the map

$$(x,y) \mapsto \theta \tag{1.6}$$

where for x and y solving $x^2 + y^2 = 1$, θ is given by (1.4). The function in (1.6) is called the *angular coordinate function* on the unit circle. As you see in this example, finding a parameterization of the solution set of some equation and finding a system of coordinates on the solutions set are two aspects of the same thing. We will make formal definitions later; for now let us continue with examples.

For $r > 0$,

$$x^2 + y^2 = r^2$$

is the equation of the centered circle of radius r in the x,y plane. Since

$$x^2 + y^2 = r^2 \qquad \iff \qquad \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1 ,$$

we can easily transform our parameterization of the unit circle into a parameterization of the circle of radius r: The parameterization is given by

$$\theta \mapsto (r\cos\theta, r\sin\theta) \tag{1.7}$$

while its inverse, the coordinate function, is given by

$$(x,y) \mapsto \theta := \begin{cases} \arccos\left(x/\sqrt{x^2+y^2}\right) & y \ge 0 \\ -\arccos\left(x/\sqrt{x^2+y^2}\right) & y < 0 \end{cases} \tag{1.8}$$

which specifies the angular coordinate as a function of x and y. Since $x^2 + y^2 = r^2 > 0$, we never divide by zero in this formula.

For our next example, let us parameterize the unit sphere; i.e., the solution set of (1.1). Note that $x^2 + y^2 + z^2 = 1$ implies that $-1 \leq z \leq 1$. Recalling (1.3) once more, we define

$$\phi = \arccos(z) \ , \tag{1.9}$$

so that $0 \leq \phi \leq \pi$, and $z = \cos\phi$.

It follows from (1.1) and (1.3) that $x^2 + y^2 = \sin^2\phi$, and then, since $\sin\phi \geq 0$ for $0 \leq \phi \leq \pi$,

$$\sin\phi = \sqrt{x^2 + y^2} \ . \tag{1.10}$$

Evidently, for x and y not both zero, (x, y) lies on the circle of radius $\sin\phi$. We already know how to parameterize this: Setting $r = \sin\phi$ in (1.7), the parameterization map is

$$\theta \mapsto (\sin\phi\cos\theta, \sin\phi\sin\theta) = (x, y) \ . \tag{1.11}$$

Since (1.9) gives us $z = \cos\phi$, we combine results to obtain the parameterization

$$(\theta, \phi) \mapsto (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) = (x, y, z) \ .$$

Note that θ is only defined when at least one of x and y is not zero, and so θ is not defined at $(0, 0, 1)$, the "North pole" and $(0, 0, -1)$, the "South pole". These points correspond to $\phi = 0$ and $\phi = \pi$ respectively. However, apart from these two points, every point on the unit sphere is of the form $(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$, for exactly one pair of values (θ, ϕ) in the rectangle $(-\pi, \pi] \times (0.\pi)$ The function

$$(\theta, \phi) \mapsto (\sin\theta\cos\phi, \sin\theta, \sin\phi, \cos\phi) \ ,$$

is therefore a parameterization of the unit sphere, take away the North and South poles.

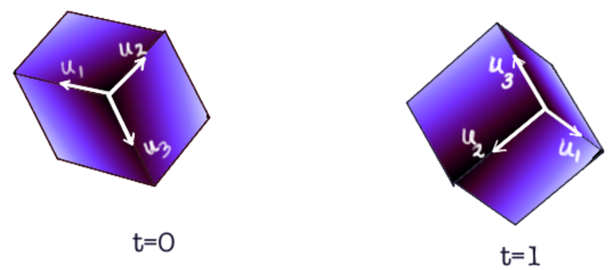
The inverse function,

$$(x, y, z) \mapsto (\theta, \phi)$$

where θ is given by (1.7) and ϕ is given by (1.9), is a standard coordinate coordinate function on the sphere. These coordinates are essentially the usual *longitude* and *latutude* coordinates, except the here we measure "latitide" from the North pole instead of the equator.

Our final example in this subsection opens perspectives on several issues that will concern us in this course. The problem of *the description of rigid body motion*, which is fundamentally important in physics, robotics, aeronautical engineering, computer graphics and other fields, and yet can be discussed without any background in any of them:

Imagine a solid, rigid object moving in three dimensional space. To keep the picture simple, suppose the object is a cube shaped box. Here is a picture showing the box shaped object at two times: $t = 0$ and $t = 1$:

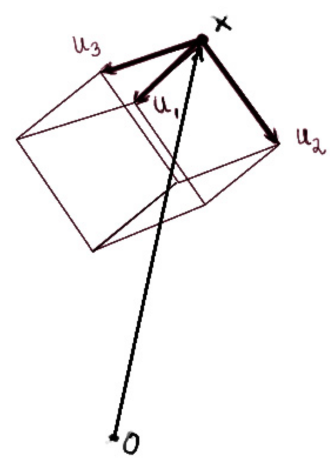


Before trying to describe the *motion* of this object, let us start with a simpler problem:

• *How can we describe, in precise mathematical terms, the way that the box is situated in three dimensional physical space at any given time, say $t = 0$?*

First of all, on the box itself, we fix one corner once and for all to be the "base-point" in the cube, and we label the three edges meeting at this corner as edge one, edge two and edge three.

Now we can specify the configuration of the box by specifying four vectors: $\mathbf{x}$, $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$.



Here is how: The vector $\mathbf{x}$ tells us the position of the base-point corner of the cube: $\mathbf{x} = (x, y, z)$ is the vector of Cartesian coordinates of location of the base point.

This is a start, but knowing where the base point is does not provide full knowledge of the *configuration* of the box; i.e., of how the box is positioned in physical space – keeping the base point fixed, one could rotate the cube around into infinitely many different configurations. This is where the three vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ come in: The box is to be positioned so that edge number one runs along the direction of $\mathbf{u}_1$, edge number two runs along the direction of $\mathbf{u}_2$, and edge number three runs along the direction of $\mathbf{u}_3$, as in the diagram.

Thus, knowing the vectors $\mathbf{x}$, $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ provides full knowledge of the configuration of the box. Notice that in our description of the configuration, both the length and the direction of $\mathbf{x}$ are important, but it is only the directions of $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ that matter, which is why we may take them to be unit vectors.

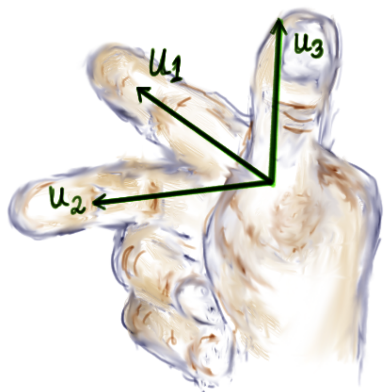
To describe the motion of the box in physical space, we then need only to give the four vectors

$\mathbf{x}$, $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ at each time t. This leads us to consider the four *vector valued functions* $\mathbf{x}(t)$, $\mathbf{u}_1(t)$, $\mathbf{u}_2(t)$ and $\mathbf{u}_3(t)$. We can specify the situation in space of our box as a function of the time t by giving four vector valued functions $\mathbf{x}(t)$, $\mathbf{u}_1(t)$, $\mathbf{u}_2(t)$ and $\mathbf{u}_3(t)$. Each of these four vectors has there entries to keep track of, so we would be have a total of 12 coordinates to keep track of. That is beginning to look complicated.

However, such a description involves quite a lot of redundant information. For example, as we explain next, essentially all of the information in $\mathbf{u}_3(t)$ is redundant, and we can reduce the number of variables we need to keep track of.

The explanation of why $\mathbf{u}_3(t)$ is redundant turns on a very important point about Cartesian coordinate systems: These all involve a reference frame of three orthonormal vectors, giving the directions of the three coordinate axes. It turns out that there a two kinds of reference frames: *right handed* and *left handed*, and if you know any two of the vectors in a reference frame, and know whether it is right handed or left handed, then you also know the third.

Here is the picture that explains the names:



Let $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ be the three orthogonal directions. If you can rigidly rotate your right hand (keeping the index finger, middle finger and thumb orthogonal) around so that your index finger points in the direction of $\mathbf{u}_1$, your middle finder points in the direction of $\mathbf{u}_2$, and your thumb points in the direction of $\mathbf{u}_3$, as in the picture, then the frame $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is right handed. Otherwise, it is left handed. We will give a mathematically precise definition later in Chapter One, and connect it with this picture in Chapter Two.

For now, let us go back to the diagram in this section showing the situation in space of our cubical box at the times $t = 0$ and $t = 1$. We have drawn in the frame $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ of unit vectors running along the edges coming out of the base point corner at both times. You should check that we have labeled them so that in both cases, the frame is right handed. Moreover, and this is the key point,

- *As the box moves, carrying the three vectors along, the frame remains right-handed.*

Making a precise mathematical statement out of this requires that we make use of the notion of *continuous motion*, which we study in Chapter Two. For now though, you will probably find this statement intuitively reasonable for any sort of physically possible rigid motion.

Now suppose you know the first two orthogonal unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ in the set of orthogonal directions for a coordinate system. Then there are only two directions that are oathogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$: Referring to the picture above, there are the "right-handed, thumbs-up" direction, and the "left-handed, thumbs-down" direction. If we know we are working with a right-handed coordinate system, $\mathbf{u}_3$ must be the "right-handed, thumbs-up" direction shown in the picture. Knowing $\mathbf{u}_1$ and $\mathbf{u}_2$, align your index finger along $\mathbf{u}_1$ and your middle finger along $\mathbf{u}_2$, as in the picture. Your thumb now points along $\mathbf{u}_3$.

- *Given that our frame is right handed, all information about the frame is contained in the first and second vectors; the third vector is determined by these, and the "right hand rule".*

Thus, all of the information in $\mathbf{u}_3(t)$ is redundant. We get a complete description of the configuration of the box in space by giving the list of three vectors $\mathbf{x}(t)$, $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$, provided we make the decision to work with right handed frames.

However, there is still redundancy in this description. As we now explain, we can *parameterize* the set of all right handed frames in three dimensional space in terms of three angles.

First, let us parameterize the choices for $\mathbf{u}_1$. The vector $\mathbf{u}_1$ corresponds to a point on the unit sphere in three dimensional space. As we have seen, to each such unit vector $\mathbf{u}_1$ (except the North and South poles), there corresponds a unique "latitude" $\phi \in (0, \pi)$ and "longitude" $\theta \in (-\pi, \pi]$ specifying the point. such that

$$\mathbf{u}_1 = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) .$$

(Even the North and South poles are included if we include the values $\phi = 0$ and $\phi = \pi$ and ignore the fact that θ is undefined, since when $\phi = 0$ or $\phi = \pi$, $\sin\phi = 0$, and the value of θ does not matter.)

Next, when we go on to specify $\mathbf{u}_2$, we do not get two additional angles, but only one: Two additional angles are what we would get if we could make an arbitrary choice in the unit sphere for $\mathbf{u}_2$, but we cannot: We must keep $\mathbf{u}_2$ orthogonal to $\mathbf{u}_1$.

To see all of the points on the unit sphere that are orthogonal to $\mathbf{u}_1$, look at the "great circle" where the plane through the origin that is orthogonal to the line through $\mathbf{u}_1$ intersects the sphere. This intersection is a circle, and all of the unit vectors that are orthogonal to $\mathbf{u}_1$ lie on this circle. Here is a picture shown a sphere that is "sliced" by the plane through the origin that is orthogonal to $\mathbf{u}_1$, with the top half "pulled away" for better visibility:

The edge of the slice is the "great circle" on which $\mathbf{u}_2$ must lie. We can parameterize this circle by choosing as a reference point the unique highest point on the circle (where we are assuming again the $\phi \neq 0, \pi$). We can then parameterize the circle by an angle $\chi \in [0, 2\pi)$ that runs counter-clockwise around the circle when viewed from above, starting at $\chi = 0$ at our reference point.

Thus, the two angles θ and ϕ determine $\mathbf{u}_1$, and then the third angle χ determines $\mathbf{u}_2$, and then the right -hand rule determines $\mathbf{u}_3$. Therefore, there is a one-to-one correspondence between right handed orthonormal frames $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ in which $\mathbf{u}_1$ is not either straight up or straight down, and triples (θ, φ, χ) in the set $(0, \pi) \times [0, 2\pi) \times [0, 2\pi)$. These angles θ, φ and χ are the parameters in our parameterization of the set of ("almost all") right-handed frames. To specify what the frame $\{\mathbf{u}_1(t), \mathbf{u}_2(t), \mathbf{u}_3(t)\}$, we only need to specify the three angles $\theta(t)$, $\varphi(t)$ and $\chi(t)$.

• *Therefore, to specify the frame* $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ *attached to the box, we only need to give the three angles* θ*,* φ *and* χ*. There is no remaining redundancy in this description.*

Thus, we can completely describe, without any redundancy, the configuration of our box as a function of time by giving a list of six numerically valued functions:

$$(x(t), y(t), z(t), \theta(t), \varphi(t), \chi(t))$$

where the first three entries are the Cartesian coordinates of the base-point, and the second three entries are the three angles specifying the orientation of the box in physical space.

We have written this list out in the same way we write three dimensional vectors; however, this list has six entries, and they are of different types: The first three represent signed distances, and the last three represent angles. In fact, it will turn out to be useful think of this more complicated object as a vector too - a vector in 6 a dimensional space. Several observation are worthwhile at this point:

(1) We have just parameterized, with a few details postponed for now, an interesting "set of mathematical objects", namely the set of right handed orthonormal frames in three dimensional space. We will be working with parameterizations of all sorts of mathematical sets in this course, and not only solutions sets of equations, and though one can often introduce an equation describing the set, it is not always the key to finding a parameterization of the set.

(2) We have just seen how a simple and very natural problem leads to the consideration of functions with values in a space of "higher dimensional vectors"; i.e., vectors with more than three entries.

However, that is not all: We will also be concerned with functions whose argument is a such a vector. For example, if various "conservative forces" are acting on our box, there will be an energy E associated to each configuration $(x, y, z, \theta, \varphi, \chi)$ of the box. This yields a function

$$E(x, y, z, \theta, \varphi, \chi)$$

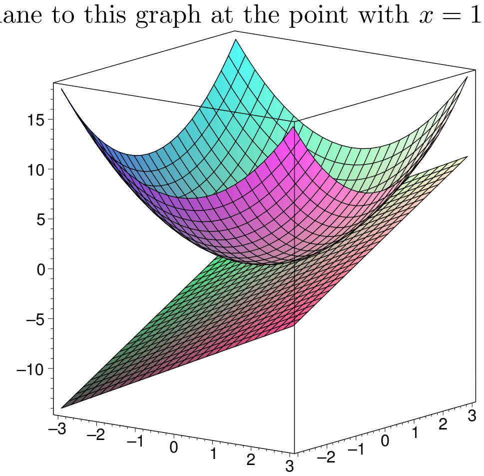
defined on a set of 6 dimensional vectors with values in the real numbers. A standard problem, important in physics, mechanical engineering and other fields, is to find the values of x, y, z, θ, φ and χ that minimize this function. One subject that we shall study in detail is the use of calculus for finding minima and maxima in this multi-variable setting.

Geometry is very helpful in finding minima and maxima for functions of several variables, and Descartes' idea is essential here. Let us briefly explain why, since otherwise you may wonder why we shall spend so much time discussing algebra and geometry at the beginning of this calculus course.

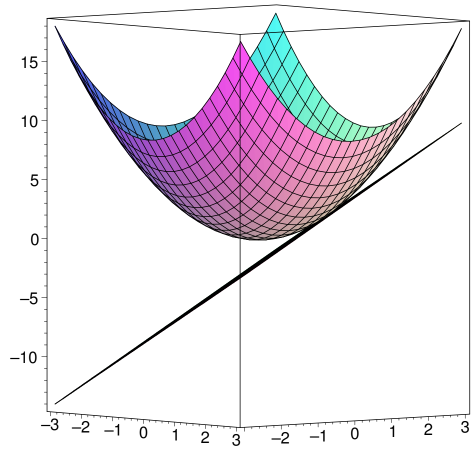
You learned in single variable calculus that the if the tangent line to the graph of $y = f(x)$ is *not* horizontal at some point x_0 in the interior (a, b) of the interval $[a, b]$, then x_0 *cannot possibly* minimize or maximize the function f, even locally: Since the graph has a non-zero slope, you can move to higher or lower values by moving a little bit to either the left or to the right. Hence the only candidates for interior maxima and minima are the *critical points*; that is, points at which the tangent line to the graph is horizontal.

Now consider a very simple real valued function $f(x, y) = x^2 + y^2$. The graph of this function is the set of points (x, y, z) for which $z = f(x, y)$; i.e., $z = x^2 + y^2$. This graph is a parabolic surface in three dimensional space. At each point on the surface there is a *tangent plane*, which is the plane that "best fits" the graph at the point in a sense quite analogous to the sense in which that tangent line provides the "best fit" to the graph of a single variable differentiable function at a given point.

Here is a three a picture showing the portion of the graph of $z = x^2 + y^2$ for $-2 \leq x, y \leq 2$, together with the tangent plane to this graph at the point with $x = 1$ and $y = 1$.



Here is another picture of the same thing from a different vantage point, giving a better view of the point of contact:



As you can see, the tangent plane is tilted, so there are both uphill and downhill directions at this point, and so $(x, y) = (1, 1)$ cannot possibly minimize or maximize $f(x, y) = x^2 + y^2$. Of course, for such a simple function, there are many ways to see this. However, for more interesting functions, this sort of reasoning in terms of tangent planes will be very useful.

To make use of this sort of reasoning, we first need effective means of working with lines and planes and such. For example, every plane has an equation that specifies it, just like $x^2 + y^2 + z^2 = 1$ specifies the unit sphere. What is the equation for the tangent plane pictures above? In this course, we will learn about a higher dimensional version of the derivative that determines tangent planes in the same way that the single variable derivative specifies tangent lines. And, as indicated above, we not only need the two or three dimensional version of this, but a version that works for any number of dimensions – though in more than two variables it will be a "tangent hyperplane" that we will be computing.

There are many other subjects we shall study involving the calculus – both integral and differential – of functions that take vectors as input, or return them as output, or even both.

• *Multivariable functions are simply functions that take an ordered list of numbers as their input, or return an ordered list as output, or both.*

In the next section, we begin developing the tools to work with them. We use these tools throughout the course.

1.1.3 The vector space $\mathbb{R}^n$

Definition 1 (Vectors in $\mathbb{R}^n$)**.** *A* vector *is an ordered list of n numbers x_j, $j = 1, 2, \ldots, n$, for some positive integer n, which is called the* dimension *of the vector. The integers $j = 1, 2, \ldots, n$ that order the list are called the* indices*, and the corresponding numbers x_j are called the* entries*. That is, for each $j = 1, 2, \ldots, n$, x_j is the jth entry on the list. The set of all n dimensional vectors is denoted by $\mathbb{R}^n$.*

As for notation, we will generally use bold face to denote vectors: We write $\mathbf{x} \in \mathbb{R}^n$ to say that $\mathbf{x}$ is a vector in $\mathbb{R}^n$. To write $\mathbf{x}$ out in terms of its entries, we often list the entries in a row, ordered left to right so that the typical vector $\mathbf{x} \in \mathbb{R}^3$ is

$$\mathbf{x} = (x_1, x_2, x_3) \ ,$$

where x_1, x_2 and x_3 are real numbers. When n is 2 or 3, it is often simpler to dispense with the subscripts, and distinguish the entries by using different letters. In this way, one would write (x, y) to denote a generic vector in $\mathbb{R}^2$ or (x, y, z) to denote a generic vector in $\mathbb{R}^3$. We use $\mathbf{0}$ to denote the vector in $\mathbb{R}^n$ with 0 in every entry.

Finally, we shall often consider sets of vectors $\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$ in $\mathbb{R}^n$ where the different vectors are distinguished by subscripts. A subscript on a boldface variable such as $\mathbf{x}_j$ *always* indicates the jth vector in a lsit of vectors and not the jth entry of a vector $\mathbf{x}$. When we need to refer to the kth entry of $\mathbf{x}_j$, we shall write $(\mathbf{x}_j)_k$.

Definition 1 may not be what you expected. After all, we began this chapter discussing three dimensional vectors that were defined as "quantities with length and direction". When the term *vector* was coined, people had in mind the description of the position and motion of points in three dimensional physical space. For such vectors, the length and the direction have a clear geometric meaning.

But what about vectors like $(x, y, z, \theta, \phi, \chi)$, which belongs to $\mathbb{R}^6$? What would we mean by the length of such a vector, and what would we mean by the angle between two such vectors?

Perhaps surprisingly, there is a useful notion of length and direction in any number of dimensions.* But until we define direction and magnitude, we cannot use these notions to define vectors themselves! Therefore, the starting point is the definition of vectors in $\mathbb{R}^n$ as ordered lists of n real numbers.

$\mathbb{R}^n$ is more than just a set; it comes equipped with an algebraic structure that makes it what is known as a *vector space*. The algebraic structure consists of two algebraic operations: *scalar multiplication* and *vector addition*. As we have already stated, Descartes' idea had such an enormous impact because it brought together what had been two quite separate branches of mathematics – algebra and geometry. Our plan for the rest of this section is to develop the algebraic aspects of Descartes' idea, and then show how the algebra may be leveraged to apply our geometric intuition about three dimensional vectors to vectors of *any* dimension.

Definition 2 (Scalar Multiplication)**.** *Given a number $a \in \mathbb{R}$ and a vector $\mathbf{x} = (x_1, x_2, \dots, x_n)$, define the product of a and $\mathbf{x}$, denoted $a\mathbf{x}$, is defined by*

$$a\mathbf{x} = (ax_1, ax_2, \dots, ax_n) \ .$$

For any vector $\mathbf{x}$, $-\mathbf{x}$ denotes the product of -1 and $\mathbf{x}$.

Example 1 (Multiplying numbers and vectors)**.** *Here are several examples:*

$$2(-1, 0, 1) = (-2, 0, 2)$$

$$\pi(-1/2, 1/2) = (-\pi/2, \pi/2) = -(\pi/2, -\pi/2)$$

*By "useful", we mean useful for solving equations, among other things In other words, useful in a practical sense, even in, say, eight dimensions.

$$0(a, b, c) = (0, 0, 0) = \mathbf{0}\ .$$

Definition 3 (Vector Addition)**.** *Given two vectors* $\mathbf{x}$ *and* $\mathbf{y}$ *in* $\mathbb{R}^n$ *for some* n*, define their vector sum,* $\mathbf{x}+\mathbf{y}$*, by summing the corresponding entries:*

$$\mathbf{x}+\mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)\ .$$

We define the vector difference of $\mathbf{x}$ *and* $\mathbf{y}$*,* $\mathbf{x}-\mathbf{y}$ *by* $\mathbf{x}-\mathbf{y} = \mathbf{x}+(-\mathbf{y})$*.*

Note that vector addition does not mix up the entries of the vectors involved at all: For each j,

$$(\mathbf{x}+\mathbf{y})_j = x_j + y_j\ .$$

The third entry, say, of the sum depends only on the third entries of the summands.

• *For this reason, vector addition inherits the commutative and associative properties of addition in the real numbers. It is just the addition of real numbers "done in parallel".*

That is: vector addition is *commutative*, meaning that $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$ and *associative*, meaning that $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$. In the same way, one sees that scalar multiplication *distributes* over vector addition:

$$a(\mathbf{x}+\mathbf{y}) = (a\mathbf{x}) + (b\mathbf{y})\ .$$

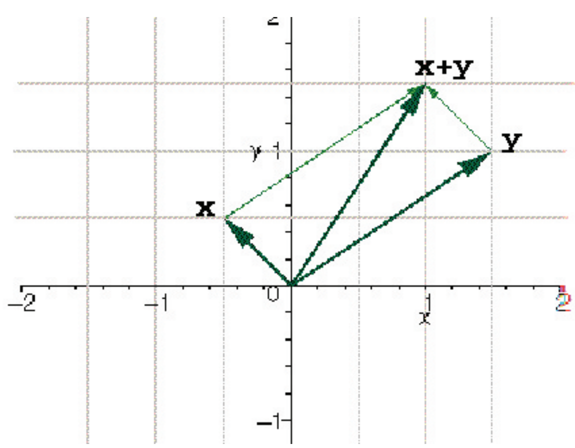
Example 2 (Vector addition)**.**

$$(\,-3, 2, 5) + (1, 1, 1) = (\,-2, 3, 6)$$

$$(8, -2, 4, -12) + (0, 0, 0, 0) = (8, -2, 4, -12)$$

$$(8, -2, 4, -12) + (\,-8, 2, -4, 12) = (0, 0, 0, 0) = \mathbf{0}\ .$$

There is a geometric way to think about vector addition in $\mathbb{R}^2$. Identify the vector $(x, y) \in \mathbb{R}^2$ with the point the Euclidean plane having these Cartesian coordinates. We can then represent this vector geometrically by drawing an arrow with its tail at the origin and its head at (x, y). The following diagram shows three vectors represented this way: $\mathbf{x} = (\,-1/2, 1/2)$, $\mathbf{y} = (3/2, 1)$ and their sum, $\mathbf{x}+\mathbf{y} = (1, 3/2)$.



The vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{x}+\mathbf{y}$ themselves are drawn in bold. There are also two arrows drawn more lightly: one is a parallel copy of $\mathbf{x}$ "transported" so its tail is at the head of $\mathbf{y}$. The other is a parallel copy of $\mathbf{y}$ "transported" so its tail is at the head of $\mathbf{x}$. These four arrows run along the sides of the parallelogram whose vertices are the origin, and the points corresponding to $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{x}+\mathbf{y}$. As you see, the arrow representing $\mathbf{x}+\mathbf{y}$ is the diagonal of this parallelogram that has its "tail end" at the origin.

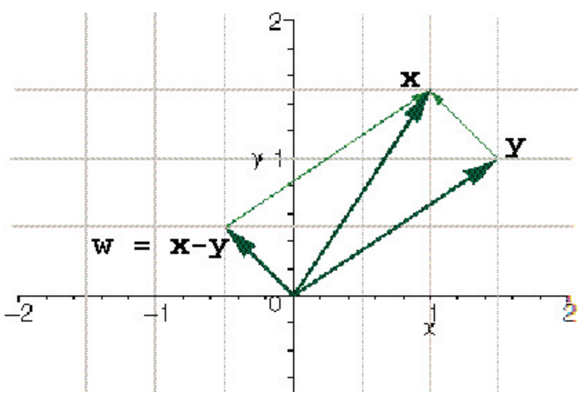
A similar diagram could be drawn for any pair of vectors and their sum, and you see that we can think of vector addition in the plane as corresponding to the following operation:

• *Represent the vectors by arrows as in the diagram. Transport one arrow without turning it – that is, in a parallel motion – to bring its tail to the other arrow's head. The head of the transported arrow is now at the point corresponding to the sum of the vectors.*

Example 3 (Subtraction of vectors)**.** *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two vectors in the plane* $\mathbb{R}^2$*, and let* $\mathbf{w} = \mathbf{x}-\mathbf{y}$*. Then, using the associative and commutative properties of vector addition,*

$$\mathbf{x} = \mathbf{x} + (\mathbf{y}-\mathbf{y}) = (\mathbf{x}-\mathbf{y}) + \mathbf{y} = \mathbf{y} + \mathbf{w} \ .$$

Using the same diagram, with the arrow labeled a bit differently, we see that $\mathbf{w} = \mathbf{x}-\mathbf{y}$ *is the arrow running from the head of* $\mathbf{y}$ *to the head of* $\mathbf{x}$*, "parallel transported" so that its tail is at the origin.*



Now let us begin to connect the algebra we have developed in this subsection with Descartes' ideas. The key is the introduction of the *standard basis for* $\mathbb{R}^n$:

Definition 4 (Standard basis for $\mathbb{R}^n$)**.** *For* $j = 1, \ldots, n$*, let* $\mathbf{e}_j$ *denote the vector in* $\mathbb{R}^n$ *whose* j*th entry is* 1*, and all of whose remaining entries are* 0*. The ordered set* $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ *is the standard basis for* $\mathbb{R}^n$*.*

For example, if $n = 3$, we have

$$\mathbf{e}_1 = (1,0,0) \qquad \mathbf{e}_2 = (0,1,0) \qquad \text{and} \qquad \mathbf{e}_3 = (0,0,1) \ .$$

In this three dimensional case, subscripts are often more of a hinderance than a help and a standard notation is

$$\mathbf{i} = \mathbf{e}_1 \qquad \mathbf{j} = \mathbf{e}_2 \qquad \text{and} \qquad \mathbf{k} = \mathbf{e}_3 \ .$$

Definition 5 (Linear combination)**.** *Let* $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ *be any set of* m *vectors in* $\mathbb{R}^n$*. A* linear combination *of these vectors is any expression of the form*

$$\sum_{j=1}^{m} a_j \mathbf{x}_j \ .$$

Theorem 1 (Fundamental property of the standard basis)**.** *Every* $\mathbf{x} = (x_1, \ldots, x_n)$ *in* $\mathbb{R}^n$ *can be expressed as a linear combination of the standard basis vectors* $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$*, and the coefficients are uniquely determined:*

$$\mathbf{x} = \sum_{j=1}^{n} x_j \mathbf{e}_j \ . \tag{1.12}$$

Proof: By definition, $\sum_{j=1}^{n} x_j \mathbf{e}_j = x_1(1, 0, \ldots, 0) + \cdots + x_n(0, 0, \ldots, 1) = (x_1, x_2, \ldots, x_n)$.

Thus, any vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ can be written as a linear combination of the standard basis vectors: Next, by the computation we just made, $\sum_{j=1}^{n} y_j \mathbf{e}_j = (y_1, y_2, \ldots, y_n)$.Thus,

$$\mathbf{x} = \sum_{j=1}^{n} y_j \mathbf{e}_j \iff y_j = x_j \quad \text{for each } j = 1, \ldots, n \ ,$$

and hence the coordinates are uniquely determined. □

The fact that every vector in $\mathbb{R}^n$ can be expressed as a unique linear combination of the standard basis vectors is a special property of this set. For many other sets sets of vectors in $\mathbb{R}^n$, there may be vectors that cannot be expressed as a linear combination at all, while others can be expressed as such in infinitely many ways.

Example 4. *Let* $n = 2$ *and let* $\mathbf{v}_0 = (3, 4)$*.* $\mathbf{v}_1 = (1, -1)$*,* $\mathbf{v}_2 = (1, 1)$ *and* $\mathbf{v}_3 = (0, 2)$*. Then as you can check,*

$$\mathbf{v}_0 = \frac{3}{2}\mathbf{v}_1 + \frac{3}{2}\mathbf{v}_2 + 2\mathbf{v}_3 = 3\mathbf{v}_1 + \frac{7}{2}\mathbf{v}_3 = 3\mathbf{v}_2 + \frac{1}{2}\mathbf{v}_3 \ .$$

Thus, in this case, $\mathbf{v}_0$ *can be expressed as a linear combination of* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$*, and there are several ways to choose the coefficients* $\{a_1, a_2, a_3\}$ *– in fact, there are infinitely many.*

One the other hand, if $\mathbf{v}_0 = (1, 1, 1)$ *and* $\mathbf{v}_1 = (1, -1, 0)$*,* $\mathbf{v}_2 = (-1, 0, 1)$ *and* $\mathbf{v}_3 = (0, 1, -1)$*, then for any numbers* a*,* b *and* c*, we have*

$$a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 = (a - b, c - a, b - c) \ .$$

Adding up the entries on the right we get $a - b + c - a + b - c = 0$*. Adding up the entries in* $\mathbf{v}_0$*, we get* 3*. Hence there is no choice of* a*,* b *and* c *for which* $\mathbf{v}_0 = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3$*. Thus, in this case,* $\mathbf{v}_0$ cannot *be expressed as a linear combination of* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$*.*

The standard basis vectors thus provide the analog of a Cartesian frame for $\mathbb{R}^n$, in that one can get to any vector in $\mathbb{R}^n$ by adding up a multiples of the standard basis vectors, just as one can get to any point by moving along the directions of the vectors in the frame. However, frames were defined in terms of orthogonality, and so far we have no notion of geometry in $\mathbb{R}^n$, only algebra. In the next subsection we bring in the geometry.

1.1.4 Geometry and the dot product

So far, we have considered $\mathbb{R}^n$ in purely algebraic terms. Indeed, the modern notion of an *abstract vector space* is a purely algebraic construct generalizing the algebraic structure on $\mathbb{R}^n$ that has been the subject of the last subsection.

Additional structure is required to make contact with the notions of length and direction that have been traditionally associated to vectors. Let us begin with length, recalling that we have identified the length of a vector with distance from the origin. We introduce the notion of a "metric" which measures the "distance" between two points".

Definition 6. *Let X be a set. A function ϱ on the Cartesian product $X \times X$ with values in $[0, \infty)$ is a* metric *on X in case:*

(1) $\varrho(x, y) = 0$ if and only if $x = y$.

(2) For all $x, y \in X$, $\varrho(x, y) = \varrho(y, x)$.

(3) For all $x, y, z \in X$,

$$\varrho(x, z) \le \varrho(x, y) + \varrho(y, z) \ . \tag{1.13}$$

When ϱ is a metric on X, the pair (X, ϱ) is called a metric space.

As noted above, a metric measures the "distance" between two points in a metric space. At some intuitive level, the distance corresponds to the "length of the shortest path connecting x and y".

On an intuitive level, the phrase in quotes motivates the three items in the definition: Two points are the same if and only if there is no distance between them. This leads to *(1)*. The distance from x to y is the same as the distance from y to x; just "go back" on the same path. This leads to *(2)*. Finally, if you insist on stopping by y on your way from x to z, the detour can only increase the total distance traveled. This leads to *(3)*.

You might be able to think of some more requirements you would like to impose on the concept of distance. However, the mathematical value of a definition lies in the applicability of the theorems one can prove using it. It turns out that the definition of metric that we have just given provides a framework in which one can prove a great many very useful theorems. It is a very fruitful *abstraction* of the notion of distance in physical space. We now prepare to introduce the *Euclidean metric* on $\mathbb{R}^n$.

Definition 7 (Euclidean distance)**.** *The Euclidean length of a vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is denoted by $\|\mathbf{x}\|$, and is defined by*

$$\|\mathbf{x}\| = \left(\sum_{j=1}^{n} x_j^2 \right)^{1/2} \ .$$

The distance between two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is defined by $\|\mathbf{x} - \mathbf{y}\|$. A vector $\mathbf{x} \in \mathbb{R}^n$ is called a unit vector *in case $\|vx\| = 1$; i.e., in case $\mathbf{x}$ has unit length.*

We sometimes think of unit vectors as reprinting "pure directions" . Given any non-zero vector $\mathbf{x}$, we can write define the unit vector $\mathbf{u} = \dfrac{1}{\|\mathbf{x}\|}\mathbf{x}$, which is called the *normalization of* $\mathbf{x}$, and then we have

$$\mathbf{x} = \|\mathbf{x}\| \left(\frac{1}{\|\mathbf{x}\|}\mathbf{x} \right) = \|\mathbf{x}\|\mathbf{u} \ .$$

This way of writing $\mathbf{x}$ expresses it as the product of its length and direction.

As you can check from the definition, for any $\mathbf{x} \in \mathbb{R}^n$ and any $t \in \mathbb{R}$,

$$\|t\mathbf{x}\| = |t|\|\mathbf{x}\| \ . \tag{1.14}$$

As we shall soon see, the function ϱ_E defined by

$$\varrho_E(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\| \tag{1.15}$$

is a metric on $\mathbb{R}^n$, and is called the *Euclidean metric*. Indeed, with this definition of ϱ_E on $\mathbb{R}^2$, the distance between two vectors (x, y) and (u, v) is $\sqrt{(x-u)^2 + (y-v)^2}$. which is of course the usual formula derived from the Pythagorean Theorem.

It is easy to see that the function ϱ_E defined in (1.15) satisfies requirements *(1)* and *(2)* in the definition of a metric. The fact that it also satisfies *(3)* is less transparent, but fundamentally important.

The first step towards this is to write $\|\mathbf{x}\|$ in terms of the dot product, which we now define:

Definition 8 (Dot product)**.** *The* dot product *of two vectors*

$$\mathbf{a} = (a_1, \dots, a_n) \ , \ \mathbf{b} = (b_1, \dots, b_n)$$

in $\mathbb{R}^n$ *is given by*

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots a_n b_n \ .$$

Note that the dot product is commutative, meaning that $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. This follows directly from the definition and the commutativity of multiplication in $\mathbb{R}$. In the same way one sees that the dot product *distributes* in the sense that for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$, and all $s, t \in \mathbb{R}$,

$$(s\mathbf{a} + t\mathbf{b}) \cdot \mathbf{c} = s(\mathbf{a} \cdot \mathbf{c}) + t(\mathbf{b} \cdot \mathbf{c}) \ .$$

However, except when $n = 1$, it does not make sense to talk about the associativity of the dot product: For $n > 1$ and $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$, $\mathbf{a} \cdot \mathbf{b} \notin \mathbb{R}^n$, so $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ is not defined.

From the definitions, we have $\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x}$. Therefore, using the distributive and commutative properties of the dot product,

$$\begin{aligned} \|\mathbf{x} - \mathbf{y}\|^2 &= (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \\ &= \mathbf{x} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} - 2\mathbf{x} \cdot \mathbf{y} \ , \\ &= \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\mathbf{x} \cdot \mathbf{y} \ . \end{aligned} \tag{1.16}$$

The formula (1.16) has an interpretation in terms of the lengths of the vectors $\mathbf{x}$ and $\mathbf{y}$, and the angle between these vectors. The key to this is the law of cosines. Recall that if the lengths of the three sides of a triangle in a Euclidean plane are A, B and C, and the angle between the sides with lengths A and B is θ, then $C^2 = A^2 + B^2 - 2AB\cos\theta$.

Now let $\mathbf{x}$ and $\mathbf{y}$ be any vectors in the plane $\mathbb{R}^2$. Consider the triangle whose vertices are $\mathbf{0}$, $\mathbf{x}$ and $\mathbf{y}$. Define the angle between $\mathbf{x}$ and $\mathbf{y}$ to be the angle between the two sides of the triangle issuing from the vertex at $\mathbf{0}$. Since the length of the side of the triangle opposite this vertex is $\|\mathbf{x}-\mathbf{y}\|$, by the law of cosines,

$$\|\mathbf{x}-\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\|\mathbf{x}\|\|\mathbf{y}\|\cos\theta \ .$$

Comparing this with (1.16), we conclude that $\mathbf{x}\cdot\mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\|\cos\theta$. Therefore, in two dimensions, we have proved that the angle θ between two non-zero vectors $\mathbf{x}$ and $\mathbf{y}$, considered as sides of a triangle in the plane $\mathbb{R}^2$, is given by the formula

$$\theta = \arccos\left(\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}\right) \ , \tag{1.17}$$

where the arccosine function is defined on $[-1,1]$ with values in $[0,\pi]$.

The same sort of reasoning applies to vectors in $\mathbb{R}^3$ since any two non-colinear vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$ lie in the plane determined by the three points $\mathbf{0}$, $\mathbf{x}$ and $\mathbf{y}$, and then the law of cosines may be applied in this plane. Thus, the formula (1.17) is valid in $\mathbb{R}^3$ as well.

Why not go on from here? It may seem intuitively clear that just as in $\mathbb{R}^3$, again in $\mathbb{R}^n$ for any $n \geq 3$, any two non-colinear vectors $\mathbf{x}$ and $\mathbf{y}$ lie in a two dimensional plane in which we can apply the law of cosines, just as we did in $\mathbb{R}^2$ and $\mathbb{R}^3$. This suggests that we use use (1.17) to *define* the angle between two vectors in $\mathbb{R}^n$:

Definition 9. *Let $\mathbf{x}$ and $\mathbf{y}$ be two non-zero vectors in $\mathbb{R}^n$. Then the angle θ between $\mathbf{x}$ and $\mathbf{y}$ is defined to be*

$$\theta = \arccos\left(\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}\right) \ , \tag{1.18}$$

Two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ are orthogonal *in case $\mathbf{x}\cdot\mathbf{y} = 0$.*

There is an important matter to be checked before going forward: Does this definition make sense? The issue is that the arccos function is defined on $[-1,1]$, so it had better be the case that

$$-1 \leq \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} \leq 1$$

for all nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$. This certainly is the case for $n = 2$ and $n = 3$, where we have proved the formula (1.17) is true with the classical definition of θ. but what about larger values of n? The following theorem shows that there is no problem with using (1.18) to define θ no matter what the dimension n is. (It has many other uses as well!)

Theorem 2 (Cauchy–Schwartz inequality)**.** *For any two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$,*

$$|\mathbf{a}\cdot\mathbf{b}| \leq \|\mathbf{a}\|\|\mathbf{b}\| \ . \tag{1.19}$$

There is equality in (1.19) if and only if $\|\mathbf{b}\|\mathbf{a} = \pm\|\mathbf{a}\|\mathbf{b}$

Proof: Clearly (1.19) is true, with equality, in case either of the vectors is the zero vector, and also in this case $\|\mathbf{b}\|\mathbf{a} = \|\mathbf{a}\|\mathbf{b} = 0$.

Hence we may assume that neither $\mathbf{a}$ nor $\mathbf{b}$ is the zero vector. Under this assumption, define $\mathbf{x} = \dfrac{\mathbf{a}}{\|\mathbf{a}\|}$ and $\mathbf{y} = \dfrac{\mathbf{b}}{\|\mathbf{b}\|}$. Now let us compute $\|\mathbf{x}-\mathbf{y}\|^2$:

$$\|\mathbf{x}-\mathbf{y}\|^2 = (\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) = \mathbf{x}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{y} - 2\mathbf{x}\cdot\mathbf{y} = 2(1-\mathbf{x}\cdot\mathbf{y}) \ ,$$

where we have used the fact that $\mathbf{x}$ and $\mathbf{y}$ are unit vectors. But since the left had side is certainly non-negative, is must be the case that $\mathbf{x}\cdot\mathbf{y} \le 1$.

Likewise, computing $\|\mathbf{x}+\mathbf{y}\|^2 = 2(1+\mathbf{x}\cdot\mathbf{y})$, we see that $\mathbf{x}\cdot\mathbf{y} \ge -1$. Thus,

$$-1 \le \mathbf{x}\cdot\mathbf{y} \le 1 \ ,$$

which is equivalent to (1.19) by the definition of $\mathbf{x}$ and $\mathbf{y}$.

As for the cases of equality, $|\mathbf{x}\cdot\mathbf{y}| = 1$ if and only if either $\|\mathbf{x}-\mathbf{y}\| = 0$ or $\|\mathbf{x}+\mathbf{y}\| = 0$, which means $\mathbf{x} = \pm\mathbf{y}$. From the definitions of $\mathbf{x}$ and $\mathbf{y}$, this is the same as $\|\mathbf{b}\|\mathbf{a} = \pm\|\mathbf{a}\|\mathbf{b}$. □

Theorem 3 (Triangle inequality)**.** *For any three vectors* $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{R}^n$,

$$\|\mathbf{x}-\mathbf{z}\| \le \|\mathbf{x}-\mathbf{y}\| + \|\mathbf{y}-\mathbf{z}\| \ . \tag{1.20}$$

Proof: Let $\mathbf{a} = \mathbf{x}-\mathbf{y}$ and $\mathbf{b} = \mathbf{z}-\mathbf{y}$ so that $\mathbf{x}-\mathbf{z} = \mathbf{a}-\mathbf{b}$. Then by (1.16), and then the Cauchy-Schwarz inequality,

$$\begin{aligned} \|\mathbf{x}-\mathbf{z}\|^2 = \|\mathbf{a}-\mathbf{b}\|^2 &= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a}\cdot\mathbf{b} \\ &\le \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + 2\|\mathbf{a}\|\|\mathbf{b}\| \\ &= (\|\mathbf{a}\| + \|\mathbf{b}\|)^2 \ . \end{aligned}$$

Taking square roots of both sides, and recalling the definitions of $\mathbf{a}$ and $\mathbf{b}$, we obtain (1.20). □

Theorem 4. *The Euclidean distance function*

$$\varrho_E(x,y) := \sqrt{(\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y})} = \|\mathbf{x}-\mathbf{y}\|$$

is a metric on $\mathbb{R}^n$.

Proof: The function $\varrho_E(\mathbf{x},\mathbf{y})$ is non-negative and clearly satisfies conditions *(1)* and *(2)* in the definition of a metric. By Theorem 3, it also satisfies *(3)*. □

1.1.5 Parallel and orthogonal components

Now that we have equipped $\mathbb{R}^n$ with geometric as well as algebraic structure, let us put this to work for something that is simple but useful – the decomposition of vectors into parallel and orthogonal components:

Definition 10 (Parallel and orthogonal components)**.** *Given some non-zero vector* $\mathbf{a} \in \mathbb{R}^n$*, let* $\mathbf{u} := \dfrac{1}{\|\mathbf{a}\|}\mathbf{a}$*, which is the unit vector in the direction of* $\mathbf{a}$*. We can decompose any vector* $\mathbf{x} \in \mathbb{R}^n$ *into two pieces,* $\mathbf{x}_{\|}$ *and* $\mathbf{x}_{\perp}$ *where*

$$\mathbf{x}_{\|} := (\mathbf{x}\cdot\mathbf{u})\mathbf{u} \qquad \text{and} \qquad \mathbf{x}_{\perp} := \mathbf{x} - (\mathbf{x}\cdot\mathbf{u})\mathbf{u} \ . \tag{1.21}$$

These two vectors are called, respectively, the parallel and orthogonal components of **x** with respect to **a**.

By the definition and simple computations,

$$\mathbf{x} = \mathbf{x}_{\|} + \mathbf{x}_{\perp} \qquad \text{and} \qquad \mathbf{x}_{\|} \cdot \mathbf{x}_{\perp} = 0 \ . \tag{1.22}$$

As a consequence of (1.21),

$$\|\mathbf{x}\|^2 = (\mathbf{x}_{\|} + \mathbf{x}_{\perp}) \cdot (\mathbf{x}_{\|} + \mathbf{x}_{\perp}) = \|\mathbf{x}_{\|}\|^2 + \|\mathbf{x}_{\perp}\|^2 \ ,$$

which is a form of the Pythagorean Theorem.

Note that $\mathbf{x}_{\|}$ is a multiple of **u**, and hence of **a**, which is why we refer to $\mathbf{x}_{\|}$ as "parallel" to **a**. (Actually, it could be a negative multiple of **a**, in which case $\mathbf{x}_{\|}$ and **a** would have the opposite direction, but would at least lie on the same line through the origin.) Also $\mathbf{x}_{\perp}$ is orthogonal to $\mathbf{x}_{\|}$, and hence to **a**, which is why we refer to $\mathbf{x}_{\perp}$ as "orthogonal" to **a**.

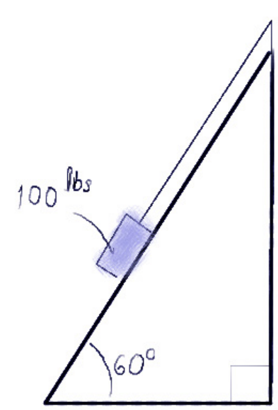
There is one final part of our terminology to be justified: the definite article "the": Suppose $\mathbf{x} = \mathbf{b} + \mathbf{c}$ where **b** is a multiple of **a**, and **c** is orthogonal to **a**. Then from this way of writing **x** as well as (1.22) we have $\mathbf{b} + \mathbf{c} = \mathbf{x}_{\|} + \mathbf{x}_{\perp}$ which can be rearranged into

$$\mathbf{b} - \mathbf{x}_{\|} = \mathbf{x}_{\perp} - \mathbf{c} \ .$$

Note that the vector on the left is a multiple of **a**, while the vector on the right is orthogonal to **a**. Since they are both the same vector, they are orthogonal to themselves. But the only vector that is orthogonal to itself is **0**, and so $\mathbf{b} - \mathbf{x}_{\|} = \mathbf{0}$ and $\mathbf{x}_{\perp} - \mathbf{c} = \mathbf{0}$. That is, $\mathbf{b} = \mathbf{x}_{\|}$ and $\mathbf{c} = \mathbf{x}_{\perp}$. In summary, there is one and only one way to decompose a vector **x** as a sum of multiple of **a** and a vector orthogonal to **a**, and this decomposition is given by (1.21)

The decomposition of vectors into parallel and orthogonal components is often useful. Here is a first example of this.

Example 5. *A 100 pound weight sits on an slick (frictionless) incline making a 60 degree angle with the horizontal. It is held in place by a rope attached to the base of the weight and tied down at the top of the ramp. The tensile strength of the rope is such that it is only guaranteed not to break for tensions of no more than 80 pounds. Is this a dangerous situation?*



To answer this we need to compute the tension in the rope. Let us use coordinates with the rope lying in the x, y *plane, and the* y *axis being vertical. The gravitational force vector, measured in pounds, is*

$$\mathbf{f} = (0, -100)\ .$$

The unit vector pointing down *the slope is*

$$\mathbf{u} = -(\cos(\pi/3), \sin(\pi/3)) = -\frac{1}{2}(1, \sqrt{3})$$

since 60 degrees is $\pi/3$ *radians. The tension in the rope must balance the component of the gravitation force in the direction* $\mathbf{u}$*; i.e., the direction of possible motion. That is, the magnitude of the tension will be* $\|\mathbf{f}_{\|}\|$ *where* $\mathbf{f}_{\|}$ *is computed with respect to* $\mathbf{u}$*. Doing the computation we find*

$$\mathbf{f}_{\|} = (\mathbf{f} \cdot \mathbf{u})\mathbf{u} = -25\sqrt{3}(1, \sqrt{3})\ ,$$

and thus $\|\mathbf{f}_{\|}\| = 50\sqrt{3} \approx 86.6$*. Look out below!*

Here is another way to think about the computation in the previous example. The gravitational force vector $\mathbf{f}$ has a simple expression in standard x, y coordinates, but these coordinates are no well adapted to the problem at hand since neither coordinate axis corresponds to a possible direction of possible motion. The direction of possible motion is given by $\mathbf{u}$.

Let us consider a coordinate system built around the direction of $\mathbf{u}$. We then take $\mathbf{v}$ to be on of the two unit vectors in $\mathbb{R}^2$ that is orthogonal to $\mathbf{u}$. Since $\mathbf{f}$ points downward, we take the one that points downward. That is $bv = \dfrac{1}{2}(\sqrt{3}, 1)$. As you can easily check

$$\mathbf{u} \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{v} = 1 \qquad \text{and} \qquad \mathbf{u} \cdot \mathbf{v} = 0\ , \tag{1.23}$$

so that $\{\mathbf{u}, \mathbf{v}\}$ is an orthogonal pair of unit vectors.

Let us write the force vector $\mathbf{f}$ in coordinates based on the $\{\mathbf{u}, \mathbf{v}\}$ frame of reference. That is,

$$\mathbf{f} = u\mathbf{u} + v\mathbf{v} \tag{1.24}$$

for some numbers u and v, which are the coordinates of $\mathbf{f}$ with respect to this frame of reference. The u, v coordinates of $\mathbf{f}$ are directly relevant to our problem. In particular u is the magnitude of the force in the direction u, and is what the tension in the rope must balance. Hence in these coordinates, our question becomes: *Is* $u > 80$*?*.

To answer this question we need to know how to compute u and v in terms of the x, y coordinates of $\mathbf{f}$, which is what we are given. Here is how: Take the dot products of both sides of (1.24) with $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{f} \cdot \mathbf{u} = (u\mathbf{u} + v\mathbf{v}) \cdot \mathbf{u} = u\mathbf{u} \cdot \mathbf{u} + v\mathbf{v} \cdot \mathbf{v} = u$$

where we have used the distributive property of the dot product and (1.23). In the same way, we find $v = \mathbf{f} \cdot \mathbf{v}$. Thus, we can re-write (1.24) as

$$\mathbf{f} = (\mathbf{f} \cdot \mathbf{u})\mathbf{u} + (\mathbf{f} \cdot \mathbf{v})\mathbf{v}\ , \tag{1.25}$$

or in other words, $u = \mathbf{f} \cdot \mathbf{u}$ and $v = \mathbf{f} \cdot \mathbf{v}$.

As we shall see:

• *The first step in solving many problems is to introduce a system of coordinates that is adapted to the problem, and in particular is "built out of" directions given in the problem.*

The most broadly useful and convenient coordinate systems in $\mathbb{R}^n$ are those constructed using a set of n mutually orthogonal unit vectors, such as the set $\{\mathbf{u}, \mathbf{v}\}$ of orthogonal unit vectors in $\mathbb{R}^2$ that we have just used to build coordinates for our inclined plane problem. The next subsection develops this idea in general.

1.1.6 Orthonormal subsets of $\mathbb{R}^n$

Definition 11 (Orthonormal vectors in $\mathbb{R}^n$). *A set $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ of m vectors in $\mathbb{R}^n$ is* orthonormal *in case for all $1 \le i, j \le m$*

$$\mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} . \tag{1.26}$$

Example 6. *The set of standard basis vectors $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is one very simple example of an orthonormal set. Also, any non-empty subset of an orthonormal set is easily seen to be orthonrmal, so we can get other examples by taking non-empty subsets of $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. These are important examples, but not the only ones. Here is a more invteresting example: Let*

$$\mathbf{u}_1 = \frac{1}{3}(1, 2, -2) \qquad \mathbf{u}_2 = \frac{1}{3}(2, 1, 2) \qquad \text{and} \qquad \mathbf{u}_3 = \frac{1}{3}(2, -2, 1) \ . \tag{1.27}$$

Then you can easily check that (1.26) is satisfied, so that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set in $\mathbb{R}^3$. In fact, it is a rather special kind of orthonormal set: It is an orthonormal basis*, as defined next.*

The main theorem concerning orthonormal sets in $\mathbb{R}^n$ is the following.

Theorem 5 (Fundamental Theorem on Orthonormal Sets in $\mathbb{R}^n$). *Let $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be any orthonormal set in $\mathbb{R}^n$ consisting of exactly n vectors. Then every vector $\mathbf{x} \in \mathbb{R}^n$ can be written as a linear combination of the the vectors $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ in exactly one way, namely*

$$\mathbf{x} = \sum_{j=1}^{n} (\mathbf{x} \cdot \mathbf{u}_j)\mathbf{u}_j \ . \tag{1.28}$$

Moreover, the squared length of $\mathbf{x}$ is the sum of the squares of the coefficients in this expansion:

$$\|\mathbf{x}\|^2 = \sum_{j=1}^{n} (\mathbf{x} \cdot \mathbf{u}_j)^2 \ .$$

The standard basis of $\mathbb{R}^n$, $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a set of n orthonormal vectors in $\mathbb{R}^n$, and so the theorem says that

$$\mathbf{x} = \sum_{j=1}^{n} x_j \mathbf{e}_j = \sum_{j=1}^{n} (\mathbf{x} \cdot \mathbf{e}_j)\mathbf{e}_j$$

is the unique way to express any $\mathbf{x}$ in $\mathbb{R}^n$ as a linear combination of the standard basis vectors. This is the content of Theorem 1. Theorem 5 generalizes this to arbitrary sets of n orthonormal vectors in

$\mathbb{R}^n$. It allows us to take *any* set of n orthonormal vectors in $\mathbb{R}^n$ as the basis of a coordinate system in $\mathbb{R}^n$. This will prove to be *very useful* in practice. It will allow us to use coordinates that are especially adapted to whatever computation we are trying to make, which will often be *much* simpler than a direct computation using coordinates based on the stanard basis. The next definitions pave the way for this.

Definition 12 (Orthonormal basis)**.** *An* orthonormal basis *in* $\mathbb{R}^n$ *is any set of* n *orthonormal vectors in* $\mathbb{R}^n$*.*

As we have noted above, the standard basis is one example, but there are many others. We have seen in Example 6 that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ with the vectors specified by (1.27) is an orthonormal set of three vectors in $\mathbb{R}^3$. Hence it is an orthonormal basis for $\mathbb{R}^3$.

Definition 13 (Coordinates with respect to an orthonormal basis)**.** *Consider an orthonormal basis* $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ *of* $\mathbb{R}^n$*, and a vector* $\mathbf{x}$ *in* $\mathbb{R}^n$*. Then the numbers* $\mathbf{x} \cdot \mathbf{u}_j$*,* $1 \leq j \leq n$*, are called the* coordinates of $\mathbf{x}$ with respect to $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$*.*

The coordinates of a vector $\mathbf{x} \in \mathbb{R}^n$ with respect to an orthonormal basis behave just like Cartesian coordinates in that they tell you how to get from $\mathbf{0}$ to $\mathbf{x}$: If the coordinates are $y_1, \ldots, y_n$, start at $\mathbf{0}$, move y_1 units in the $\mathbf{u}_1$ direction, then move y_2 units in the $\mathbf{u}_2$ direction and so forth.

The hard part of the proof of Theorem 5 is contained in the following lemma:

Lemma 1 (No $n+1$ orthonormal vectors in $\mathbb{R}^n$)**.** *There does not exist any set of* $n+1$ *orthonormal vectors in* $\mathbb{R}^n$*.*

The proof of this simple statement is very instructive, and very important, but somewhat involved. We give it in full in the next subsection. For now, let us take it on faith, and see how we may use it to prove Theorem 5. Most of the applications we make in the next section will be in $\mathbb{R}^2$ and $\mathbb{R}^3$, and you will probably agree that the lemma is "geometrically obvious" in these cases where you can easily visualize things.

Proof of Theorem 5: Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be any set of n orthonormal vectors in $\mathbb{R}^n$, and let $\mathbf{x}$ be ano non-zero vector in $\mathbb{R}^n$. Define the vector $\mathbf{z}$ by

$$\mathbf{z} := \mathbf{x} - \sum_{j=1}^{n} (\mathbf{x} \cdot \mathbf{u}_j)\mathbf{u}_j \ .$$

for each $i = 1, \ldots, n$, we have

$$\mathbf{z} \cdot \mathbf{u}_i = \left(\mathbf{x} - \sum_{j=1}^{n} (\mathbf{x} \cdot \mathbf{u}_j)\mathbf{u}_j \right) \cdot \mathbf{u}_i = \mathbf{x} \cdot \mathbf{u}_i - \sum_{j=1}^{n} (\mathbf{x} \cdot \mathbf{u}_j)\mathbf{u}_j \cdot \mathbf{u}_i = \mathbf{x} \cdot \mathbf{u}_i - \mathbf{x} \cdot \mathbf{u}_i = 0 \ .$$

Thus, $\mathbf{z}$ is orthogonal to each $\mathbf{u}_i$.

Now suppose that $\mathbf{z} \neq \mathbf{0}$, Then we may define a unit vector $\mathbf{u}_{n+1}$ by $\mathbf{u}_{n+1} = \dfrac{1}{\|\mathbf{z}\|}\mathbf{z}$. Since this unit vector is orthogonal to each unit vector in the the orthonormal set $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$, the augmented set $\{\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{u}_{n+1}\}$ would be a set of $n+1$ orthonormal vectors in $\mathbb{R}^n$. By Lemma 1, this is impossible. Therefore, $\mathbf{z} = 0$. By the definition of $\mathbf{z}$, this means that $\mathbf{x} = \sum_{j=1}^{n} (\mathbf{x} \cdot \mathbf{u}_j)\mathbf{u}_j$, which is (1.28). Thus, every vector $\mathbf{x} \in \mathbb{R}^n$ can be written as a linear combination of the vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$. Moreover, there is only one way to do this, since if $\mathbf{x} = \sum_{j=1}^{n} y_j \mathbf{u}_j$, taking the dot product of both sides with $\mathbf{u}_i$ yields

$$\mathbf{x} \cdot \mathbf{u}_i = \left(\sum_{j=1}^{n} y_j \mathbf{u}_j \right) \cdot \mathbf{u}_i = \sum_{j=1}^{n} y_j (\mathbf{u}_j \cdot \mathbf{u}_i) = y_i \ .$$

That is, each y_i must equal $\mathbf{x} \cdot \mathbf{u}_i$.

Finally, going back to (1.28), we compute

$$\begin{aligned} \|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x} &= \left(\sum_{j=1}^{n} (\mathbf{x} \cdot \mathbf{u}_j)\mathbf{u}_j \right) \cdot \left(\sum_{k=1}^{n} (\mathbf{x} \cdot \mathbf{u}_k)\mathbf{u}_k \right) \\ &= \sum_{j,k=1}^{n} (\mathbf{x} \cdot \mathbf{u}_j)(\mathbf{x} \cdot \mathbf{u}_k)\mathbf{u}_j \cdot \mathbf{u}_k = \sum_{j=1}^{n} (\mathbf{x} \cdot \mathbf{u}_j)^2 \ . \end{aligned}$$

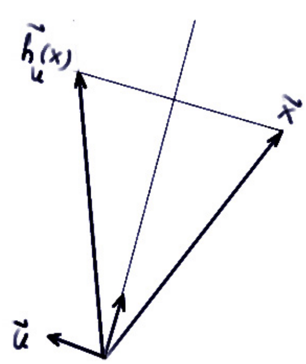
□

1.1.7 Householder reflections and orthonormal bases

In this subsection we shall prove Lemma 1. The proof is very interesting, and introduces many techniques and ideas that will be important later on.

We begin by introducing an extremely useful class of functions $\mathbf{f}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$: the *Householder reflections*.

First, for $n = 2$, fix a unit vector $\mathbf{u} \in \mathbb{R}^2$ and consider the line $\ell_\mathbf{u}$ through the origin that is *orthogonal* to $\mathbf{u}$. Then, for any $\mathbf{x} \in \mathbb{R}^2$, define $\mathbf{h}_\mathbf{u}(\mathbf{x})$ to be the *mirror image* of $\mathbf{x}$ across the line $\ell_\mathbf{u}$. That it, $\mathbf{h}_\mathbf{u}(\mathbf{x})$ is the *reflection* of $\mathbf{x}$ across the line $\ell_\mathbf{u}$. Here is a picture illustrating the transformation form $\mathbf{x}$ to $\mathbf{h}_\mathbf{u}(\mathbf{x})$:



The transformation from $\mathbf{x}$ to $\mathbf{h}_\mathbf{u}(\mathbf{x})$ is geometrically well defined, and you could easily plot the output point $\mathbf{h}_\mathbf{u}(\mathbf{x})$ for any given input point $\mathbf{x}$. But to do computations, we need a formula. Let us derive a formula.

The key thing to realize, which you can see in the picture, is that both $\mathbf{x}$ and $\mathbf{h}_\mathbf{u}(\mathbf{x})$ have the *same* component orthogonal to $\mathbf{u}$ (that is, along the line $\ell_\mathbf{u}$) and have *opposite* components parallel to $\mathbf{u}$. In formulas, with respect to the direction $\mathbf{u}$,

$$(\mathbf{h_u}(\mathbf{x}))_\perp = \mathbf{x}_\perp \qquad \text{and} \qquad (\mathbf{h_u}(\mathbf{x}))_\parallel = -\mathbf{x}_\parallel \ .$$

Therefore, since $\mathbf{h_u}(\mathbf{x}) = (\mathbf{h_u}(\mathbf{x}))_\perp + (\mathbf{h_u}(\mathbf{x}))_\parallel$, we have the formula

$$\mathbf{h_u}(\mathbf{x}) = \mathbf{x}_\perp - \mathbf{x}_\parallel \ . \tag{1.29}$$

Then since $\mathbf{x}_\perp = \mathbf{x} - (\mathbf{x}\cdot\mathbf{u})\mathbf{u}$ and $\mathbf{x}_\parallel = (\mathbf{x}\cdot\mathbf{u})\mathbf{u}$, we deduce the more explicit formula

$$\mathbf{h_u}(\mathbf{x}) = \mathbf{x} - 2(\mathbf{x}\cdot\mathbf{u})\mathbf{u} \ . \tag{1.30}$$

We have derived the formula (1.30) for $n = 2$. However, the formula does not explicitly involve the dimension and makes sense in any dimension. Now, given any unit vector $\mathbf{u} \in \mathbb{R}^n$, for any positive integer n, we use this formula to *define* the transformation $\mathbf{h_u}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ that we shall call the *Householder reflection on* $\mathbb{R}^n$ *in the direction* $\mathbf{u}$:

Definition 14 (Householder reflection in the direction $\mathbf{u}$)**.** *For any unit vector $\mathbf{u} \in \mathbb{R}^n$, the function $\mathbf{h_u}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ is defined by (1.30).*

Example 7. *Let $\mathbf{u} = \frac{1}{\sqrt{3}}(-1, 1, -1)$. As you can check, this is a unit vector. Let $\mathbf{x} = (x, y, z)$ denote the generic vector in $\mathbb{R}^3$. Let us compute $\mathbf{h_u}(\mathbf{x})$. First, we find*

$$\mathbf{x}\cdot\mathbf{u} = \frac{y - x - z}{\sqrt{3}} \qquad \text{and hence} \qquad (\mathbf{x}\cdot\mathbf{u})\mathbf{u} = \frac{y - x - z}{3}(-1, 1, -1) \ .$$

Notice that the square roots have (conveniently) gone away! Now, from the definition of $\mathbf{h_u}$,

$$\mathbf{h_u}(x, y, z) = (x, y, x) - \frac{2y - 2x - 2z}{3}(-1, 1, -1) = \frac{1}{3}(x + 2y - 2z, 2x + y + 2z, -2x + 2y + z) \ .$$

This is an example of a function, or, what is the same thing, transformation *from $\mathbb{R}^3$ to $\mathbb{R}^3$. If the input vector is (x, y, z), the output vector is*

$$\mathbf{h_u}(x, y, z) = \left(\frac{x + 2y - 2z}{3}, \frac{2x + y + 2z}{3}, \frac{-2x + 2y + z}{3}\right) \ . \tag{1.31}$$

To conclude the example, let us evaluate the transformation at particular vector. We choose, more or less at random,

$$\mathbf{x} = (1, 2, 3) \ .$$

Plugging this into our formula (1.31) we find

$$\mathbf{h_u}(1, 2, 3) = \frac{1}{3}(-1, 10, 5) \ .$$

Let us compute the length of $\mathbf{h_u}(\mathbf{x})$:

$$\|\mathbf{h_u}(\mathbf{x})\| = \frac{1}{3}\|(-1, 10, 5)\| = \frac{1}{3}\sqrt{1 + 100 + 25} = \sqrt{14} \ .$$

Notice that $\|\mathbf{x}\| = \sqrt{1 + 4 + 9} = \sqrt{14}$, so $\|\mathbf{h_u}(\mathbf{x})\| = \|\mathbf{x}\|$. This will always be the case, as we explain next – and as should be the case: reflection preserves the lengths of vectors.

Householder reflections have a number of special properties that will prove to be very useful to us. Here is the first of these:

Lemma 2 (Householder reflections preserve dot products)**.** *For any two vectors* $\mathbf{x}$ *and* $\mathbf{y}$ *in* $\mathbb{R}^n$*, and any unit vector* $\mathbf{u}$ *in* $\mathbb{R}^n$*,*

$$(\mathbf{h_u}(\mathbf{x})) \cdot (\mathbf{h_u}(\mathbf{y})) = \mathbf{x} \cdot \mathbf{y} \ . \tag{1.32}$$

In particular,

$$\|\mathbf{h_u}(\mathbf{x})\| = \|\mathbf{x}\| \ . \tag{1.33}$$

Proof: We use (1.29) and the fact that $\mathbf{x}_\perp$ is orthogonal to $\mathbf{y}_\parallel$ and that $\mathbf{x}_\parallel$ is orthogonal to $\mathbf{y}_\perp$ to compute:

$$\begin{aligned} (\mathbf{h_u}(\mathbf{x})) \cdot (\mathbf{h_u}(\mathbf{y})) &= \left(\mathbf{x}_\perp - \mathbf{x}_\parallel\right) \cdot \left(\mathbf{y}_\perp - \mathbf{y}_\parallel\right) \\ &= \mathbf{x}_\perp \cdot \mathbf{y}_\perp + \mathbf{x}_\parallel \cdot \mathbf{y}_\parallel - \mathbf{x}_\perp \cdot \mathbf{y}_\parallel - \mathbf{x}_\parallel \cdot \mathbf{y}_\perp \\ &= \mathbf{x}_\perp \cdot \mathbf{y}_\perp + \mathbf{x}_\parallel \cdot \mathbf{y}_\parallel \ , \end{aligned}$$

and

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} &= \left(\mathbf{x}_\perp - +\mathbf{x}_\parallel\right) \cdot \left(\mathbf{y}_\perp + \mathbf{y}_\parallel\right) \\ &= \mathbf{x}_\perp \cdot \mathbf{y}_\perp + \mathbf{x}_\parallel \cdot \mathbf{y}_\parallel + \mathbf{x}_\perp \cdot \mathbf{y}_\parallel + \mathbf{x}_\parallel \cdot \mathbf{y}_\perp \\ &= \mathbf{x}_\perp \cdot \mathbf{y}_\perp + \mathbf{x}_\parallel \cdot \mathbf{y}_\parallel \ , \end{aligned}$$

Comparing the two computations proves (1.32). Then (1.33) follows by considering the special case $\mathbf{y} = \mathbf{x}$. □

In fact, since angles between vectors as well as the lengths of vectors are defined in terms of the dot product, Householder reflections preserve angles between vectors as well as the lengths of vectors, as you would expect from the diagram. In particular, *Householder reflections preserve orthogonality.*

Householder reflections are invertible transformations. *In fact, they are their own inverses*: For all $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{h_u}(\mathbf{h_u}(\mathbf{x})) = \mathbf{x}$, That is, $\mathbf{h_u} \circ \mathbf{h_u}$ is the identity function.

To see this from the defining formula, we compute

$$\mathbf{h_u}(\mathbf{h_u}(\mathbf{x})) = \mathbf{h_u}(\mathbf{x}_\perp - \mathbf{x}_\parallel) = \mathbf{x}_\perp - (-\mathbf{x}_\parallel) = \mathbf{x}_\perp + \mathbf{x}_\parallel = \mathbf{x} \ .$$

That is, reflecting a vector twice (about the same direction) leaves you with the vector you started with.

Since reflection does not alter the length of a vector, if we are given vectors $\mathbf{x}$ and $\mathbf{y}$ with $\|\mathbf{x}\| \neq \|\mathbf{y}\|$, then we cannot possibly find a unit vector $\mathbf{u}$ such that $\mathbf{h_u}(\mathbf{x}) \neq \mathbf{y}$. However, if $\|\mathbf{x}\| = \|\mathbf{y}\|$, but $\mathbf{x} \neq \mathbf{y}$, then there is always a “standard” Householder reflection $\mathbf{h_u}$ such that $\mathbf{h_u}(\mathbf{x}) = \mathbf{y}$:

Lemma 3. *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$*,* $n \geq 2$*, satisfy* $\|\mathbf{x}\| = \|\mathbf{y}\|$*, but* $\mathbf{x} \neq \mathbf{y}$*. Then there is a unit vector* $\mathbf{u} \in \mathbb{R}^n$ *so that for the corresponding Householder reflection* $\mathbf{h_u}$*,*

$$\mathbf{h_u}(\mathbf{x}) = \mathbf{y} \ .$$

In particular, one may always choose

$$\mathbf{u} = \frac{1}{\|\mathbf{x}-\mathbf{y}\|}(\mathbf{x}-\mathbf{y}) \ . \tag{1.34}$$

Moreover, with this choice of $\mathbf{u}$*, for any* $\mathbf{z} \in \mathbb{R}^n$ *that is orthogonal to both* $\mathbf{x}$ *and* $\mathbf{y}$

$$\mathbf{h_u}(\mathbf{z}) = \mathbf{z} \ . \tag{1.35}$$

Proof: Define $\mathbf{u}$ by (1.34), and compute

$$2(\mathbf{x}\cdot\mathbf{u})\mathbf{u} = \frac{2\mathbf{x}\cdot(\mathbf{x}-\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|^2}(\mathbf{x}-\mathbf{y}) \ . \tag{1.36}$$

Since $\|\mathbf{x}\| = \|\mathbf{y}\|$,

$$\|\mathbf{x}-\mathbf{y}\|^2 = \|\mathbf{x}|^2 + \|\mathbf{y}\|^2 - 2\mathbf{x}\cdot\mathbf{y} = 2(\|\mathbf{x}\|^2 - \mathbf{x}\cdot\mathbf{y}) = 2\mathbf{x}\cdot(\mathbf{x}-\mathbf{y}) \ .$$

Therefore, from (1.36) we have $2(\mathbf{x}\cdot\mathbf{u})\mathbf{u} = \mathbf{x}-\mathbf{y}$, and so $\mathbf{h_u}(\mathbf{x}) = \mathbf{x} - (\mathbf{x}-\mathbf{y}) = \mathbf{y}$.

The final part is simple: If $\mathbf{z}$ is orthogonal to both $\mathbf{x}$ and $\mathbf{y}$, then it is orthogonal to $\mathbf{u}$, and then (1.35) follows from the definition of $\mathbf{h_u}$. □

Example 8. *Let* $\mathbf{x} = \frac{1}{3}(1,2,-2)$ *and* $\mathbf{y} = \mathbf{e}_1 = (1,0,0)$*. These are both unit vectors, and hence* $\|\mathbf{x}\| = \|\mathbf{y}\|$*, so there is a unit vector* $\mathbf{u}$ *such that* $\mathbf{h_u}(\mathbf{x}) = \mathbf{y}$*, and* $\mathbf{u}$ *is given by (1.34). We compute* $\mathbf{u}$*:*

$$\mathbf{x}-\mathbf{y} = \frac{1}{3}(1,2,-2) - (1,0,0) = \frac{1}{3}[(1,2,-2) - (3,0,0)] = \frac{2}{3}(-1,1,-1) \ .$$

Normalizing, we find

$$\mathbf{u} = \frac{1}{\sqrt{3}}(-1,1,-1) \ .$$

Now simple computations verify that, as claimed, $\mathbf{h_u}(\mathbf{x}) = \mathbf{y}$*.*

We are now ready to prove Lemma 1, which says that there does not exist any orthonormal set of $n+1$ vectors in $\mathbb{R}^n$.

Proof of Lemma 1. We first observe that for $n = 1$, there are exactly two unit vectors, namely (1) and (-1). Since these vectors are not orthogonal, there are exactly two orthonormal sets in $\mathbb{R}^1$, namely $\{(1)\}$ and $\{(-1)\}$, and each consists of exactly one vector. This proves the Lemma for $n = 1$.

We now proceed by induction. For any $n \geq 2$ we suppose it is proved that there does not exist any orthonormal set of n vectors in $\mathbb{R}^{n-1}$. We shall show that it follows from this that there does not exist any orthonormal set of $n+1$ vectors in $\mathbb{R}^n$.

Suppose on the contrary that $\{\mathbf{u}_1, \ldots, \mathbf{u}_{n+1}\}$ is an orthonormal set of vectors in $\mathbb{R}^n$. Then there exists an an orthonormal set $\{\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}\}$ of vectors in $\mathbb{R}^n$ such that $\mathbf{v}_{n+1} = \mathbf{e}_n$. To see this, note that if $\mathbf{u}_{n+1} = \mathbf{e}_n$, we already have the desired orthonormal set. Otherwise, by Lemma 3 there exists a unit vector in $\mathbb{R}^n$ such that

$$\mathbf{h_u}(\mathbf{u}_{n+1}) = \mathbf{e}_n \ .$$

Then, since householder reflections preserve lengths and orthogonality, if we define $\mathbf{v}_j = \mathbf{h_u}(\mathbf{u}_j)$, $j = 1, \dots, n+1$ $\{\mathbf{v}_1, \dots, \mathbf{v}_{n+1}\}$ is also an orthonormal set in $\mathbb{R}^n$, and by construction, $\mathbf{v}_{n+1} = \mathbf{e}_n$.

Therefore, for each $j = 1, \dots, n$,

$$0 = \mathbf{v}_j \cdot \mathbf{v}_{n+1} = \mathbf{v}_j \cdot \mathbf{e}_n \ .$$

Since $\mathbf{v}_j \cdot \mathbf{e}_n$ is simply the final entry of $\mathbf{v}_j$, this means that for each $j = 1, \dots, n$, $\mathbf{v}_j$ has the form

$$\mathbf{v}_j = (\mathbf{w}_j, 0)$$

where $\mathbf{w}_j$ is a unit vector in $\mathbb{R}^{n-1}$.

Since $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is orthonormal in $\mathbb{R}^n$, $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ is orthonormal in $\mathbb{R}^{n-1}$, since the final zero coordinate simply "goes along for the ride". However, this is impossible, since we know that there does not exist any orthonormal set of n vectors in $\mathbb{R}^{n-1}$. We arrived at this contradiction by assuming that there existed and orthonormal set of $n+1$ vectors in $\mathbb{R}^n$. Hence this must be false. □

1.2 Lines and planes in $\mathbb{R}^3$

In this section we shall study the geometry of lines and planes in $\mathbb{R}^3$. We shall see that if we use *coordinates based on a well-chosen chosen orthonormal basis*, it is very easy to compute many geometric qunatities such as, for example, the distance between two lines in $\mathbb{R}^3$. Of course, to do this, we need a systematic method for constructing orthonormal bases. In $\mathbb{R}^3$, the *cross product* provides such a method, and has many other uses as well. In the next subsection, we introduce the cross product, starting from a question about area that the cross product is designed to answer.

1.2.1 The cross product in $\mathbb{R}^3$

Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors in $\mathbb{R}^3$, neither a multiple of the other, and consider the triangle with vertices at $\mathbf{0}$, $\mathbf{a}$, $\mathbf{b}$, which naturally lies in the plane through these three points. The cross product gives the answer to the following question, and a number of other geometric questions as well:

- *How can we express the area of this triangle in terms of the Cartesian coordinates of* $\mathbf{a}$ *and* $\mathbf{b}$*?*

The classical formula for the area of a triangle in a plane is that it is one half the length of the base times the height. Let us take the side running from $\mathbf{0}$ to $\mathbf{a}$ as the base, so that the length of the base is $\|\mathbf{a}\|$. Then, using θ to denote the angle between $\mathbf{a}$ and $\mathbf{b}$, the height is $\|\mathbf{b}\|\sin\theta$. Thus, the area A of the triangle is

$$A := \frac{1}{2}\|\mathbf{a}\|\|\mathbf{b}\|\sin\theta \ .$$

(Note that since, by definition, $\theta \in [0, \pi]$ $\sin\theta \geq 0$.)

Using the identity $\sin^2\theta + \cos^2\theta = 1$, we can write this as

$$4A^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2\sin^2\theta = \|\mathbf{a}\|^2\|\mathbf{b}\|^2(1-\cos^2\theta) = \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2 \ .$$

Now calculate the right hand side, taking $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$.

We find, after a bit of algebra,

$$\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \ .$$

The square root of the right hand side is the twice area in question. Notice that the right hand side is also square of the length of a vector in $\mathbb{R}^3$, namely, the vector $\mathbf{a}\times\mathbf{b}$, defined as follows:

Definition 15 (Cross product in $\mathbb{R}^3$)**.** *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be two vectors in* $\mathbb{R}^3$*. Then the* cross product of $\mathbf{a}$ and $\mathbf{b}$ *is the vector* $\mathbf{a}\times\mathbf{b}$ *where*

$$\mathbf{a}\times\mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{e}_1 + (a_3b_1 - a_1b_3)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3 \ . \tag{1.37}$$

Example 9. *Computing from the definition, we find*

$$\mathbf{e}_1\times\mathbf{e}_2 = \mathbf{e}_3 \qquad \mathbf{e}_2\times\mathbf{e}_3 = \mathbf{e}_1 \qquad \text{and} \qquad \mathbf{e}_3\times\mathbf{e}_1 = \mathbf{e}_2 \ . \tag{1.38}$$

By the computations that led to the definition, we have that

$$\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta \ . \tag{1.39}$$

This tells us the magnitude of $\mathbf{a}\times\mathbf{b}$. What is its direction? Before dealing with this geometric question, it will help to first establish a few algebraic properties of the cross product.

Notice from the defining formula (1.37) that

$$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a} \ .$$

Thus the cross product is not commutative; instead, it is *anticommutative*. In particular, for any $\mathbf{a}\in\mathbb{R}^3$,

$$\mathbf{a}\times\mathbf{a} = -\mathbf{a}\times\mathbf{a} = \mathbf{0} \ . \tag{1.40}$$

Also, introducing a third vector $\mathbf{c} = (c_1, c_2, c_3)$, we have from the definition that

$$\begin{aligned}
& \mathbf{a}\times(\mathbf{b}+\mathbf{c}) \\
= \ & [a_2(b_3+c_3) - a_3(b_2+c_2))]\mathbf{e}_1 + [a_3(b_1+c_1) - a_1(b_3+c_3)]\mathbf{e}_2 + [a_1(b_2+c_2) - a_2)b_1+c_1)]\mathbf{e}_3 \\
= \ & (a_2b_3 - a_3b_2)\mathbf{e}_1 + (a_3b_1 - a_1b_3)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3 \\
+ \ & (a_2c_3 - a_3c_2)\mathbf{e}_1 + (a_3c_1 - a_1c_3)\mathbf{e}_2 + (a_1c_2 - a_2c_1)\mathbf{e}_3 \\
= \ & \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c} \ .
\end{aligned}$$

Thus, $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}$, which means that the cross product *distributes* over vector addition. From this identity and the anticommutivity, we see that $(\mathbf{b}+\mathbf{c})\times\mathbf{a} = \mathbf{b}\times\mathbf{a} + \mathbf{c}\times\mathbf{a}$; i.e., the distributivitiy holds on both sides of the product.

Finally, a similar but simpler proof shows that for any number t, $(t\mathbf{a})\times\mathbf{b} = t(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times(t\mathbf{b})$. We summarize our conclusions in a theorem:

Theorem 6 (Algebraic properties of the cross product)**.** *Let* $\mathbf{a}$*,* $\mathbf{b}$ *and* $\mathbf{c}$ *be any three vectors in* $\mathbb{R}^3$*, and let* t *be any number. Then*

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \ .$$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} \ .$$

$$(t\mathbf{a}) \times \mathbf{b} = t(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (t\mathbf{b}) \ .$$

We now return to the geometric information contained in the cross product. The following result, which relates the cross product and the dot product, is useful for this.

Theorem 7. *Let* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$*. Then*

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \ .$$

Proof: We compute:

$$\begin{aligned}
(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} &= (a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3 \\
&= (b_2c_2 - b_3c_2)a_1 + (b_3c_1 - b_1c_3)a_2 + (b_1c_2 - b_2c_1)a_3 \\
&= (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} \\
&= \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \ .
\end{aligned}$$

□

Therefore, by (1.40) and Theorem 7, for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$,

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{b}) = \mathbf{0} \ .$$

Likewise, using also the anticommutivity of the cross product,

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = -(\mathbf{b} \times \mathbf{a}) \cdot \mathbf{a} = -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{a}) = \mathbf{0} \ .$$

We have proved:

Theorem 8. *Let* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$*. Then* $\mathbf{a} \times \mathbf{b}$ *is orthogonal to both* $\mathbf{a}$ *and* $\mathbf{b}$*.*

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be two vectors such that neither is a multiple of the other. Then by Theorem 8, $\mathbf{v}_1 \times \mathbf{v}_2$ is a non-zero vector orthogonal to every vector of the form

$$s\mathbf{v}_1 + t\mathbf{v}_2 \qquad s, t \in \mathbb{R} \ ,$$

which is to say that $\mathbf{a} := \mathbf{v}_1 \times \mathbf{v}_2$ is orthogonal to every vector in the plane in $\mathbb{R}^3$ determined by (passing through) the 3 points $\mathbf{0}$, $\mathbf{v}_1$ and $\mathbf{v}_2$. In other words:

- *The cross product* $\mathbf{v}_1 \times \mathbf{v}_2$ *gives the direction of the normal line to the plane through* $\mathbf{0}$*,* $\mathbf{v}_1$ *and* $\mathbf{v}_2$*, provided these are non-colinear, so that they do determine a plane.*

Theroem 8 says when $\mathbf{a} \times \mathbf{b} \neq \mathbf{0}$, the unit vector

$$\frac{1}{\|\mathbf{a} \times \mathbf{b}\|} \mathbf{a} \times \mathbf{b}$$

is one of the two unit vectors orthogonal to the plane through $\mathbf{0}$, $\mathbf{a}$ and $\mathbf{b}$. Which one is it?

Definition 16. *An orthonormal basis* $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ *of* $\mathbb{R}^3$ *is a* right-handed orthonormal basis *in case* $\mathbf{u}_1 \times \mathbf{u}_2 = \mathbf{u}_3$ *and is a* left-handed orthonormal basis *in case* $\mathbf{u}_1 \times \mathbf{u}_2 = -\mathbf{u}_3$*.*

Note that every orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ of $\mathbb{R}^3$ is either left-handed or right-handed, since $\mathbf{u}_1 \times \mathbf{u}_2$ must be a unit vector orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$, so that $\pm\mathbf{u}_3$ are the only possibilities. Also note that the standard basis of $\mathbb{R}^3$ is right-handed by (1.38).

Theorem 9. *Let* $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ *be any orthonormal basis of* $\mathbb{R}^3$*. Then*

$$\mathbf{u}_1 \times \mathbf{u}_2 = \mathbf{u}_3 \quad \Longleftrightarrow \quad \mathbf{u}_2 \times \mathbf{u}_3 = \mathbf{u}_1 \quad \Longleftrightarrow \quad \mathbf{u}_3 \times \mathbf{u}_1 = \mathbf{u}_2 \ . \tag{1.41}$$

In particular, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ *is right handed if and only if any one of the identities in (1.41) is valid, and in that case, all of them are valid.*

Proof: Suppose that $\mathbf{u}_1 \times \mathbf{u}_2 = \mathbf{u}_3$. Then by Theorem 5,

$$\mathbf{u}_2 \times \mathbf{u}_3 = (\mathbf{u}_2 \times \mathbf{u}_3 \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{u}_2 \times \mathbf{u}_3 \cdot \mathbf{u}_2)\mathbf{u}_2 + (\mathbf{u}_2 \times \mathbf{u}_3 \cdot \mathbf{u}_3)\mathbf{u}_3 \ .$$

Since $\mathbf{u}_2 \times \mathbf{u}_3$ orthogonal to $\mathbf{u}_2$ and $\mathbf{u}_3$, this reduces to

$$\mathbf{u}_2 \times \mathbf{u}_3 = (\mathbf{u}_2 \times \mathbf{u}_3 \cdot \mathbf{u}_1)\mathbf{u}_1 \ , \tag{1.42}$$

and we only need to compute $\mathbf{u}_2 \times \mathbf{u}_3 \cdot \mathbf{u}_1 = \mathbf{u}_1 \cdot \mathbf{u}_2 \times \mathbf{u}_3$. By Theorem 7 and the hypothesis that $\mathbf{u}_1 \times \mathbf{u}_2 = \mathbf{u}_3$,

$$\mathbf{u}_1 \cdot \mathbf{u}_2 \times \mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2 \cdot \mathbf{u}_3 = \mathbf{u}_3 \cdot \mathbf{u}_3 = 1 \ .$$

Summarizing, under the assumption that $\mathbf{u}_1 \times \mathbf{u}_2 = \mathbf{u}_3$, we have $\mathbf{u}_2 \times \mathbf{u}_3 \cdot \mathbf{u}_1 = 1$, and hence from (1.42), we have

$$\mathbf{u}_1 \times \mathbf{u}_2 = \mathbf{u}_3 \ \Rightarrow \ \mathbf{u}_2 \times \mathbf{u}_3 = \mathbf{u}_1 \ .$$

The same sort of computation also shows that

$$\mathbf{u}_1 \times \mathbf{u}_2 = \mathbf{u}_3 \ \Rightarrow \ \mathbf{u}_3 \times \mathbf{u}_1 = \mathbf{u}_2 \ . \tag{1.43}$$

Indeed, by Theorem 5

$$\begin{aligned} \mathbf{u}_3 \times \mathbf{u}_1 &= (\mathbf{u}_3 \times \mathbf{u}_1 \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{u}_3 \times \mathbf{u}_1 \cdot \mathbf{u}_2)\mathbf{u}_2 + (\mathbf{u}_3 \times \mathbf{u}_1 \cdot \mathbf{u}_3)\mathbf{u}_3 \\ &= (\mathbf{u}_3 \times \mathbf{u}_1 \cdot \mathbf{u}_2)\mathbf{u}_2 \end{aligned}$$

We then compute

$$\mathbf{u}_3 \times \mathbf{u}_1 \cdot \mathbf{u}_2 = \mathbf{u}_3 \cdot \mathbf{u}_1 \times \mathbf{u}_2 = \mathbf{u}_3 \cdot \mathbf{u}_3 = 1 \ ,$$

from which we conclude (1.43). Thus, the first of the identities in (1.41) implies the other two. The same sort of computations, which are left to the reader, show each of them implies the other two, so that they are all equivalent. □

Why is the distinction between right and left handed orthonormal bases useful? One consequence of Theorem 9 is that one can use a formula just like (1.37) to compute the Cartesian components of

$\mathbf{a} \times \mathbf{b}$ in terms of the Cartesian components of $\mathbf{a}$ and $\mathbf{b}$ for *any* coordinate system based on a *any* right handed orthonormal basis, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, and not only the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

Indeed, if we write $\mathbf{a} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3$, so that a_1, a_2 and a_3 are the Cartesian coordinates of $\mathbf{a}$ with respect to the orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, and similarly for $\mathbf{b}$, we find using Theorem 6 and Theorem 9 that

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{u}_1 + (a_3b_1 - a_1b_3)\mathbf{u}_2 + (a_1b_2 - a_2b_1)\mathbf{u}_3 \ .$$

The next identity, for the cross product of three vectors, has many uses. For example, we shall use it later on to deduce Kepler's Laws from Newton's Universal Theory of Gravitation. It was for exactly this purpose that Lagrange proved the identity, though he stated it in a different form.

Theorem 10 (Lagrange's Identity). *Let* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$. *Then*

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \ . \tag{1.44}$$

One way to prove Theorem 10 is to write $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$ and $\mathbf{c} = (c_1, c_2, c_3)$, and then to compute both sides of (1.44), and check that they are equal. It is much more enlightening, and much less work, to do the computation using coorindates based on an orthonormal basis built out of the given vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. This will be our first example of the strategy of using a "well adapted" coordinate system. In the next subsection, we provide many more examples.

Proof of Theorem 10: Note first of all that if $\mathbf{b}$ is a multiple of $\mathbf{c}$, then $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{0}$. So let us assume for now that this is not the case. Let $\mathbf{b}_{\|}$ and $\mathbf{b}_{\perp}$ be the components of $\mathbf{b}$ that are parallel and orthogonal to $\mathbf{c}$ respectively. Our assumption is that $\mathbf{b}_{\perp} \neq \mathbf{0}$, and we can therefore build an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ of $\mathbb{R}^3$ as follows: Define

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{b}_{\perp}\|}\mathbf{b}_{\perp} \ , \quad \mathbf{u}_2 = \frac{1}{\|\mathbf{c}\|}\mathbf{c} \qquad \text{and} \qquad \mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2 \ .$$

By the properties of the cross product, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a right handed orthonormal basis for $\mathbb{R}^3$. We now compute

$$\mathbf{b} \times \mathbf{c} = \mathbf{b}_{\perp} \times \mathbf{c} = (\|\mathbf{b}_{\perp}\|\mathbf{u}_1) \times (\|\mathbf{c}\|\mathbf{u}_2) = \|\mathbf{b}_{\perp}\|\|\mathbf{c}\|\mathbf{u}_1 \times \mathbf{u}_2 = \|\mathbf{b}_{\perp}\|\|\mathbf{c}\|\mathbf{u}_3 \ ,$$

where we have used Theorem 9 in the final step.

Now using Theorem 5, we write

$$\mathbf{a} = (\mathbf{a} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{a} \cdot \mathbf{u}_2)\mathbf{u}_2 + (\mathbf{a} \cdot \mathbf{u}_3)\mathbf{u}_3 \ ,$$

and compute $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. By our computation of $\mathbf{b} \times \mathbf{c}$ above, this gives

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \|\mathbf{b}_{\perp}\|\|\mathbf{c}\|[(\mathbf{a} \cdot \mathbf{u}_1)\mathbf{u}_1 \times \mathbf{u}_3 + (\mathbf{a} \cdot \mathbf{u}_2)\mathbf{u}_2 \times \mathbf{u}_3 + (\mathbf{a} \cdot \mathbf{u}_3)\mathbf{u}_3 \times \mathbf{u}_3],$$

and by Theorem 9 once more, this gives

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \|\mathbf{b}_{\perp}\|\|\mathbf{c}\|[(-\mathbf{a} \cdot \mathbf{u}_1)\mathbf{u}_2 + (\mathbf{a} \cdot \mathbf{u}_2)\mathbf{u}_1 \\ &= (\mathbf{a} \cdot \mathbf{c})\mathbf{b}_{\perp} - (\mathbf{a} \cdot \mathbf{b}_{\perp})\mathbf{c} \ . \end{aligned} \tag{1.45}$$

This is *almost* the formula we want. The final step is to observe that since $\mathbf{b}_\|$ is some multiple of $\mathbf{c}$, say, $\mathbf{b}_\| = t\mathbf{c}$, then

$$(\mathbf{a}\cdot\mathbf{c})\mathbf{b}_\| - (\mathbf{a}\cdot\mathbf{b}_\|)\mathbf{c} = t[(\mathbf{a}\cdot\mathbf{c})\mathbf{c} - (\mathbf{a}\cdot\mathbf{c})\mathbf{c}] = 0 \ , \tag{1.46}$$

and therefore the parallel component of $\mathbf{b}$ drops out so that ,

$$(\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b}_\perp - (\mathbf{a}\cdot\mathbf{b}_\perp)\mathbf{c} \ . \tag{1.47}$$

Comparing (1.45) and (1.47) we see that (1.44) is true whenever $\mathbf{b}_\perp \neq \mathbf{0}$. On the other hand, when $\mathbf{b}_\perp = \mathbf{0}$, (1.46) and the fact that in this case $\mathbf{b}\times\mathbf{c} = \mathbf{b}_\perp\times\mathbf{c} = \mathbf{0}$ show that (1.44) is true also when $\mathbf{b}_\perp = \mathbf{0}$. □

The identity we get from Theorem 10 in the special case $\mathbf{a} = \mathbf{b}$ is often useful:

Corollary 1. *Let* $\mathbf{u}$ *be any unit vector in* $\mathbb{R}^3$*. Then for all* $\mathbf{c}\in\mathbb{R}^3$

$$\mathbf{c}_\perp \ = (\mathbf{u}\times\mathbf{c})\times\mathbf{u} \tag{1.48}$$

where $\mathbf{c}_\perp$ *is the component of* $\mathbf{c}$ *orthogonal to* $\mathbf{b}$*.*

Note the resemblance of (1.48) to the formula $\mathbf{c}_\| = (\mathbf{u}\cdot\mathbf{c})\mathbf{u}$.

Proof: Applying (1.44) with $\mathbf{a} = \mathbf{b} = \mathbf{u}$, we obtain

$$\mathbf{u}\times(\mathbf{u}\times\mathbf{c}) = (\mathbf{c}\cdot\mathbf{u})\mathbf{u} - (\mathbf{u}\cdot\mathbf{u})\mathbf{c} = -(\mathbf{c} - (\mathbf{c}\cdot\mathbf{u})\mathbf{u}) = -\mathbf{c}_\perp \ .$$

Now using the anticommutivity of the cross product, we obtain (1.48). □

Thus, one can readily compute orthogonal components by computing cross products. The magnitude of $\mathbf{c}_\perp$ is even simpler to compute: Since $\mathbf{b}\times\mathbf{c} = \mathbf{b}\times\mathbf{c}_\perp$, and since $\|\mathbf{b}\times\mathbf{c}_\perp\| = \|\mathbf{b}\|\|\mathbf{c}_\perp\|$,

$$\|\mathbf{c}_\perp\| = \frac{\|\mathbf{b}\times\mathbf{c}\|}{\|\mathbf{b}\|} \ .$$

The cross product is not only useful for checking whether a given orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is right handed or not; it is useful for *constructing* such bases. The next example concerns a problem that often arises when working with lines and planes in $\mathbb{R}^3$.

Example 10 (Constructing a right-handed orthonormal basis containing a given direction)**.** *Given a nonzero vector* $\mathbf{v}\in\mathbb{R}^3$*, how can we find an orthonormal basis* $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ *in which*

$$\mathbf{u}_3 = \frac{1}{\|\mathbf{v}\|}\mathbf{v} \ ?$$

Here is one way to do it using the cross product. First, $\mathbf{u}_3$ *is given. Let us next choose* $\mathbf{u}_1$*. This has to be some unit vector that is orthogonal to* $\mathbf{v}$*.*

If $\mathbf{v} = (v_1, v_2, v_3)$ *and* $v_j = 0$ *for any* j*, then* $\mathbf{v}\cdot\mathbf{e}_j = v_j = 0$*, and we may take* $\mathbf{u}_1 = \mathbf{e}_j$ *for this* j*. On the other hand, if each* v_j *is non-zero, define*

$$\mathbf{w} := \mathbf{e}_3\times\mathbf{v} = (-v_2, v_1, 0) \ .$$

This is orthogonal to $\mathbf{v}$ *by Theorem 8, and* $\|\mathbf{w}\| \neq 0$*. Now define*

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{w}\|}\mathbf{w} \ .$$

Finally, define $\mathbf{u}_2 = \mathbf{u}_3 \times \mathbf{u}_1$*, so that* $\mathbf{u}_2$ *is orthogonal to both* $\mathbf{u}_3$ *and* $\mathbf{u}_1$*, and is a unit vector. Thus,* $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ *is an orthonormal basis, and since* $\mathbf{u}_3 \times \mathbf{u}_1 = \mathbf{u}_2$*, Theorem 9 says that this basis is right handed.*

Now let us do this for specific vectors. Let $\mathbf{v} = (2, 1, 2)$*. We will now find an orthonormal basis* $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ *in which* $\mathbf{u}_3$ *points in the same direction as* $\mathbf{v}$*.*

For this, we must have $\mathbf{u}_3 = \dfrac{1}{\|\mathbf{v}\|}\mathbf{v} = \dfrac{1}{3}(2, 1, 2)$*. Next, since none of the entries of* $\mathbf{v}$ *is zero, we define*

$$\mathbf{w} := \mathbf{e}_3 \times \mathbf{v} = (\,-1, 2, 0),$$

and then

$$\mathbf{u}_1 := \frac{1}{\|\mathbf{w}\|}\mathbf{w} = \frac{1}{\sqrt{5}}(\,-1, 2, 0) \ ,$$

Finally we define

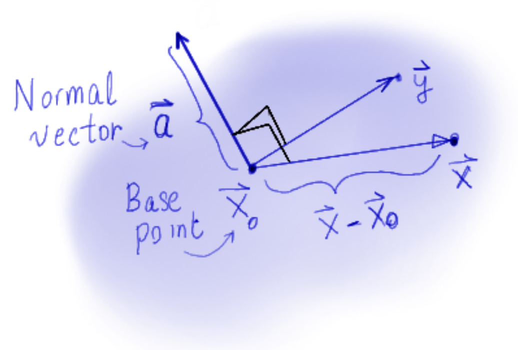
$$\mathbf{u}_2 := \mathbf{u}_3 \times \mathbf{u}_1 = \frac{1}{3\sqrt{5}}(\,-4, -2, 5) \ .$$

The method in the last example can be used to find a nice system of equation for the line through two distinct points in $\mathbb{R}^3$.

1.2.2 Equations for planes in $\mathbb{R}^3$

A plane in three dimensional space may be specified in several ways: One is by specifying a point $\mathbf{x}_0$ in the plane, and a direction vector $\mathbf{a}$. The corresponding plane consists of all points $\mathbf{x}$ such that the vector $\mathbf{x} - \mathbf{x}_0$ (which carries $\mathbf{x}_0$ to $\mathbf{x}$) is orthogonal to $\mathbf{a}$.

In this description, we refer to the vector $\mathbf{x}_0$ as the *base point* and the vector $\mathbf{a}$ as the *normal vector*. Here is a diagram showing a plane with a given base point and normal vector, and also two vectors $\mathbf{x}$ and $\mathbf{y}$ in the plane:



We can translate this verbal and pictorial description into an equation specifying the plane by taking advantage of the fact that orthogonality can be expressed in terms of the dot product: The vector $\mathbf{x}$ belongs to the plane if and only if

$$\mathbf{a} \cdot (\mathbf{x} - \mathbf{x}_0) = 0 \ .$$

Example 11 (The equation of a plane specified in terms of a base point and normal direction)**.** *Consider the plane passing through the point* $\mathbf{x}_0 = (3, 2, 1)$ *that is orthogonal to the vector* $\mathbf{a} = (1, 2, 3)$*. Then* $\mathbf{x} = (x, y, z)$ *belongs to this plane if and only if* $(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{a} = 0$*. Doing the computations,*

$$((x, y, z) - (3, 2, 1)) \cdot (1, 2, 3) = 0 \qquad \Longleftrightarrow \qquad x + 2y + 3z = 10 \ .$$

The solution set of the equation $x+2y+3z = 10$ *consists of the Cartesian coordinates (with respect to the standard basis) of the points in this plane. We say that* $x+2y+3z = 10$ *is an equation specifying this plane.*

As we have just seen in the last example, an equation specifying a plane can also be written in the form $\mathbf{a} \cdot \mathbf{x} = d$. Indeed, defining $d := \mathbf{a} \cdot \mathbf{x}_0$,

$$(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{a} = 0 \qquad \Longleftrightarrow \qquad \mathbf{a} \cdot \mathbf{x} = d \ .$$

Example 12 (Parameterizing a plane specified by an equation)**.** *Consider the equation* $\mathbf{a} \cdot \mathbf{x} = d$ *where* $\mathbf{a} = (1, 2, 1)$*, and* $d = 10$*. Writing* $\mathbf{x} = (x, y, z)$*, the equation becomes* $x + 2y + z = 10$*.*

Parameterizing solution sets means solving euations. Solving equations means eliminating variables. All three variables are present in this equation, so we can choose to eliminate any of them. Let us eliminate z*:*

$$z = 10 - 2y - x \ .$$

Thus, $\mathbf{x} = (x, y, z)$ *belong to the plane if and only if*

$$\mathbf{x} = (x, y, 10 - 2y - x) = (0, 0, 10) + x(1, 0, -1) + y(0, 1, -2) \ .$$

We have expanded the left hand side, and collected terms into a constant vector, a second constant vector times x *and a third constant vector times* y*. The point of this is that defining*

$$\mathbf{x}_0 := (0, 0, 10) \ , \quad \mathbf{v}_1 := (1, 0, -1) \quad \text{and} \quad \mathbf{v}_2 := (0, 1, -2) \ ,$$

we conclude that a vector $\mathbf{x}$ *belongs to the plane if and only if it can be written in the form* $\mathbf{x}_0 + x\mathbf{v}_1 + y\mathbf{v}_2$ *for some numbers* x *and* y*. Moreover, whenever* $\mathbf{x}$ *can be written in the form* $\mathbf{x}_0 + x\mathbf{v}_1 + y\mathbf{v}_2$*, the numbers* x *and* y *are uniquely determined, since direct computation yields*

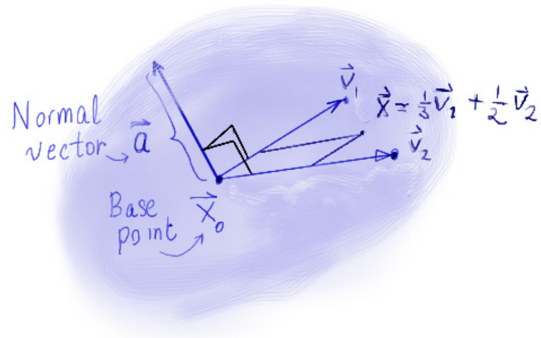
$$\mathbf{x}_0 + s\mathbf{v}_1 + t\mathbf{v}_2 = (s, t, 10 - 2s - t) \ .$$

If this is to equal (x, y, z)*, we must have* $s = x$ *and* $y = t$*.*

Thus there is a one-to-one correspondence between points $\mathbf{x}$ *in the plane and vectors* $(s, t) \in \mathbb{R}^2$ *given by*

$$\mathbf{x}(s, t) = s\mathbf{v}_1 + t\mathbf{v}_2 \ .$$

As (s, t) *varies over* $\mathbb{R}^2$*,* $\mathbf{x}(s, t)$ *varies over the plane in question in a one-to one way. Thus, we have parameterized the plane.*



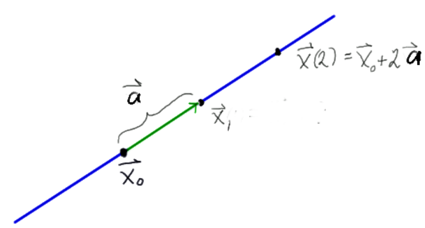
Later in this course, we shall be studying more general parametrized surfaces in $\mathbb{R}^3$. Parameterized planes are our first example, and also one of the most important: One thing that is often useful in solving problems involving more general parameterized surfaces is to compute parameterizations of their "tangent planes". Hence, what we are doing now is fundamental for what follows.

1.2.3 Equations for lines in $\mathbb{R}^3$

Given two distinct points $\mathbf{x}_0$ and $\mathbf{x}_1$ in $\mathbb{R}^3$, define $\mathbf{a} = \mathbf{x}_1 - \mathbf{x}_0$, and then for all $t \in \mathbb{R}$, define

$$\mathbf{x}(t) := \mathbf{x}_0 + t\mathbf{a} \ :$$

As illustrated below, as t varies, $\mathbf{x}(t)$ varies over the set of points that one can reach starting from $\mathbf{x}_0$, and moving only in the direction of $\mathbf{a}$ (or its opposite).



We have arrived at a parameterization of the line without first finding an equation for it. We now ask: What equation has this line as its solution set? One way to get an equation for the line is to take the cross product of $\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{a}$ with $\mathbf{a}$: Since $\mathbf{x}(t) - \mathbf{x}_0 = t\mathbf{a}$, and $\mathbf{a} \times \mathbf{a} = \mathbf{0}$,

$$\mathbf{a} \times (\mathbf{x}(t) - \mathbf{x}_0) = \mathbf{0} \ .$$

That is, every point on the line parameterized by $\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{a}$ solves the equation

$$\mathbf{a} \times (\mathbf{x} - \mathbf{x}_0) = \mathbf{0} \ . \tag{1.49}$$

We now show that: *A vector* $\mathbf{x} \in \mathbb{R}^3$ *satisfies this equation if and only if it is on the line parameterized by* $\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{a}$.

Having done this we shall have shown that the solution set of the equation (1.49) is exactly the line parameterized by $\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{a}$. Since $\mathbf{x}_0$ can be any vector in $\mathbb{R}^3$, and $\mathbf{a}$ can be any non-zero vector in $\mathbb{R}^3$, this shows that every line in $\mathbb{R}^3$ is given by an equation of the form (1.49) just as every plane in $\mathbb{R}^3$ is given by an equation of the form $\mathbf{a} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$.

To see that the solution set of (1.49) is exactly the line parameterized by $\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{a}$, choose a right handed orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ in which $\mathbf{u}_3$ is a multiple of $\mathbf{a}$. In Example 10, we have shown how to construct such an orthonormal basis, but for our present purpose, we only need to know that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ exists.

We then expand $\mathbf{x} - \mathbf{x}_0$ with respect to this basis using Theorem 5:

$$\mathbf{x} - \mathbf{x}_0 = (\mathbf{u}_1 \cdot (\mathbf{x} - \mathbf{x}_0))\mathbf{u}_1 + (\mathbf{u}_2 \cdot (\mathbf{x} - \mathbf{x}_0))\mathbf{u}_2 + (\mathbf{u}_3 \cdot (\mathbf{x} - \mathbf{x}_0))\mathbf{u}_3 \ . \tag{1.50}$$

Taking the cross product with $\mathbf{a} = \|\mathbf{a}\|\mathbf{u}_3$, and using Theorem 9 we obtain:

$$\begin{aligned} \mathbf{a} \times (\mathbf{x} - \mathbf{x}_0) &= \|\mathbf{a}\|[(\mathbf{u}_1 \cdot (\mathbf{x} - \mathbf{x}_0))\mathbf{u}_3 \times \mathbf{u}_1 + (\mathbf{u}_2 \cdot (\mathbf{x} - \mathbf{x}_0))\mathbf{u}_3 \times \mathbf{u}_2] \\ &= \|\mathbf{a}\|[(\mathbf{u}_1 \cdot (\mathbf{x} - \mathbf{x}_0))\mathbf{u}_2 - (\mathbf{u}_2 \cdot (\mathbf{x} - \mathbf{x}_0))\mathbf{u}_1] \ . \end{aligned}$$

By Theorem 5 once more,

$$\|\mathbf{a} \times (\mathbf{x} - \mathbf{x}_0)\|^2 = \|\mathbf{a}\|^2[(\mathbf{u}_1 \cdot (\mathbf{x} - \mathbf{x}_0))^2 + (\mathbf{u}_2 \cdot (\mathbf{x} - \mathbf{x}_0))^2] \ .$$

Therefore, $\mathbf{a} \times (\mathbf{x} - \mathbf{x}_0) = \mathbf{0}$ if and only if $\mathbf{u}_1 \cdot (\mathbf{x} - \mathbf{x}_0) = 0$ and $\mathbf{u}_2 \cdot (\mathbf{x} - \mathbf{x}_0) = 0$. But in this case, (1.50) reduces to

$$\mathbf{x} - \mathbf{x}_0 = (\mathbf{u}_3 \cdot (\mathbf{x} - \mathbf{x}_0))\mathbf{u}_3 = (\mathbf{u}_3 \cdot (\mathbf{x} - \mathbf{x}_0))\frac{1}{\|\mathbf{a}\|}\mathbf{a} \ .$$

Thus, $\mathbf{a} \times (\mathbf{x} - \mathbf{x}_0) = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{x}_0 + t\mathbf{a}$ where $t = \|\mathbf{a}\|^{-1}(\mathbf{u}_3 \cdot (\mathbf{x} - \mathbf{x}_0))$. In particular, as claimed above, the solution set of (1.49) is exactly the line parameterized by $\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{a}$.

Defining $\mathbf{d} := \mathbf{a} \times \mathbf{x}_0$, we can re-write (1.49) as

$$\mathbf{a} \times \mathbf{x} = \mathbf{d} \ . \tag{1.51}$$

However, given two generic non-zero vectors $\mathbf{a}$ and $\mathbf{d}$ in $\mathbb{R}^3$, it is not always the case that the solution set of the equation $\mathbf{a} \times \mathbf{x} = \mathbf{d}$ is a line: *It can also be the empty set.* Indeed, for all $\mathbf{x}$, $\mathbf{a} \times \mathbf{x}$ is orthogonal to both $\mathbf{x}$ and $\mathbf{a}$. Hence, if $\mathbf{d}$ is not orthogonal to $\mathbf{a}$, there is no vector $\mathbf{x}$ satisfying $\mathbf{a} \times \mathbf{x} = \mathbf{d}$. On the other hand, if $\mathbf{d}$ is orthogonal to $\mathbf{a}$, then by (1.48) of the corollary to Theorem 10,

$$\mathbf{a} \times (\mathbf{a} \times \mathbf{d}) = -\|\mathbf{a}\|^2\mathbf{d}$$

and hence

$$\mathbf{x}_0 := -\frac{1}{\|\mathbf{a}\|^2}\mathbf{a} \times \mathbf{d}$$

is a solution of $\mathbf{a} \times \mathbf{x} = \mathbf{d}$, and then evidently (1.51) is equivalent to (1.49) with this choice of $\mathbf{x}_0$.

• *Given two non-zero vectors* $\mathbf{a}$ *and* $\mathbf{d}$ *in* $\mathbb{R}^3$*, the equation* $\mathbf{a} \times \mathbf{x} = \mathbf{d}$ *has no solution if* $\mathbf{a}$ *is not orthogonal to to* $\mathbf{a}$*, and otherwise, its solution set is the line parameterized by*

$$-\frac{1}{\|\mathbf{a}\|^2}\mathbf{a} \times \mathbf{d} + t\mathbf{a} \ .$$

Example 13 (Solving $\mathbf{a}\times\mathbf{x} = \mathbf{d}$)**.** *Let* $\mathbf{a} = (1,-2,1)$ *and* $\mathbf{d} = (\,-1/2, 1, 5/2)$*. We check that* $\mathbf{a}\cdot\mathbf{d} = 0$*, and hence the solutions set of* $\mathbf{a}\times\mathbf{x} = \mathbf{d}$ *is a line. We then compute* $\|\mathbf{a}\|^2 = 6$ *and*

$$\mathbf{a}\times\mathbf{d} = (\,-6, -3, 0)\ .$$

Hence

$$\mathbf{x}_0 := -\frac{1}{\|\mathbf{a}\|^2}\mathbf{a}\times\mathbf{d}) = -\frac{1}{6}(\,-6,-3,0) = (1, 1/2, 0)\ .$$

Thus, the solution set is the line parameterized by

$$\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{a} = (\,-1, 1/2, 0) + t(1,-2,1) = (1+t, 1/2-2t, t)\ .$$

Example 14 (The equation $\mathbf{a}\times\mathbf{x} = \mathbf{d}$ as a system of equations)**.** *Let* $\mathbf{a} = (1,-2,1)$ *and* $\mathbf{b} = (\,-1/2, 1, 5/2)$ *as in the previous example. Computing* $\mathbf{a}\times\mathbf{x}$ *for* $\mathbf{x} = (x,y,z)$*, we find*

$$(1,-2,1)\times(x,y,z) = (\,-y-2z, x-z, 2x+y)\ .$$

Hence, $\mathbf{a}\times\mathbf{x} = \mathbf{d}$ *is equivalent to the system of three equations*

$$\begin{aligned} -y-2z &= -1/2 \\ x-z &= 1 \\ 2x+y &= 5/2\ . \end{aligned} \tag{1.52}$$

Each of the equations in (1.52) is the equation of a plane. Defining

$$\mathbf{a}_1 := (0,-1,-2)\quad,\quad \mathbf{a}_2 := (1,0,-1)\quad\text{and}\quad \mathbf{a}_3 := (2,1,0)\ ,$$

and defining

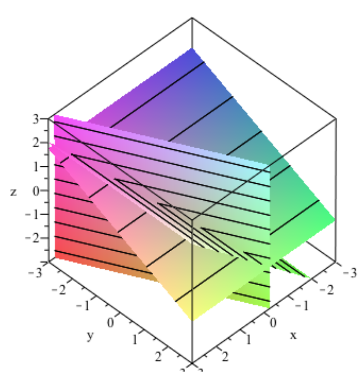
$$d_1 := -1/2\quad,\quad d_2 := 1\quad\text{and}\quad d_3 := 5/2\ ,$$

The system of equations (1.52) becomes

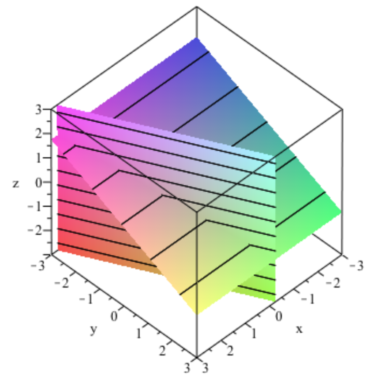
$$\begin{aligned} \mathbf{a}_1\cdot\mathbf{x} &= d_1 \\ \mathbf{a}_2\cdot\mathbf{x} &= d_2 \\ \mathbf{a}_3\cdot\mathbf{x} &= d_3\ . \end{aligned} \tag{1.53}$$

Thus, geometrically, writing the equation for a line in the form $\mathbf{a}\times\mathbf{x} = \mathbf{d}$ expresses the line as the intersection of *three planes*.

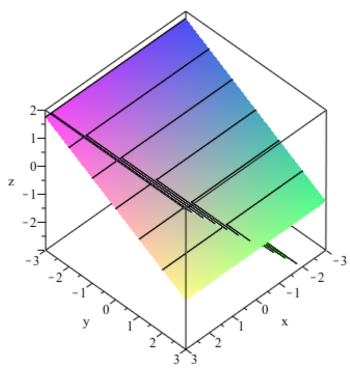
Here is a plot showing these three planes intersecting in the line parameterized by $\mathbf{x}_0 + t\mathbf{a}$:

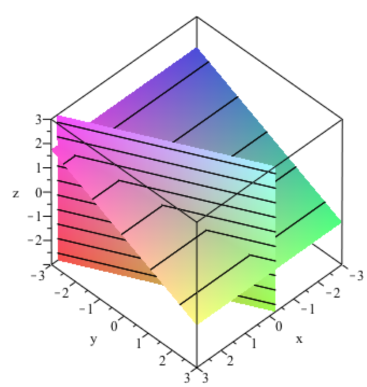


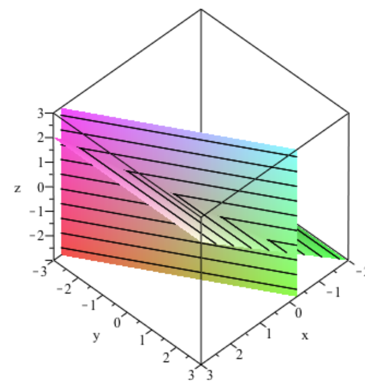
A line in $\mathbb{R}^3$ can always be represented as the intersection of only *two* planes, as illustrated below:



In (1.52), which represents a line as the intersection of *three* planes, each pair of the three planes intersect in the *same line*. Here is is a plot of the planes one pair at a time:







Each pair of the three planes intersects in the same line.

In fact, there are *infinitely many* ways to represent any particular line as the intersection of two planes: Given a line represented as the intersection of two planes, one can rotate either plane about the line, and as long as the two planes are kept distinct, the intersection of the two planes is still the *same line*.

Thus, in the system (1.52), any two of the equations suffice to specify the line, and the third equation only provides redundant information. In general, the single vector equation $\mathbf{a}\times(\mathbf{x}-\mathbf{x}_0) = \mathbf{0}$ corresponds to the system of scalar equations obtained by setting each components of $\mathbf{a}\times(\mathbf{x}-\mathbf{x}_0)$ equal to zero:

$$\mathbf{e}_j \cdot [\mathbf{a}\times(\mathbf{x}-\mathbf{x}_0)] = 0 \qquad \text{for} \quad j = 1,2,3\ . \tag{1.54}$$

But by the triple product formula,

$$\mathbf{e}_j \cdot [\mathbf{a}\times(\mathbf{x}-\mathbf{x}_0)] = [\mathbf{e}_j \times \mathbf{a}]\cdot(\mathbf{x}-\mathbf{x}_0)\ ,$$

so that if we define $\mathbf{a}_j = \mathbf{e}_j \times \mathbf{a}$, $j = 1,2,3$, the system (1.54) is equivalent to the system

$$\mathbf{a}_j \cdot (\mathbf{x}-\mathbf{x}_0) = 0 \qquad \text{for} \quad j = 1,2,3\ . \tag{1.55}$$

But writing $\mathbf{a} = (a_1, a_2, a_3)$, we have

$$\sum_{j=1}^{3} a_j \mathbf{a}_j = \sum_{j=1}^{3} a_j (\mathbf{e}_j \times \mathbf{a}) = \left(\sum_{j=1}^{3} a_j \mathbf{e}_j \right) \times \mathbf{a} = \mathbf{a} \times \mathbf{a} = \mathbf{0} \ .$$

Therefore, if for example $a_1 \neq 0$, we have

$$\mathbf{a}_1 = -\frac{1}{a_1}(\mathbf{a}_2 + \mathbf{a}_3)$$

and hence if $\mathbf{a}_2 \cdot (\mathbf{x} - \mathbf{x}_0) = 0$ and $\mathbf{a}_3 \cdot (\mathbf{x} - \mathbf{x}_0) = 0$ then *automatically* we have $\mathbf{a}_1 \cdot (\mathbf{x} - \mathbf{x}_0) = 0$, showing that the first equation in the system (1.55) is redundant. Since at least one entry of $\mathbf{a}$ is non-zero, there is alway one redundant equation in the system (1.55).

Because of this, it is often advantageous instead to specify lines as the solution set of a system of two equations, each of which describes a plane. Since every line in $\mathbb{R}^3$ is the intersection of two planes, we can always specify a line as the solution set of a system of two equations specifying planes:

$$\begin{aligned} \mathbf{a}_1 \cdot \mathbf{x} &= d_1 \\ \mathbf{a}_2 \cdot \mathbf{x} &= d_2 \ . \end{aligned} \tag{1.56}$$

For example, if $\mathbf{a}_1 = (1, 2, 3)$ and $\mathbf{a}_2 = (3, 2, 1)$ and $d_1 = 1$ and $d_2 = -1$, then the system of equations (1.56) becomes

$$\begin{aligned} x + 2y + 3z &= 1 \\ 3x + 2y + z &= -1 \end{aligned} \tag{1.57}$$

Its solution set is the line formed by the intersection of the two planes given by

$$x + 2y + 3z = 1 \qquad \text{and} \qquad 3x + 2y + z = -1 \ . \tag{1.58}$$

Since there are infinitely many pairs of planes one can use to specify the line, there are infinitely many systems of equations that have the line as their solution set. There is an important caveat to keep in mind, though: *Not every pair of equations of the from (1.56) defines a line.* If the planes are parallel, then either they do not intersect at all, or they are the same plane In the first case, the intersection is the empty set, and in the second case it is a plane. But it is easy to identify when the planes specified in ((1.56)) are parallel: This is when $\mathbf{a}_1$ is a multiple of $\mathbf{a}_2$ which is the case if and only if $\mathbf{a}_1 \times \mathbf{a}_2 = \mathbf{0}$.

Example 15 (Parameterizing a line given by a pair of equations)**.** *Consider the system of equations*

$$\begin{aligned} x + 2y + 3z &= 2 \\ 3x + 2y + z &= 4 \end{aligned}$$

All three variables are present in these equations, so we can start by eliminating any one of them, using either equation. Let us eliminate x using the first equation:

$$x = 2 - 2y - 3z \ . \tag{1.59}$$

Substituting this into the second equation, it becomes

$$3(2 - 2y - 3z) + 2y + z = 4 \qquad \text{which yields} \quad y = \frac{1}{2} - 2z \ . \tag{1.60}$$

This expresses y as a function of z. Now going back to (1.59), we may use (1.60) to eliminate y, and thus to express x as a functions of z alone. We obtain

$$x = 1 + z \ .$$

Thus, $\mathbf{x} = (x, y, z)$ belongs to the line if and only if

$$\mathbf{x} = (1 + z, 1/2 - 2z, z) = (1, 1/2, 0) + z(1, -2, 1) \ .$$

Definng $\mathbf{x}_0 = (1, 1/2, 0)$ and $\mathbf{v}_2 = (1, -2, 1)$, $\mathbf{x}(t) := \mathbf{x}_0 + t\mathbf{v}$ is a parameterization of the line.

Notice that it takes two parameters to parameterize a plane in $\mathbb{R}^3$, and one parameter to parameterize a line in $\mathbb{R}^3$, as you might expect. Every line in $\mathbb{R}^3$ has a parameterization of the form

$$\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{v} \ ,$$

just as in the example we worked out above.

As with planes, there are infinitely many ways to parameterize a given line with this scheme: Any point on the line can serve a the base point $\mathbf{x}_0$, and any non-zero vector that runs parallel to the line can serve as the direction vector.

There are also other ways of specifying planes and lines in $\mathbb{R}^3$ that are more geometric and less algebraic. For example, there is a unique plane passing through any three points that do not lie on a common line in $\mathbb{R}^3$, and there there is a unique line passing through any two points in $\mathbb{R}^3$. This brings us to the following questions:

- *Given three non collinear points $\mathbf{p}_!$, $\mathbf{p}_2$ and $\mathbf{p}_3$ in $\mathbb{R}^3$, how do we find the equation of the unique plane in $\mathbb{R}^3$ passing through these points?*

- *Given two distinct points $\mathbf{z}_1$ and $\mathbf{z}_2$ in $\mathbb{R}^3$, how do we find a system of equations for the unique line passing through these points?*

The next examples answer these questions, and explain why we might want to know the answers.

Example 16 (When do four points belong to one plane?). *Consider the points*

$$\mathbf{p}_1 = (1, 2, 3) \qquad \mathbf{p}_2 = (3, 2, 1) \quad \mathbf{p}_3 = (1, 3, 2) \quad \text{and} \quad \mathbf{p}_4 = (4, -1, 3) \ . \tag{1.61}$$

Do all of these points lie in the same plane?

It is easy to answer this question once we know the equation for the plane determined by the first three points: Simply plug the fourth point into this equation. If the equation is satisfied, the point is on the plane, and otherwise it is not.

To find the equation, choose $\mathbf{p}_1$ as the base point, and define

$$\mathbf{x}_0 := (1, 2, 3) \qquad \mathbf{v}_1 := \mathbf{p}_2 - \mathbf{x}_0 = (2, 0, -2) \qquad \text{and} \qquad \mathbf{v}_2 := \mathbf{p}_3 - \mathbf{x}_0 = (0, 1, -1) \ .$$

Writing the equation of the plane in the form $\mathbf{a}\cdot(\mathbf{x}-\mathbf{x}_0) = 0$*, and plugging in* $\mathbf{p}_2 = \mathbf{x}_0 + \mathbf{v}_1$ *and* $\mathbf{p}_3 = \mathbf{x}_0 + \mathbf{v}_2$*, we have*

$$\mathbf{a}\cdot\mathbf{v}_1 = 0 \qquad \text{and} \qquad \mathbf{a}\cdot\mathbf{v}_2 = 0\ .$$

Thus $\mathbf{a}$ *must be orthogonal to* $\mathbf{v}_1$ *and* $\mathbf{v}_2$*. We get such a vector by taking the cross product of* $\mathbf{v}_1$ *and* $\mathbf{v}_2$*. Thus we define*

$$\mathbf{a} := \mathbf{v}_1 \times \mathbf{v}_2 = (2,0,-2)\times(0,-1,1) = (2,2,2)\ .$$

We then compute $\mathbf{a}\cdot\mathbf{x} = 2x+2y+2z$ *and* $\mathbf{a}\cdot\mathbf{x}_0 = 12$ *so the equation* $\mathbf{a}\cdot(\mathbf{x}-\mathbf{x}_0) = 0$ *written out in terms of* x*,* y *and* z *is* $2x+2y+2z = 12$*, or, what is the same thing,*

$$x+y+z = 6\ . \tag{1.62}$$

This is the equation for the plane passing through the first three points in the list (1.61). You should check that these points do satisfy the equation.

We can now easily decide whether $\mathbf{p}_4$ *lies in the same plane as the first three points. With* $x = 4$*,* $y = -1$ *and* $z = 3$*, the equation (1.62) is satisfied, so it is in the plane.*

Example 17 (A system of equations for the line through two points)**.** *Let* $\mathbf{x}_0 = (1,1,-1)$ *and let* $\mathbf{x}_1 = (3,2,1)$*. Let us find a system of equations*

$$\begin{aligned}\mathbf{a}_1\cdot\mathbf{x} &= d_1\\ \mathbf{a}_2\cdot\mathbf{x} &= d_2\end{aligned} \tag{1.63}$$

that has this line as its solution set.

To do this, we first find a parameterization $\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{v}$ *of the line taking the base point to be* $\mathbf{x}_0 = (1,1,-1)$*. Since* $\mathbf{x}_1$ *is also on the line, the vector* $\mathbf{v} := \mathbf{x}_1 - \mathbf{x}_0 = (2,1,2)$ *points along the line, and we can choose it to be the direction vector in our parametrization.*

Now consider some plane containing the line through $\mathbf{x}_0$ *and* $\mathbf{x}_1$*. We can use any point in the plane, and therefore any point in the line as the base point. Hence the equation of the line can be written as* $\mathbf{a}\cdot(\mathbf{x}-\mathbf{x}_0) = 0$*, But since* $\mathbf{x}_1$ *is also in the plane*

$$\mathbf{a}\cdot(\mathbf{x}_1-\mathbf{x}_0) = 0\ .$$

Thus, $\mathbf{a}$ *must be orthogonal to* $\mathbf{v}$*, and then the equation of the plane is* $\mathbf{a}\cdot\mathbf{x} - \mathbf{a}\cdot\mathbf{x}_0$*.*

Thus, to find our system of equations, what we need to do is to find two non-zero vecotrs $\mathbf{a}_1$ *and* $\mathbf{a}_2$ *that are not multiples of one another and such that*

$$\mathbf{a}_1\cdot\mathbf{v} = \mathbf{a}_2\cdot\mathbf{v} = 0\ .$$

Here is one way to do this. Given any vector $\mathbf{v} = (a,b,c)$*, if* $b = c = 0$*, we can take* $\mathbf{a}_1 = (0,0,1)$*. Otherwise, we can take* $\mathbf{a}_1 = (0,-c,b)$*. Either way,* $\mathbf{a}_1$ *is a non-zero vector orthogonal to* $\mathbf{v}$*.*

Now define $\mathbf{a}_2 := \mathbf{a}_1 \times \mathbf{v}$*. By Theorem 8 and (1.39),* $\mathbf{a}_2$ *is a non-zero vecotr that is orthogonal to both* $\mathbf{v}$ *and* $\mathbf{a}_1$*. Evidently then,* $\mathbf{a}_2$ *is not a multiple of* $\mathbf{a}_1$*. Let us now do the numbers:*

Since $\mathbf{v} = (2,1,2)$*, we take* $\mathbf{a}_1 = (0,-2,1)$*. We then compute*

$$\mathbf{a}_2 = \mathbf{a}_1 \times \mathbf{v} = (0,-2,1)\times(2,1,2) = (\,-5,2,4)\ .$$

Next,

$$d_1 = \mathbf{a}_1 \cdot \mathbf{x}_0 = (0, -2, 1) \cdot (1, 1, -1) = -3$$

and

$$d_2 = \mathbf{a}_2 \cdot \mathbf{x}_0 = (-5, 2, 4) \cdot (1, 1, -1) = -7\ .$$

Thus, (1.63) can be taken to be

$$\begin{aligned} -2y + z &= -3 \\ -5x + 2y + 4z &= -7\ . \end{aligned}$$

We can now check our work: computing, we find that $\mathbf{x}_0$ *and* $\mathbf{x}_1$ *both satisfy this system of equations.*

Example 18 (Finding an orthonormal parameterization of a plane)**.** *Consider the plane given by*

$$2x + y + 2z = 1\ .$$

We will now find a parameterization of this plane of the form

$$\mathbf{x}(s,t) = \mathbf{x}_0 + s\mathbf{u}_1 + t\mathbf{u}_2$$

where $\{\mathbf{u}_1, \mathbf{u}_2\}$ *is orthonormal. As we shall see in the next subsection, such parameterizations are especially convenient for many computations.*

First, to find a base point, we need one solution of the equation. Choosing $x = z = 0$*, the equation reduces to* $y = 1$*, and so* $\mathbf{x}_0 = (0, 1, 0)$ *lies on the plane, and we choose it as our base point. The normal vector is* $\mathbf{a} = (2, 1, 2)$*, and so* $\mathbf{u}_1$ *and* $\mathbf{u}_2$ *must be orthonormal vectors that are each orthogonal to* $\mathbf{a}$*. In Example 10 we found such vectors when we found a right handed orthonormal basis* $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ *in which*

$$\mathbf{u}_3 = \frac{1}{3}(2, 1, 2) = \frac{1}{\|\mathbf{a}\|}\mathbf{a}\ .$$

Thus, the parameterization we seek is given by

$$\mathbf{x}(s,t) = (0, 1, 0) + s\frac{1}{\sqrt{5}}(-1, 2, 0) + t\frac{1}{3\sqrt{5}}(-4, -2, 5)\ .$$

This may seem like a cumbersome sort of parameterization, due to the square roots. however, we shall see the advantages of orthonormality in the next subsection.

1.2.4 Distance problems

Consider a point $\mathbf{p} \in \mathbb{R}^3$ and a line parameterized by $\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{v}$. Which point on the line comes closest to $\mathbf{p}$? To answer this question, we seek to find the value of t that minimizes the distance between $\mathbf{x}(t)$ and $\mathbf{p}$. Actually, we can avoid a square root if instead we seek to minimize the square of the distance, and this will produce the same value of t. (Think about why this is.)

Therefore, we seek to minimize the function

$$f(t) := \|\mathbf{p} - \mathbf{x}_0 - t\mathbf{v}\|^2 = (\mathbf{p} - \mathbf{x}_0 - t\mathbf{v}) \cdot (\mathbf{p} - \mathbf{x}_0 - t\mathbf{v}) = \|\mathbf{p} - \mathbf{x}_0\|^2 - 2t(\mathbf{p} - \mathbf{x}_0) \cdot \mathbf{v} + t^2\|\mathbf{v}\|^2\ .$$

Thus, $f(t)$ has the form $a - 2bt + ct^2$ for some numbers a, b and c. Since $c > 0$, the graph is an upwards parabola, and there is a unique value t_0 such that $f(t_0) < f(t)$ for all $t \neq t_0$.

To find it, we differentiate and solve $0 = f'(t_0) = -2b + 2ct_0$, which yields

$$t_0 = \frac{b}{v} = \frac{\mathbf{p} - \mathbf{x}_0) \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \ .$$

Thus, the point on the line that is closest to $\mathbf{p}$ is

$$\mathbf{q} := \mathbf{x}_0 + \frac{(\mathbf{p} - \mathbf{x}_0) \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} \ ,$$

and $\mathbf{p} - \mathbf{q} = (\mathbf{p} - \mathbf{x}_0) - \dfrac{\mathbf{p} - \mathbf{x}_0) \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = (\mathbf{p} - \mathbf{x}_0)_\perp$ where $(\mathbf{p} - \mathbf{x}_0)_\perp$ is the orthogonal component of $\mathbf{p} - \mathbf{x}_0$ with respect to $\mathbf{v}$. Thus, the distance is $\|(\mathbf{p} - \mathbf{x}_0)_\perp\|$. This is intuitively natural: To move from $\mathbf{p}$ directly to the line, one should move in a direction that is orthogonal to $\mathbf{v}$. We have proved the following result:

Theorem 11 (Closest point to a line). *Consider any line in $\mathbb{R}^3$ parameterized by $\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{v}$. Let $\mathbf{p}$ be any point in $\mathbb{R}^3$. Then there is a unique point $\mathbf{q}$ in the line such that for all other points $\mathbf{x}$ in the line,*

$$\|\mathbf{p} - \mathbf{q}\| < \|\mathbf{p} - \mathbf{x}\| \ .$$

That is, the distance from $\mathbf{p}$ to $\mathbf{q}$ is less than the distance from $\mathbf{p}$ to any other point in the line. Moreover, we have

$$\|\mathbf{p} - \mathbf{q}\| = \left\| (\mathbf{p} - \mathbf{x}_0) - \frac{1}{\|\mathbf{v}\|^2}((\mathbf{p} - \mathbf{x}_0) \cdot \mathbf{v})\mathbf{v} \right\| = \|(\mathbf{p} - \mathbf{x}_0)_\perp\| \ , \tag{1.64}$$

and

$$\mathbf{q} := \mathbf{x}_0 + \frac{1}{\|\mathbf{v}\|^2}((\mathbf{p} - \mathbf{x}_0) \cdot \mathbf{v})\mathbf{v} \ .$$

where $(\mathbf{p} - \mathbf{x}_0)_\perp$ is the orthogonal component of $\mathbf{p} - \mathbf{x}_0$ with respect to $\mathbf{v}$.

Definition 17 (Distance from a point to a line). *Given any line in $\mathbb{R}^3$ and any point $\mathbf{p} \in \mathbb{R}^3$, the distance from $\mathbf{p}$ to the line is defined to be $\|\mathbf{p} - \mathbf{q}\|$ where $\mathbf{q}$ is the unique point on the line that comes closest to $\mathbf{p}$.*

If we consider the analogous problem for a plane, we will be led to try and find minimizing values of a function

$$f(s,t) = \|\mathbf{p} - (\mathbf{x}_0 + s\mathbf{v}_1 + t\mathbf{v}_2)\|^2 \ .$$

Finding minima and maxima of functions of several variables is a subject we shall study later in this course, so it might appear that we would be getting ahead of ourselves by discussing this problem now. However, it turns out that by making a judicious coice of coordinates we can set the problem up so that it can be solved "by inspection". This works in many problems. Let us first use this approach to give an alternative solution to the problem of finding the distance from a point to a line.

Let us choose an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ that is well-adapted to the geometry of our line. We have seen in Example 10 that we can always construct such an orthonormal basis of $\mathbb{R}^3$ with $\mathbf{u}_3 = \dfrac{1}{\|\mathbf{v}\|}\mathbf{v}$.

Theorem 5 tells us that the length of any vector $\mathbf{w}$ can be computed in terms of this basis through

$$\|\mathbf{w}\|^2 = (\mathbf{w}\cdot\mathbf{u}_1)^2 + (\mathbf{w}\cdot\mathbf{u}_2)^2 + (\mathbf{w}\cdot\mathbf{u}_3)^2 \ .$$

Applying this with $\mathbf{w} = \mathbf{p} - \mathbf{x}(t) = \mathbf{p} - \mathbf{x}_0 - t\mathbf{v}$, we and remembering that $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal to $\mathbf{v}$, we have

$$\|\mathbf{p} - \mathbf{x}(t)\|^2 = ((\mathbf{p}-\mathbf{x}_0)\cdot\mathbf{u}_1)^2 + ((\mathbf{p}-\mathbf{x}_0)\cdot\mathbf{u}_2)^2 + ((\mathbf{p}-\mathbf{x}_0 - t\mathbf{v})\cdot\mathbf{u}_3)^2 \ .$$

Notice that t only enters in the third term of this sum of squares. *This is because we chose our orthonormal basis so that* $\mathbf{u}_1$ *and* $\mathbf{u}_2$ *are orthogonal to* $\mathbf{v}$. Therefore, we can make this sum of squares as small as possible by choosing $t = t_0$ so that the third term is zero. This requires

$$(\mathbf{p} - \mathbf{x}_0 - t_0\mathbf{v})\cdot\mathbf{v} = 0 \quad \text{so that} \quad t_0 = \frac{(\mathbf{p}-\mathbf{x}_0)\cdot\mathbf{v}}{\|\mathbf{v}\|^2} \ ,$$

which is what we found before by differentiating. From here onwards, the rest of the analysis is the same.

Example 19. *Consider the line parameterized by* $\mathbf{x}_0 + t\mathbf{v}$ *with* $\mathbf{x}_0 = (2,-2,3)$ *and* $\mathbf{v} = (1,2,1)$. *Let* $\mathbf{p} = (1,2,3)$. *What is the distance from* $\mathbf{p}$ *to this line? We obtain* $\mathbf{u}_3$ *by normalizing* $\mathbf{v}$, *and then we apply (1.64), finding*

$$\left(\|(-1,4,0)\|^2 - \frac{1}{6}[(-1,4,0)\cdot(1,2,1)]^2\right)^{1/2} = \left(17 - \frac{49}{6}\right)^{1/2} \ .$$

We now turn to the problem of finding the distance between a point and a plane. We know how to find a parameterization of a plane given an equation for it, so we may as well specify the plane by an equation. We will prove:

Theorem 12 (Closest point to a plane)**.** *Consider any plane in* $\mathbb{R}^3$ *given by an equation of the form* $\mathbf{a}\cdot(\mathbf{x}-\mathbf{x}_0) = 0$. *Let* $\mathbf{p}$ *be any point in* $\mathbb{R}^3$. *Then there is a unique point* $\mathbf{q}$ *in the plane such that for all other points* $\mathbf{x}$ *in the plane,*

$$\|\mathbf{p} - \mathbf{q}\| < \|\mathbf{p} - \mathbf{x}\| \ .$$

That is the distance from $\mathbf{p}$ *to* $\mathbf{q}$ *is less than the distance from* $\mathbf{p}$ *to any other point in the plane. Moreover, we have*

$$\|\mathbf{p} - \mathbf{q}\| = \frac{1}{\|\mathbf{a}\|}|\mathbf{a}\cdot(\mathbf{p}-\mathbf{x}_0)| \ , \tag{1.65}$$

$$\mathbf{q} = \mathbf{x}_0 + (\mathbf{p}-\mathbf{x}_0)_\perp = \mathbf{p} - \frac{1}{\|\mathbf{a}\|^2}((\mathbf{p}-\mathbf{x}_0)\cdot\mathbf{a})\mathbf{a} \ . \tag{1.66}$$

where $(\mathbf{p}-\mathbf{x}_0)_\perp$ *is the orthogonal component of* $\mathbf{p}-\mathbf{x}_0$ *with respect to* $\mathbf{a}$.

Proof: This is easy if one uses an orthonormal basis that is adapted to the geometry of the plane. As we have seen in Example 10, it is always possible to find an orthonormal basis $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ of $\mathbb{R}^3$ such that $\mathbf{u}_3$ is the unit vector in the direction of $\mathbf{a}$.

Then, as seen in Example 18, the plane is parameterized by $\mathbf{x}(s,t) = \mathbf{x}_0 + s\mathbf{u}_1 + t\mathbf{u}_2$. By Theorem 5, the square of the Euclidean distance from $\mathbf{p}$ to the general point $\mathbf{x}(s,t)$ in the plane is given by

$$\begin{aligned} & \|\mathbf{p} - \mathbf{x}(s,t)\|^2 \\ = & \|(\mathbf{p} - \mathbf{x}_0) - s\mathbf{u}_2 - t\mathbf{u}_3\|^2 \\ = & [(\mathbf{p} - \mathbf{x}_0 - s\mathbf{u}_2 - t\mathbf{u}_3)\cdot\mathbf{u}_1]^2 + [(\mathbf{p} - \mathbf{x}_0 - s\mathbf{u}_1 - t\mathbf{u}_2)\cdot\mathbf{u}_2]^2 + [(\mathbf{p} - \mathbf{x}_0 - s\mathbf{u}_1 - t\mathbf{u}_2)\cdot\mathbf{u}_3]^2 \ . \\ = & [(\mathbf{p} - \mathbf{x}_0)\cdot\mathbf{u}_1 - s]^2 + [(\mathbf{p} - \mathbf{x}_0)\cdot\mathbf{u}_2 - t]^2 + [(\mathbf{p} - \mathbf{x}_0)\cdot\mathbf{u}_3]^2 \ . \end{aligned}$$

The result is a sum of three squares. The variable s enters only in the first term, and the variable t enters only in the second. We make the sum of the three terms as small as possible by choosing s and t to make the first and second terms zero. Thus, since $\mathbf{u}_3 = \frac{1}{\|\mathbf{a}\|}\mathbf{a}$,

$$\|\mathbf{p} - \mathbf{x}(s,t)\|^2 \geq [(\mathbf{p} - \mathbf{x}_0)\cdot\mathbf{u}_3]^2 = \frac{1}{\|\mathbf{a}\|}|(\mathbf{p} - \mathbf{x}_0)\cdot\mathbf{a}| \ ,$$

which proves (1.65), and there is equality if and only if $s = (\mathbf{p} - \mathbf{x}_0)\cdot\mathbf{u}_1$ and $t = (\mathbf{p} - \mathbf{x}_0)\cdot\mathbf{u}_2$. Thus, the closest point $\mathbf{q}$ is given by $\mathbf{x}(s,t)$ for these values of s and t:

$$\mathbf{q} := \mathbf{x}_0 + [(\mathbf{p} - \mathbf{x}_0)\cdot\mathbf{u}_1]\mathbf{u}_1 + [(\mathbf{p} - \mathbf{x}_0)\cdot\mathbf{u}_2]\mathbf{u}_2 \ . \tag{1.67}$$

Then since, by Theorem 5,

$$\mathbf{p} - \mathbf{x}_0 = [(\mathbf{p} - \mathbf{x}_0)\cdot\mathbf{u}_1]\mathbf{u}_1 + [(\mathbf{p} - \mathbf{x}_0)\cdot\mathbf{u}_2]\mathbf{u}_2 + [(\mathbf{p} - \mathbf{x}_0)\cdot\mathbf{u}_3]\mathbf{u}_3 \ ,$$

and

$$[(\mathbf{p} - \mathbf{x}_0)\cdot\mathbf{u}_1]\mathbf{u}_1 + [(\mathbf{p} - \mathbf{x}_0)\cdot\mathbf{u}_2]\mathbf{u}_2 = (\mathbf{p} - \mathbf{x}_0) - \frac{1}{\|\mathbf{a}\|^2}((\mathbf{p} - \mathbf{x}_0)\cdot\mathbf{a})\mathbf{a} \ .$$

we can rewrite (1.67) as

$$\mathbf{q} = \mathbf{p} - \frac{1}{\|\mathbf{a}\|^2}((\mathbf{p} - \mathbf{x}_0)\cdot\mathbf{a})\mathbf{a} \ ,$$

and then since

$$(\mathbf{p} - \mathbf{x}_0)_\perp = (\mathbf{p} - \mathbf{x}_0) - \frac{1}{\|\mathbf{a}\|^2}((\mathbf{p} - \mathbf{x}_0)\cdot\mathbf{a})\mathbf{a} \ ,$$

this proves (1.66). □

Definition 18 (Distance from a point to a plane)**.** *We define the distance from a point $\mathbf{p} \in \mathbb{R}^3$ to a plane in $\mathbb{R}^3$ to be the distance between $\mathbf{p}$ and the point $\mathbf{q}$ in the plane that comes closest to $\mathbf{p}$.*

Theorem 12 shows the distance between the plane given by $\mathbf{a}\cdot(\mathbf{x} - \mathbf{x}_0) = 0$ is $\frac{1}{\|\mathbf{a}\|}|\mathbf{a}\cdot(\mathbf{p} - \mathbf{x}_0)|$. Note that this is zero if and only if $\mathbf{p}$ belongs to the plane.

Example 20. *Let $\mathbf{p} = (1,-1,1)$ and consider the plane given by $x + 2y + 3z = 4$. What is the distance between $\mathbf{p}$ and the plane?*

To solve this problem ,we would first like to write the equation for the plane in the form $\mathbf{a}\cdot(\mathbf{x} - \mathbf{x}_0) = 0$. To find a base point $\mathbf{x}_0$, let us look for a solution of $x + 2y + 3z = 4$ with $y = z = 0$. Then we must have $x = 4$, and so $\mathbf{x}_0 = (4,0,0)$).

We can write $x + 2y + 3z = 4$ in the form $\mathbf{a} \cdot \mathbf{x} = d$ by defining $\mathbf{a} = (1, 2, 3)$ and $d = 4$, and thus we may take $\mathbf{a} = (1, 2, 3)$ as the normal vector.

We then apply (1.65), to find that the distance is:

$$\frac{1}{\sqrt{14}}|(1, -1, 1) - (4, 0, 0)) \cdot (1, 2, 3)| = \frac{1}{\sqrt{14}}|(- 3, -1, 1)) \cdot (1, 2, 3)| = \frac{2}{\sqrt{14}} \ .$$

To find the point $\mathbf{q}$ in the plane that is closest to $\mathbf{p}$, we use the second formula in (1.66):

$$\mathbf{q} = (1, -1, 1) - \frac{1}{14}((- 3, -1, 1)) \cdot (1, 2, 3))(1, 2, 3) = \frac{1}{7}(8, -5, 10) \ .$$

You can check the computations by verifying that this point does satisfy $x + 2y + 3z = 4$.

We will close this subsection by considering another distance problem that involves minimizing a function of two variables: Finding the distance between two lines. Again, if we work with a judiciously chosen system of coordinates, the minimization problem can be solved "by inspection".

Consider two lines, ℓ_1 and ℓ_2, parameterized by $\mathbf{x}_1(s) = \mathbf{x}_1 + s\mathbf{v}_1$ and $\mathbf{x}_2(t) = \mathbf{x}_2 + t\mathbf{v}_2$. Let us assume that $\mathbf{v}_1$ and $\mathbf{v}_2$ are not multiples of one another, so that the lines are not parallel. (The parallel case is easier; we shall come back to it.) What values of s and t minimize $\|\mathbf{x}_1(s) - \mathbf{x}_2(t)\|$?

To answer this, let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be an orthonormal basis of $\mathbb{R}^3$ in which $\mathbf{u}_1$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$, and in which $\mathbf{u}_2$ is orthogonal to $\mathbf{v}_1$ (and of course to $\mathbf{u}_1$). To produce this basis, define

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{v}_1 \times \mathbf{v}_2\|}\mathbf{v}_1 \times \mathbf{v}_2 \quad \text{and then} \quad \mathbf{u}_2 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 \times \mathbf{u}_1 \ ,$$

By the properties of the cross product, $\mathbf{u}_1$ is a unit vector orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$, and $\mathbf{u}_2$ is a unit vector orthogonal to both $\mathbf{u}_1$ and $\mathbf{v}_1$. Finally, we define $\mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2$, and this gives us the orthonormal basis we seek. Since $\mathbf{v}_1$ is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$, $\mathbf{u}_3$ must be a non-zero multiple of $\mathbf{v}_1$.

We compute $\|\mathbf{x}_1(s) - \mathbf{x}_2(t)\|^2$ in terms of coordinate for this basis. To simplify the notation, define $\mathbf{b}$ by $\mathbf{b} := \mathbf{x}_1 - \mathbf{x}_2$. Then

$$\begin{aligned}
\|\mathbf{x}_1(s) - \mathbf{x}_2(t)\|^2 &= \|\mathbf{b} + s\mathbf{v}_1 - t\mathbf{v}_2\|^2 \\
&= [(\mathbf{b} + s\mathbf{v}_1 - t\mathbf{v}_2) \cdot \mathbf{u}_1]^2 + [(\mathbf{b} + s\mathbf{v}_1 - t\mathbf{v}_2) \cdot \mathbf{u}_2)]^2 + [(\mathbf{b} + s\mathbf{v}_1 - t\mathbf{v}_2) \cdot \mathbf{u}_3)]^2 \\
&= [\mathbf{b} \cdot \mathbf{u}_1]^2 + [\mathbf{b} \cdot \mathbf{u}_2 - t(\mathbf{v}_2 \cdot \mathbf{u}_2)]^2 + [\mathbf{b} \cdot \mathbf{u}_3 + s(\mathbf{v}_1 \cdot \mathbf{u}_3) - t(\mathbf{v}_2 \cdot \mathbf{u}_3)]^2
\end{aligned}$$

This is a sum of three squares. The first does not depend on s or t. The second depends only on t, and we can make it zero by choosing

$$t = t_0 := \frac{\mathbf{b} \cdot \mathbf{u}_2}{\mathbf{v}_2 \cdot \mathbf{u}_2} \tag{1.68}$$

With this choice of t, we can then make the third term zero by choosing

$$s = s_0 := \frac{t_0(\mathbf{v}_2 \cdot \mathbf{u}_3) - \mathbf{b} \cdot \mathbf{u}_3}{\mathbf{v}_1 \cdot \mathbf{u}_3} \ . \tag{1.69}$$

This then leaves only the first term, which is then the square of the minimal distances. However, we have to first verify that we are not dividing by zero in (1.68) and (1.69).

First, since $\mathbf{v}_2$ is orthogonal to $\mathbf{u}_1$,

$$\mathbf{v}_2 = (\mathbf{v}_2 \cdot \mathbf{u}_2)\mathbf{u}_2 + (\mathbf{v}_2 \cdot \mathbf{u}_3)\mathbf{u}_3 \ .$$

If $\mathbf{v}_2 \cdot \mathbf{u}_2$ were zero, then $\mathbf{v}_2$ would be a multiple of $\mathbf{u}_3$, and hence of $\mathbf{v}_1$, since $\mathbf{u}_3$ is a multiple of $\mathbf{u}_1$. However, since we are assuming that the lines are not parallel, $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$, and so $\mathbf{v}_2 \cdot \mathbf{u}_2 \neq 0$. This shows that t_0 is well defined.

Even more simply, since $\mathbf{u}_3$ is a non-zero multiple of $\mathbf{v}_1$, $\mathbf{v}_1 \cdot \mathbf{u}_3 \neq 0$, which shows that s_0 is well defined.

Thus, for any choices of s and t,

$$\|\mathbf{x}_1(s) - \mathbf{x}_2(t)\| \geq \|\mathbf{x}_1(s_0) - \mathbf{x}_2(t_0)\| = |\mathbf{b} \cdot \mathbf{u}_1| \ , \tag{1.70}$$

and there is equality on the left if and only if $s = s_0$ and $t = t_0$. Thus, (1.70) gives the distance between the two lines. We then define the distance between the two lines to be the distance between these two closest points.

Example 21 (The distance between two lines in $\mathbb{R}^3$). *Consider the two lines parameterized by*

$$(1,2,3) + s(1,4,5) \qquad \text{and} \qquad (2,-1,1) + t(-2,-1,2) \ .$$

To calculate the distance between these two lines, we first need to compute a unit vector $\mathbf{u}_1$ *that is orthogonal to both* $(1,4,5)$ *and* $(-2,-1,2)$*. Computing the cross product, we find*

$$(1,4,5) \times (-2,-1,2) = (13,-12,7) \ .$$

Then since $\|(13,-12,7)\| = \sqrt{13^2+12^2+7^2} = \sqrt{362}$*,*

$$\mathbf{u}_1 = \frac{1}{\sqrt{362}}(13,-12,7) \ .$$

Next, we compute $\mathbf{b}$ *in (1.70), which is the difference of the base points:*

$$\mathbf{b} = (1,2,3) - (2,-1,1) = (-1,3,2) \ .$$

Finally we compute

$$\mathbf{b} \cdot \mathbf{u}_1 = \frac{(-1,3,2)\cdot(13,-12,7)}{\sqrt{362}} = \frac{-13-36+14}{\sqrt{362}} = -\frac{35}{\sqrt{362}} \ .$$

Thus, the distance between he two lines is $35/\sqrt{362}$*. If we had wanted to find the point on the first line that comes closes to the second, and the point of the second line that comes closest to the first, we would compute the rest of the orthonormal basis. But to compute the distance, we need only* $\mathbf{u}_1$*.*

Now, what about the case of parallel lines? It is left to the reader to show that if the lines are parallel, the distance from any point on the first line to the second line is independent of the choice of the point on the first line. Thus, this problem reduces to the problem of computing the distance between a point and a line.

Altogether, we have proved:

Theorem 13 (The distance between two lines)**.** *The distance between two non-parallel lines in* $\mathbb{R}^3$ *parameterized by* $\mathbf{x}_1(s) = \mathbf{x}_0 + s\mathbf{v}_1$ *and* $\mathbf{x}_2(s) = \mathbf{x}_0 + t\mathbf{v}_2$ *is*

$$\frac{|(\mathbf{x}_1 - \mathbf{x}_2) \cdot (\mathbf{v}_1 \times \mathbf{v}_2)|}{\|\mathbf{v}_1 \times \mathbf{v}_2\|} .$$

If the two lines are parallel, so that $\mathbf{v}_1 \times \mathbf{v}_2 = 0$*, the distance is the distance from* $\mathbf{x}_1$ *to the second line; i.e.,*

$$\|(\mathbf{x}_1 - \mathbf{x}_2)_\perp\|$$

where $(\mathbf{x}_1 - \mathbf{x}_2)_\perp$ *is the component of* $\mathbf{x}_1 - \mathbf{x}_2$ *orthogonal to* $\mathbf{v}_1$*, or, what is the same thing, orthogonal to* $\mathbf{v}_2$*.*

Notice that each of the three distance problems solved in this subsection involved minimizing a squared distance over the certain parameters, and that in each case, this minimization problem was rendered transparent by an appropriate choice of an orthonormal basis.

1.3 Subspaces of $\mathbb{R}^n$

1.3.1 Dimension

In this section we study the generalization to higher dimensions of lines and planes through the origin in $\mathbb{R}^3$. The results proved here will be very useful to us in our study of multivariable calculus.

Definition 19 (Subspaces of $\mathbb{R}^n$)**.** *A non-empty subset* $V \subset \mathbb{R}^n$ *is a* subspace of $\mathbb{R}^n$ *in case for every* $\mathbf{x}, \mathbf{y} \in V$ *and every pair of numbers* a, b*,*

$$a\mathbf{x} + b\mathbf{y} \in V \ . \tag{1.71}$$

Note that the condition in the definition is satisfied in a trivial way in case either $V = \mathbb{R}^n$ or $V = \{\mathbf{0}\}$. These are the *trivial subspaces* of $\mathbb{R}^n$. By a *non-zero subspace* of $\mathbb{R}^n$, we mean a subspace that is not simply $\{\mathbf{0}\}$.

Taking $a = b = 0$, we see that $\mathbf{0}$ belongs to every subspace $V \subset \mathbb{R}^n$. For $n = 3$, lines and planes through the origin are subspaces: Consider a line through the origin with the parametric representation $\mathbf{x}(t) = t\mathbf{v}$. Then

$$a(t_1\mathbf{v}) + b(t_2\mathbf{v}) = (at_1 + bt_2)\mathbf{v} \ ,$$

so that (1.71) is satisfied. The same reasoning using the parametric representation $\mathbf{x}(s, t) = s\mathbf{u} + t\mathbf{v}$ applies to planes through the origin.

Definition 20 (Orthonormal basis of a subspace)**.** *Let* V *be a non-zero subspace of* $\mathbb{R}^n$*. An* orthonormal basis *of* V *is an orthonormal set* $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ *of vectors in* V *such that there does not exist any non-zero vector in* V *that is orthogonal each vector in* $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$*.*

In other words and orthonormal basis of V is a *maximal* orthonormal set in V; i.e., one that is not a subset of any larger orthonormal subset of V.

Lemma 4. *Every non-zero subspace V of $\mathbb{R}^n$ contains at least one orthonormal basis.*

Proof: Let $\mathbf{v}_1$ be any non-zero vector in V. Define $\mathbf{u}_1 := (1/\|\mathbf{v}_1\|)\mathbf{v}_1$. Since V is a subspace, $\mathbf{u}_1 \in V$. If there is no non-zero vector in in V that is orthogonal to $\mathbf{u}_1$, then $\{\mathbf{u}_1\}$ is an orthonormal basis, and we are done.

Otherwise, let $\mathbf{v}_2$ be a non-zero vector in V that is orthogonal to $\mathbf{u}_1$, and define $\mathbf{u}_2 := (1/\|\mathbf{v}_2\|)\mathbf{v}_2$. Since V is a subspace, $\mathbf{u}_2 \in V$. Thus, $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal set in V. If there is no non-zero vector in in V that is orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$, then $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal basis.

Otherwise, we can add another vector. This process must stop at most when we have n vectors in the set, since $V \subset \mathbb{R}^n$, and by Lemma 1, there does not exist any set of $n+1$ orthonormal vectors in $\mathbb{R}^n$. □

Lemma 5. *Let V be a non-zero subspace of $\mathbb{R}^n$, and let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be an orthonormal basis of V. Then for each $\mathbf{v} \in V$,*

$$\mathbf{v} = \sum_{j=1}^{k} (\mathbf{v} \cdot \mathbf{u}_j)\mathbf{u}_j \ . \tag{1.72}$$

Moreover, all orthonormal bases of V consist of the same number k of unit vectors.

Proof: Let $\mathbf{v} \in V$, and define $\mathbf{z} := \mathbf{v} - \sum_{i=1}^{k} (\mathbf{v} \cdot \mathbf{u}_i)\mathbf{u}_i$. Since V is a subspace, $\mathbf{z} \in V$. We compute

$$\mathbf{z} \cdot \mathbf{u}_j = \left(\mathbf{v} - \sum_{i=1}^{k} (\mathbf{v} \cdot \mathbf{u}_i)\mathbf{u}_i\right) \cdot \mathbf{u}_j = \mathbf{v} \cdot \mathbf{u}_j - \mathbf{v} \cdot \mathbf{u}_j = 0 \ ,$$

where we have used the orthonormality of $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$.

If $\mathbf{z} \neq \mathbf{0}$, there is a non-zero vector in V orthogonal to each vector in $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$. This is not possible since $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthonormal basis. Hence $\mathbf{z} = 0$, and (1.72) is proved.

Next, Consider the function $\mathbf{c}$ from V to $\mathbb{R}^k$ given by

$$\mathbf{c}(\mathbf{v}) := (\mathbf{v} \cdot \mathbf{u}_1, \ldots, \mathbf{v} \cdot \mathbf{u}_k) \ .$$

We now claim that for any $\mathbf{v}, \mathbf{w} \in V$, $\mathbf{v} \cdot \mathbf{w} = \mathbf{c}(\mathbf{v}) \cdot \mathbf{c}(\mathbf{w})$. Indeed, by the first part:

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{w} &= \left(\sum_{i=1}^{k} (\mathbf{v} \cdot \mathbf{u}_i)\mathbf{u}_i\right) \cdot \left(\sum_{j=1}^{k} (\mathbf{v} \cdot \mathbf{u}_j)\mathbf{u}_j\right) \\
&= \sum_{i,j=1}^{k} (\mathbf{v} \cdot \mathbf{u}_i)(\mathbf{v} \cdot \mathbf{u}_j)\mathbf{u}_i \cdot \mathbf{u}_j \\
&= \sum_{i,=1}^{k} (\mathbf{v} \cdot \mathbf{u}_i)(\mathbf{v} \cdot \mathbf{u}_i) = \mathbf{c}(\mathbf{v}) \cdot \mathbf{c}(\mathbf{w}) \ .
\end{aligned}$$

Now suppose that $\{\mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$ is another orthonormal basis of V. Then for each $i, j = 1 \ldots, \ell$,

$$\mathbf{c}(\mathbf{w}_i) \cdot \mathbf{c}(\mathbf{w}_j) = \mathbf{w}_i \cdot \mathbf{w}_j = \begin{cases} 1 & i = i \\ 0 & i \neq j \end{cases} \ .$$

Thus $\{\mathbf{c}(\mathbf{w}_1), \dots, \mathbf{c}(\mathbf{w}_\ell)\}$ is an orthonormal set in $\mathbb{R}^k$. By Lemma 1, $\ell \leq k$. Since our original orthonormal basis was arbitrary, we have seen that no orthonormal basis of V can contain more vectors than any other. Hence, they all contain the same number of vectors. □

Definition 21 (Dimension)**.** *Let V be a non-zero subspace of $\mathbb{R}^n$. We define the* dimension *of V,* $\dim(V)$*, to be the number of vectors in any orthonormal basis of V. By convention, we define the dimension of the zero subspace to be zero.*

1.3.2 Orthogonal complements

Definition 22 (Orthogonal complement)**.** *Let S be any subset of $\mathbb{R}^n$, finite or infinite. The* orthogonal complement of S*, $S^\perp$, is the subset of vectors $\mathbf{x} \in \mathbb{R}^n$ that are orthogonal to each $\mathbf{v} \in S$.*

Note that if $S_1 \subset S_2$, then a vector $\mathbf{y}$ has to satsify more conditions of the form $\mathbf{x} \cdot \mathbf{y} = 0$ to belong to $S_2^\perp$ than to $S_1^\perp$. Hence:

$$S_1 \subset S_2 \quad \Rightarrow \quad S_2^\perp \subset S_1^\perp \ . \tag{1.73}$$

Example 22 (Orthogonal complements)**.** *Let $\{\mathbf{a}\}$ be a non-zero vector in $\mathbb{R}^3$, and let $S = \{\mathbf{a}\}$. Then $S^\perp$ is the set of all vectors orthogonal to $\mathbf{a}$, i.e., the set of all $\mathbf{x} \in \mathbb{R}^3$ such that*

$$\mathbf{a} \cdot \mathbf{x} = 0 \ .$$

This is the equation of the plane through the origin that is normal to $\mathbf{a}$.

Likewise, let $S = \{\mathbf{a}_1, \mathbf{a}_2\}$ be a set of two non-zero vectors in $\mathbb{R}^3$ that are not multiples of one another. Then $S^\perp$ is the line specified by

$$\begin{aligned} \mathbf{a}_1 \cdot \mathbf{x} &= 0 \\ \mathbf{a}_2 \cdot \mathbf{x} &= 0 \ . \end{aligned}$$

In both of these examples, $S^\perp$ is a subspace. This is always the case:

Theorem 14. *Let S be any subset of $\mathbb{R}^n$. Then $S^\perp$ is a subspace. For any subspace V of $\mathbb{R}^n$,*

$$\dim(V) + \dim(V^\perp) = n \tag{1.74}$$

and

$$(V^\perp)^\perp = V \ . \tag{1.75}$$

Finally, with $k := \dim(V)$, there exists an orthonormal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ of $\mathbb{R}^n$ such that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthonormal basis of V, and $\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an orthonormal basis of $V^\perp$.

Proof: Let $\mathbf{x}, \mathbf{y} \in S^\perp$, and let a, b be any two numbers. For any $\mathbf{v} \in S$,

$$(a\mathbf{x} + b\mathbf{y}) \cdot \mathbf{v} = a(\mathbf{x} \cdot \mathbf{v}) + b(\mathbf{y} \cdot \mathbf{v}) = a0 + b0 = 0 \ .$$

Thus $a\mathbf{x} + b\mathbf{y} \in S^\perp$. This shows that $S^\perp$ is a subspace.

Now let V be a non-trivial subspace of $\mathbb{R}^n$, and let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be an orthonormal basis of V, so that $\dim(V) = k$. Since V is non-trivial, $0 < k < n$.

While $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthonormal basis of V, it is not an orthonormal basis of $\mathbb{R}^n$ itself. Thus, as in the proof of Lemma 4, we can continue adding unit vectors to the set until we arrive at an orthonormal basis of $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ of $\mathbb{R}^n$, necessarily consisting of n vectors, and each vector in $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is orthogonal to each vector in $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$, and hence to each vector in V.

Now, every vector $\mathbf{x} \in \mathbb{R}^n$ can be written as $\mathbf{x} = \sum_{i=1}^{n} (\mathbf{x} \cdot \mathbf{u}_i)\mathbf{u}_i$. Every vector $\mathbf{x} \in V^\perp$ is orthogonal to each vector in V, and thus to each vector in $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$. Hence,

$$\mathbf{x} \in V^\perp \quad \Rightarrow \quad \mathbf{x} = \sum_{i=k+1}^{n} (\mathbf{x} \cdot \mathbf{u}_i)\mathbf{u}_i \ .$$

Thus, $\mathbf{x}$ is a linear combination of the vectors in $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$, and cannot be orthogonal to all of them unless $\mathbf{x} = \mathbf{0}$, and consequently, $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$, is anl orthonormal basis of $V^\perp$. It follows that $\dim(V^\perp) = n - k$. This proves (1.74).

Next by (1.74) applied with $V^\perp$ in place of V,

$$\dim(V^\perp) + \dim((V^\perp)^\perp) = n = \dim(V) + \dim(V^\perp) \ .$$

It follows that

$$\dim((V^\perp)^\perp) = \dim(V) \ .$$

Also since by definition, every vector in V is orthogonal to every vector in $V^\perp$, every vector in V belongs to $(V^\perp)^\perp$. That is,

$$V \subset (V^\perp)^\perp \ .$$

Now let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be an orthonormal basis of V. Then $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthonormal subset of $(V^\perp)^\perp$, and since $\dim(V^\perp)^\perp) = k$, no more vectors can be added to it, and it is an orthonormal basis of $(V^\perp)^\perp$. By Lemma 5 again, every vector in $(V^\perp)^\perp$ is a linear combination of the vectors in $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$, and hence belongs to V. Thus $V \subset (V^\perp)^\perp$. Altogether, $V = (V^\perp)^\perp$. The final statement is now evident. □

Definition 23 (Span of a set of vectors)**.** *Let $S \subset \mathbb{R}^n$. Then the* span of S, Span(S), *is the set of all linear combinations of finite subsets of S; i.e., of all vectors of the form $\sum_{j=1}^{k} x_j\mathbf{x}_j$ with $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \subset S$.*

Since a linear combination of two linear combinations of vectors in $S \subset \mathbb{R}^n$ is a again a linear combination of vectors in S, Span(S) is a subspace of $\mathbb{R}^n$. In fact:

Theorem 15 (Span and orthogonal complements)**.** *Let S be any subset of $\mathbb{R}^n$. Then $(S^\perp)^\perp =$ Span(S).*

Proof: Let W be any subspace of $\mathbb{R}^n$ such that $S \subset W$. By (1.73), $W^\perp \subset S^\perp$, and then by (1.73) again and (1.75), $(S^\perp)^\perp \subset (W^\perp)^\perp = W$. In particular, since $S \subset \text{Span}(S)$, $(S^\perp)^\perp \subset \text{Span}(S)$.

On the other hand, since subspaces are closed under taking linear combinations, and since $(S^\perp)^\perp$ contains S, $(S^\perp)^\perp$ contains every finite linear combination of vectors in S; i.e., $\text{Span}(S) \subset (S^\perp)^\perp$.

Having proved both $(S^\perp)^\perp \subset \text{Span}(S)$ and $\text{Span}(S) \subset (S^\perp)^\perp$, we have the equality. □

The answers to most questions concerning a k dimensional subspace V of $\mathbb{R}^n$ can be answered by considering the orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ of $\mathbb{R}^n$ such that $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthonormal basis of V, and $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthonormal basis of $V^\perp$ that was constructed in the previous theorem.

For example, since $\mathbf{x} \in \mathbb{R}^n$ belongs to V is and only if $\mathbf{u}_j \cdot \mathbf{x} = 0$ for each $j = k+1, \ldots, n$, V is the solution set of the $n - k$ equations

$$\begin{aligned} \mathbf{u}_{k+1} \cdot \mathbf{x} &= 0 \\ \vdots &= \vdots \\ \mathbf{u}_n \cdot \mathbf{x} &= 0 \end{aligned}$$

Moreover, the function

$$(t_1, \ldots, t_k) \mapsto \mathbf{x}(t_1, \ldots, t_k) := \sum_{j=1}^{k} t_j \mathbf{u}_k$$

is a one-to-one function from $\mathbb{R}^k$ onto V, and hence is a parameterization of V. The inverse function, which we denote by $\mathbf{c}$, is the corresponding *coordinate map* on V, and is given by

$$\mathbf{c}(\mathbf{x}) = (\mathbf{u}_1 \cdot \mathbf{x}, \ldots, \mathbf{u}_k \cdot \mathbf{x}) \ .$$

As you can readily check,

$$\mathbf{c}(\mathbf{x}(t_1, \ldots, t_k)) = (t_1, \ldots, t_k) \ .$$

Indeed, we call $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ an orthonormal basis of V since it is the basis of a system of orthonormal coordinate on V.

Next, let us consider the higher dimensional analog of the problem of finding the distance between a point and a line or a plane in $\mathbb{R}^3$. Let $\mathbf{b}$ be any point in $\mathbb{R}^n$. Can we find a point $\mathbf{x} \in V$ that comes closer to $\mathbf{b}$ than any other point in V? Yes, using this "well-adapted basis": For any $\mathbf{x} \in V$,

$$\begin{aligned} \|\mathbf{b} - \mathbf{x}\|^2 &= \sum_{j=1}^{n} ((\mathbf{b} - \mathbf{x}) \cdot \mathbf{u}_j)^2 \\ &= \sum_{j=1}^{n} (\mathbf{b} \cdot \mathbf{u}_j - \mathbf{x} \cdot \mathbf{u}_j)^2 \\ &= \sum_{j=1}^{k} (\mathbf{b} \cdot \mathbf{u}_j - \mathbf{x} \cdot \mathbf{u}_j)^2 + \sum_{j=k+1}^{n} (\mathbf{b} \cdot \mathbf{u}_j)^2 \end{aligned}$$

where in the last line we have used the orthogonality of every vector in V to each $\mathbf{u}_j$, $j > k$. Evidently we make this sum of squares as small as possible by choosing

$$\mathbf{x} = \sum_{j=1}^{k} (\mathbf{b} \cdot \mathbf{u}_j) \mathbf{u}_j \tag{1.76}$$

since this choice, and this choice alone, makes $\sum_{j=1}^{k}(\mathbf{b}\cdot\mathbf{u}_j - \mathbf{x}\cdot\mathbf{u}_j)^2 = 0$. The conclusion is that the vector $\mathbf{x}$ defined in (1.76) comes closer to $\mathbf{b}$ than any other vector in V. We define the distance between $\mathbf{b}$ and V to be $\|\mathbf{b}-\mathbf{x}\|$ for this choice of $\mathbf{x}$, and we see from the calculation above that this distance is

$$\left(\sum_{j=k+1}^{n}(\mathbf{b}\cdot\mathbf{u}_j)^2\right)^{1/2} .$$

This result generalizes our previous results on the distances between points and lines and points and planes to arbitrarily many dimensions. Of course, to use these formulas, one has to be able to find the "well-adapted basis" $\{\mathbf{u}_1,\dots,\mathbf{u}_k,\mathbf{u}_{k+1},\dots,\mathbf{u}_n\}$ of $\mathbb{R}^n$. We have effective means for doing this for $n = 3$, and we shall develop effective means for doing this in every dimension as we go along. The main point that we want to make here, with which we conclude this chapter, is that *"well-adpated" orthonormal bases provide the key to a great many geometric problem that we shall consider, no matter how many variables are involved.*

1.4 Exercises

1.1 Let $\mathbf{a} = (3,-1)$, $\mathbf{b} = (2,1)$ and $\mathbf{c} = (1,3)$. Express $\mathbf{a}$ as a linear combination of $\mathbf{b}$ and $\mathbf{c}$. That is, find numbers s and t so that $\mathbf{a} = s\mathbf{b} + t\mathbf{c}$.

1.2 Let $\mathbf{a} = (5,2)$, $\mathbf{b} = (2,-1)$ and $\mathbf{c} = (1,1)$. Express each of these three vectors as a linear combination of the other two.

1.3 Let $\mathbf{x} = (1,4,8)$ and $\mathbf{y} = (1,2,-2)$. Compute the lengths of each of these vectors, and the angle between them.

1.4 Let $\mathbf{x} = (4,7,-4,1,2,-2)$ and $\mathbf{y} = (2,1,2,2,-1,-1)$. Compute the lengths of each of these vectors, and the angle between them.

1.5 Let $\mathbf{x} = (4,7,4)$ and $\mathbf{y} = (2,1,2)$. Compute the lengths of each of these vectors, and the angle between them.

1.6 Let $\mathbf{x} = (-5,2,-5)$ and $\mathbf{y} = (1,2,1)$. Is the angle between $\mathbf{x}$ and $\mathbf{y}$ acute or obtuse? Justify your answer.

1.7 Let

$$\mathbf{u}_1 = \frac{1}{9}(1,-4,-8) \qquad \mathbf{u}_2 = \frac{1}{9}(8,4,-1) \quad \text{and} \quad \mathbf{u}_3 = \frac{1}{9}(4,-7,4) .$$

(a) Show that $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ is an orthonormal basis of $\mathbb{R}^3$. Is it a right-handed orthonormal basis? Justify your answer.

(b) Find numbers y_1, y_2 and y_3 such that

$$y_1\mathbf{u}_1 + y_2\mathbf{u}_2 + y_3\mathbf{u}_3 = (10,11,-11) .$$

What are the lengths of the vectors $(10,11,-11)$ and (y_1,y_2,y_3)? give calculations or an explanation in each case.

1.8 Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be any three vectors in $\mathbb{R}^3$ with $\mathbf{a} \neq \mathbf{0}$. Show that $\mathbf{b} = \mathbf{c}$ if and only if

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} \quad \text{and} \quad \mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c} \ .$$

1.9 Let $\mathbf{a} = (1, 1, 1)$

(a) Find a vector $\mathbf{x}$ such that

$$\mathbf{x} \times \mathbf{a} = (\,-7, 2, 5) \qquad \text{and} \qquad \mathbf{x} \cdot \mathbf{a} = 0 \ .$$

(b) There is no vector $\mathbf{x}$ such that

$$\mathbf{x} \times \mathbf{a} = (1, 0, 0) \qquad \text{and} \qquad \mathbf{x} \cdot \mathbf{a} = 0 \ .$$

Show that no such vector exists.

1.10 (a) Let $\mathbf{a} = (\,-1, 1, 2)$ and $\mathbf{b} = (2, -1, 1)$. Find all vectors $\mathbf{x}$, if any exist, such that

$$\mathbf{a} \times \mathbf{x} = (\,-2, 4, -3) \qquad \text{and} \qquad \mathbf{b} \cdot \mathbf{x} = 2 \ .$$

If none exist, explain why this is the case.

(b) Let $\mathbf{a} = (\,-1, 1, 2)$ and $\mathbf{b} = (2, -1, 1)$. Find all vectors $\mathbf{x}$, if any exist, such that

$$\mathbf{a} \times \mathbf{x} = (2, 4, 3) \qquad \text{and} \qquad \mathbf{b} \cdot \mathbf{x} = 2 \ .$$

If none exist, explain why this is the case.

(c) Among all vectors $\mathbf{x}$ such that $(\,-1, 1, 2) \times \mathbf{x} = (\,-2, 4, -3)$, find the one that is closest to $(1, 1, 1)$.

1.11 (a) Let $\mathbf{a}$ and $\mathbf{b}$ be orthogonal vectors. Define a sequence of vectors $\{\mathbf{b}_n\}$ by

$$\mathbf{b}_{n+1} = \mathbf{a} \times \mathbf{b}_n \qquad \text{and} \qquad \mathbf{b}_0 = \mathbf{b} \ .$$

Show that for all positive integers m

$$\mathbf{b}_{2m} = (-1)^m \|\mathbf{a}\|^{2m} \mathbf{b} \ .$$

How do you have to adjust the formula if the hypothesis that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal is dropped?

(b) Let $\mathbf{a} = \frac{1}{3}(2, -1, 2)$ and $\mathbf{b} = (1, 1, 1)$. With $\mathbf{b}_n$ defined as in part **(a)**, compute $\mathbf{b}_{99}$.

1.12 (a) Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three non-zero vectors in $\mathbb{R}^3$. Define a transformation $\mathbf{f}$ from $\mathbb{R}^3$ to $\mathbb{R}^3$ by

$$\mathbf{f}(\mathbf{x}) = \mathbf{a} \times (\mathbf{b} \times (\mathbf{c} \times \mathbf{x}))) \ .$$

Show that $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^3$ if and only if $\mathbf{b}$ is orthogonal to $\mathbf{c}$, and $\mathbf{a}$ is a multiple of $\mathbf{c}$.

1.13 (a) Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three non-zero vectors in $\mathbb{R}^3$. Show that

$$|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| \leq \|\mathbf{a}\| \|\mathbf{b}\| \|\mathbf{c}\|$$

and there is equality if and only if $\left\{ \frac{1}{\|\mathbf{a}\|}\mathbf{a}, \frac{1}{\|\mathbf{b}\|}\mathbf{b}, \frac{1}{\|\mathbf{c}\|}\mathbf{c} \right\}$ is orthonormal.

1.14 Let P_1 the plane through the three points $\mathbf{a}_1 = (1, 2, 1)$ $\mathbf{a}_2 = (-1, 2, -3)$ and $\mathbf{a}_3 = (2, -3, -2)$. Let P_2 denote the plane through the three points $\mathbf{b}_1 = (1, 1, 0)$ $\mathbf{b}_2 = (1, 0, 1)$ and $\mathbf{b}_3 = (0, 1, 1)$.

(a) Find equations for the planes P_1 and P_2.

(b) Parameterize the line given by $P_1 \cap P_2$, and find the distance between this line and the point $\mathbf{a}_1$.

(c) Consider the line through $\mathbf{b}_1$ and $\mathbf{b}_2$. Determine the point of intersection of this line with the plane P_1.

1.15 Consider the vector $\mathbf{v} = (1, 4, 3)$. Find an orthonormal basis of $\mathbb{R}^3$ whose third vector is a multiple of $\mathbf{v}$.

1.16 Consider the vector $\mathbf{a} = (1, 4, 3)$ and $\mathbf{b} = (3, 2, 1)$. Find an orthonormal basis of $\mathbb{R}^3$ whose third vector is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

1.17 Consider the plane passing through the three points

$$\mathbf{p}_1 = (-2, 0, 2) \quad \mathbf{p}_2 = (1, -2, 2) \qquad \text{and} \qquad \mathbf{p}_3 = (3, -1, -2)$$

and the line passing through

$$\mathbf{z}_0 = (1, 4, -2) \qquad \text{and} \qquad \mathbf{z}_1 = (0, -3, 1)$$

(a) Find a parametric representation $\mathbf{x}(s, t) = \mathbf{x}_0 + s\mathbf{v}_1 + t\mathbf{v}_2$ for the plane.

(b) Find a parametric representation $\mathbf{z}(u) = \mathbf{z}_0 + u\mathbf{w}$ for the line.

(c) Find an equation for the plane.

(d) Find a system of equations for the line.

(e) Find the points, if any, where the line intersects the plane.

(f) Find the distance from $\mathbf{p}_1$ to the line.

(g) Find the distance from $\mathbf{z}_0$ to the plane.

1.18 Consider the plane passing through the three points

$$\mathbf{p}_1 = (-1, -3, 0) \qquad \mathbf{p}_2 = (5, 1, 2) \qquad \text{and} \qquad \mathbf{p}_3 = (0, -3, 4)$$

and the line passing through

$$\mathbf{z}_0 = (1, 1, -1) \qquad \text{and} \qquad \mathbf{z}_1 = (1, -2, 2)$$

(a) Find a parametric representation $\mathbf{x}(s, t) = \mathbf{x}_0 + s\mathbf{v}_1 + t\mathbf{v}_2$ for the plane.

(b) Find a parametric representation $\mathbf{z}(u) = \mathbf{z}_0 + u\mathbf{w}$ for the line.

(c) Find an equation for the plane.

(d) Find a system of equations for the line.

(e) Find the points, if any, where the line intersects the plane.

(f) Find the distance from $\mathbf{p}_1$ to the line.

(g) Find the distance from $\mathbf{z}_0$ to the plane.

1.19 Consider the two lines parameterized by

$$(1,1,0)+t(1,-1,2) \qquad \text{and} \qquad (2,0,2)+s(-1,1,0)\ .$$

(a) These lines intersect. Find the point of intersection.

(b) Find an equation for the plane P containing these two lines.

1.20 Consider the plane given by

$$2x-y+3z=4\ .$$

Let $\mathbf{p}=(-1,-3,0)$. What is the distance from $\mathbf{p}$ to the plane?

1.21 Consider the plane given by

$$x-3y+z=2\ .$$

Let $\mathbf{p}=(-2,-5,1)$. What is the distance from $\mathbf{p}$ to the plane?

1.22 Consider the line ℓ given by

$$\begin{aligned} 2x-y+3z &= 4 \\ x+y+z &= 2\ . \end{aligned}$$

Find a parametric representation of the line obtained by reflecting this line through the plane $x+3y-z=1$. That is; the outgoing line should have as its base point the intersection of the incoming line and the plane, and its direction vector should be $\mathbf{h_u}(\mathbf{v})$ where $\mathbf{v}$ is the incoming direction vector, and $\mathbf{u}$ is a unit normal vector to the plane.

1.23 Consider the line ℓ given by

$$\begin{aligned} x-3y+z &= 2 \\ 2y+z &= 3\ . \end{aligned}$$

Find a parametric representation of the line obtained by reflecting this line through the plane $x+2y-z=1$. Find a parametric representation of the line obtained by reflecting this line through the plane $x+3y-z=1$. (See the previous exercise.)

1.24 Let $\mathbf{x}=(5,2,4,2)$. Let $\mathbf{u}$ be a unit vector such that $\mathbf{h_u}(\mathbf{x})$ is a multiple of $\mathbf{e}_1$. What are the possible values of this multiple? Find four such unit vectors $\mathbf{u}$.

1.25 Consider two lines in $\mathbb{R}^3$ given parametrically by $\mathbf{x}_1(s)=\mathbf{x}_1+s\mathbf{v}_1$ and $\mathbf{x}_2(t)=\mathbf{x}_2+t\mathbf{v}_2$ where

$$\mathbf{x}_1=(1,2,1) \qquad \mathbf{x}_2=(1,-1,0) \qquad \mathbf{v}_1=(1,0,-1) \qquad \text{and} \qquad \mathbf{v}_2=(2,1,1)\ .$$

Compute the distance between these two lines.

1.26 Consider two lines in $\mathbb{R}^3$ given parametrically by $\mathbf{x}_1(s)=\mathbf{x}_1+s\mathbf{v}_1$ and $\mathbf{x}_2(t)=\mathbf{x}_2+t\mathbf{v}_2$ where

$$\mathbf{x}_1=(1,2,3) \qquad \mathbf{x}_2=(2,0,2) \qquad \mathbf{v}_1=(1,2,2) \qquad \text{and} \qquad \mathbf{v}_2=(-2,1,1)\ .$$

Compute the distance between these two lines.

1.27 Consider two lines in $\mathbb{R}^3$ given parametrically by $\mathbf{x}_1(s) = \mathbf{x}_1 + s\mathbf{v}_1$ and $\mathbf{x}_2(t) = \mathbf{x}_2 + t\mathbf{v}_2$ where

$$\mathbf{x}_1 = (3,2,1) \qquad \mathbf{x}_2 = (1,1,-1) \qquad \mathbf{v}_1 = (3,-5,-1) \qquad \text{and} \qquad \mathbf{v}_2 = (\,-1,3,3)\ .$$

Find the point on the first line that is closest to the second line, the point on the second line that is closest to the first line, and the distance between these two lines.

1.28 Consider two lines in $\mathbb{R}^3$ given parametrically by $\mathbf{x}_1(s) = \mathbf{x}_1 + s\mathbf{v}_1$ and $\mathbf{x}_2(t) = \mathbf{x}_2 + t\mathbf{v}_2$ where

$$\mathbf{x}_1 = (1,2,-1) \qquad \mathbf{x}_2 = (2,1,-5) \qquad \mathbf{v}_1 = (1,-4,-2) \qquad \text{and} \qquad \mathbf{v}_2 = (1,1,-2).$$

Find the point on the first line that is closest to the second line, the point on the second line that is closest to the first line, and the distance between these two lines.

1.29 Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be given by

$$\mathbf{u}_1 = \frac{1}{3}(1,2,-2) \qquad \mathbf{u}_2 = \frac{1}{3}(2,1,2) \qquad \text{and} \qquad \mathbf{u}_3 = \frac{1}{3}(2,-2,-1)\ .$$

(a) Verify whether $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is, or is not, an orthonormal basis of $\mathbb{R}^3$.

(b) Find a unit vector $\mathbf{u}$ so that $\mathbf{h}_{\mathbf{u}}(\mathbf{u}_1) = \mathbf{e}_1$.

(c) With this came choice of $\mathbf{u}$, compute $\mathbf{h}_{\mathbf{u}}(\mathbf{u}_2)$ and $\mathbf{h}_{\mathbf{u}}(\mathbf{u}_3)$.

1.30 Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be given by

$$\mathbf{u}_1 = \frac{1}{9}(1,4,8) \qquad \mathbf{u}_2 = \frac{1}{9}(8,-4,1) \qquad \text{and} \qquad \mathbf{u}_3 = \frac{1}{9}(4,7,-4))\ .$$

(a) Verify whether $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is, or is not, an orthonormal basis of $\mathbb{R}^3$.

(b) Find a unit vector $\mathbf{u}$ so that $\mathbf{h}_{\mathbf{u}}(\mathbf{u}_1) = \mathbf{e}_1$.

(c) With this came choice of $\mathbf{u}$, compute $\mathbf{h}_{\mathbf{u}}(\mathbf{u}_2)$ and $\mathbf{h}_{\mathbf{u}}(\mathbf{u}_3)$.

1.31 Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be given by

$$\mathbf{u}_1 = \frac{1}{3}(1,2,-2) \qquad \mathbf{u}_2 = \frac{1}{\sqrt{2}}(0,1,1) \qquad \text{and} \qquad \mathbf{u}_3 = \frac{1}{3\sqrt{2}}(4,1,-1))\ .$$

(a) Verify whether $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is, or is not, an orthonormal basis of $\mathbb{R}^3$.

(b) Find a unit vector $\mathbf{u}$ so that $\mathbf{h}_{\mathbf{u}}(\mathbf{u}_1) = \mathbf{e}_1$.

(c) With this same choice of $\mathbf{u}$, compute $\mathbf{h}_{\mathbf{u}}(\mathbf{u}_2)$ and $\mathbf{h}_{\mathbf{u}}(\mathbf{u}_3)$.

1.32 Let V_1 and V_2 be two subspaces of $\mathbb{R}^n$.

(a) Show that $V_1 \cap V_2$ is a subspace of $\mathbb{R}^n$.

(b) Define $V_1 + V_2$ to be the set of all vectors $\mathbf{z} \in \mathbb{R}^n$ such that $\mathbf{z} = \mathbf{x} + \mathbf{y}$ for some $\mathbf{x} \in V_1$ and some $\mathbf{y} \in V_2$. Show that $V_1 + V_2$ is a subspace of $\mathbb{R}^n$.

1.33 Let V_1 and V_2 be two subspaces of $\mathbb{R}^n$. Using the results and notation from the previous exercise, show that

$$\dim(V_1 \cap V_2) + \dim(V_1 + V_2) = \dim(V_1) + \dim(V_2)\ .$$

1.34 For $n > 3$, an $n-1$ dimensional V subspace of $\mathbb{R}^n$ is called a *hyperplane through the origin.* The orthogonal complement $V^\perp$ is a one dimensional subspace. In this case, starting from the equation of the hyperplane, it is easy to write down an orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_{n-1}, \mathbf{u}_n\}$ such that $\{\mathbf{u}_1, \ldots, \mathbf{u}_{n-1}\}$ is an orthonormal basis of V, and such that $\{\mathbf{u}_n\}$ is an orthonormal basis of $V^\perp$:

Let $\mathbf{a}$ be a non-zero vector in $\mathbb{R}^n$, and let V be the solution set of $\mathbf{a}\cdot\mathbf{x} = 0$. Define the uni vector $\mathbf{w} = (1/\|\mathbf{a}\|)\mathbf{a}$. Let $\mathbf{u}$ be a unit vector such that the Householder reflection $\mathbf{h}_\mathbf{u}$ satisfies

$$\mathbf{h}_\mathbf{u}(\mathbf{w}) = \mathbf{e}_n \ .$$

Define

$$\mathbf{u}_j = \mathbf{h}_\mathbf{u}(\mathbf{e}_j) \qquad \text{for} \quad j = 1, \ldots, n \ .$$

Show that with these definitions, $\{\mathbf{u}_1, \ldots, \mathbf{u}_{n-1}\}$ is an orthonormal basis of V, and such that $\{\mathbf{u}_n\}$ is an orthonormal basis of $V^\perp$.

1.35 We use the notation and results of the previous exercise. Consider the hyperplane V through the origin in $\mathbb{R}^4$ given by

$$2x + 2y - 7z + 4w = 0 \ .$$

Let $\mathbf{b} = (1, 2, 0, 2)$. Find the point $\mathbf{x} \in V$ that is closest to $\mathbf{b}$ and find the distance between $\mathbf{b}$ and V.

1.36 Show that for all vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ in $\mathbb{R}^3$,

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \ .$$

1.37 Show that for all $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^3$,

$$(\mathbf{b} \times \mathbf{c}) \cdot [(\mathbf{c} \times \mathbf{a}) \times (\mathbf{a} \times \mathbf{b})] = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|^2 \ .$$

Chapter 2

DESCRIPTION AND PREDICTION OF MOTION

2.1 Functions from $\mathbb{R}$ to $\mathbb{R}^n$ and the description of motion

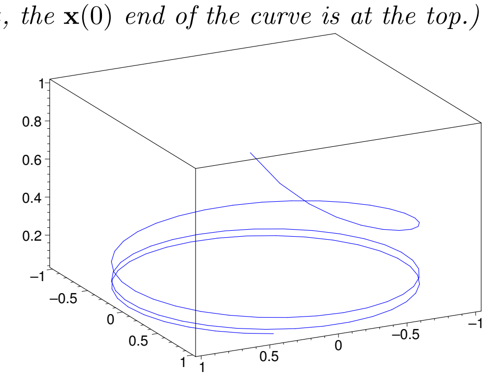
In many ways, the simplest multivariable functions are functions from $\mathbb{R}$ to $\mathbb{R}^n$ for some $n \geq 1$. These are functions that have one input variable (one independent variable), and n output variables (n dependent variables).

If $n = 2$ or if $n = 3$, we can think of the output variable $\mathbf{x}$ as the coordinate vector of a point moving in $\mathbb{R}^2$ or $\mathbb{R}^3$, and we can think of the input variable t as the time so that the function gives us the location of a moving point at time t. We then write the function as $\mathbf{x}(t)$. Such functions are also called *vector valued functions of a real variable.*

Example 23. *Consider the function $\mathbf{x}(t)$ of the real variable t with values in $\mathbb{R}^3$ given by*

$$\mathbf{x}(t) = (\cos(t) \ , \sin(t) \ , 1/t) \ . \tag{2.1}$$

Here is a three dimensional plot of the curve traced out by $\mathbf{x}(t)$ as for $t \leq 1 \leq 20$. (Since $x_3(t) = 1/t$ is a decreasing function of t, the $\mathbf{x}(0)$ end of the curve is at the top.)



2.1.1 Continuity of functions from $\mathbb{R}$ to $\mathbb{R}^n$

Vector valued functions of one real variable that describe particle motion usually have certain *regularity properties*: For example, particle motions are usually at least *continuous*:

Definition 24 (Convergence and continuity in $\mathbb{R}^n$)**.** *A sequence of vectors* $\{\mathbf{x}_j\}$ *in* $\mathbb{R}^n$ converges to $\mathbf{z} \in \mathbb{R}^n$ *in case for each* $\epsilon > 0$*, there is a natural number* N_ϵ *so that*

$$j \geq N_\epsilon \qquad \Rightarrow \qquad \|\mathbf{x}_j - \mathbf{z}\| \leq \epsilon \ ,$$

in which case we say that $\mathbf{z}$ *is the limit of the sequence and write*

$$\lim_{n\to\infty} \mathbf{x}_j = \mathbf{z} \ .$$

A function $\mathbf{x}(t)$ *defined on an open interval* $(a,b) \subset \mathbb{R}$ *with values in* $\mathbb{R}^n$ *is* continuous at $t_0 \in (a,b)$ *in case for each* $\epsilon > 0$*, there is a real number* $\delta_\epsilon > 0$ *so that*

$$|t - t_0| \leq \delta_\epsilon \qquad \Rightarrow \qquad \|\mathbf{x}(t) - \mathbf{x}(t_0)\| \leq \epsilon \ , \tag{2.2}$$

in which case we write $\lim_{t\to t_0} \mathbf{x}(t) = \mathbf{x}(t_0)$*. The function* $\mathbf{x}(t)$ *is said to be* continuous *if it is continuous at each point in its domain. Such a function is often called a* curve *in* $\mathbb{R}^n$*.*

We begin with several observations. First, a sequence $\{\mathbf{x}_j\}$ cannot have more than one limit, and so our reference to "the limit" in the definition above does make sense. To see this, suppose that

$$\lim_{n\to\infty} \mathbf{x}_n = \mathbf{y} \qquad \text{and} \qquad \lim_{n\to\infty} \mathbf{x}_n = \mathbf{z} \ .$$

Fix any $\epsilon > 0$. Then there is a natural number N so that for all $j \geq N$, we have $\|\mathbf{x}_j - \mathbf{y}\| \leq \epsilon/2$ and $\|\mathbf{x}_j - \mathbf{z}\| \leq \epsilon/2$. Hence by the triangle inequality, Theorem 3, for any such j,

$$\|\mathbf{y} - \mathbf{z}\| \leq \|\mathbf{y} - \mathbf{x}_j\| + \|\mathbf{x}_j - \mathbf{z}\| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \ .$$

Thus, for every $\epsilon > 0$, $\|\mathbf{y} - \mathbf{z}\| \leq \epsilon$. This is only possible in case $\mathbf{y} = \mathbf{z}$.

Next, notice that by definition,

$$\lim_{t\to t_0} \mathbf{x}(t) = \mathbf{x}(t_0) \qquad \Longleftrightarrow \qquad \lim_{t\to t_0} \|\mathbf{x}(t) - \mathbf{x}(t_0)\| = 0 \ .$$

Then since for each $k = 1, \ldots, n$,

$$|x_k(t) - x_k(t_0)|^2 \leq \|\mathbf{x}(t) - \mathbf{x}(t_0)\|^2 = \sum_{j=1}^n |x_j(t) - x_j(t_0)|^2 \ .$$

$\lim_{t\to t_0} \|\mathbf{x}(t) - \mathbf{x}(t_0)\| = 0$ iff and only if $\lim_{t\to t_0} |x_k(t) - x_k(t_0)| = 0$ for each $k = 1, \ldots, n$. That is,

$$\lim_{t\to t_0} \mathbf{x}(t) = \mathbf{x}(t_0) \qquad \Longleftrightarrow \qquad \text{for each } k \ , \lim_{t\to t_0} x_k(t) = x_k(t_0) \ .$$

This shows:

• *A vector valued function* $\mathbf{x}(t)$ *of a real variable* t *is continuous at* t_0 *if and only if each of its entry functions* $x_j(t)$ *is a continuous at* t_0*. To check the continuity of a vector valued function* $\mathbf{x}(t)$ *at* t_0*, it suffices to check the continuity of each of its entry functions* $x_j(t)$ *at* t_0*.*

Therefore, everything we already know about continuity for single variable functions applies directly to vector valued functions of a real variable: Since $\sin(t)$, $\cos(t)$ and $1/t$ are all continuous functions on the interval $(1, 20)$, the vector valued function in Example 23 is continuous.

2.1.2 Differentiability of functions from $\mathbb{R}$ to $\mathbb{R}^n$

Physical motions of particles are usually not only continuous, but differentiable. In fact, as we shall explain later, as a consequence of Newton's second law, as long as no infinite forces act on a particle, its motion will be described by a curve that is at least twice differentiable.

Before we define differentiability, let us explain the idea of the derivative:

• *To say that $\mathbf{x}(t)$ is differentiable at $t = t_0$ means that if we look at the graph of this function, and "zoom in" close enough on the graph near $\mathbf{x}(t_0)$, it will look indistinguishable from some line through $\mathbf{x}(t_0)$. That is, under "sufficiently high magnification" every segment of the graph of a differentiable vector valued function looks like a straight line segment.*

Consider a vector valued function $\mathbf{x}(t) = (x(t), y(t), z(t))$ with values in $\mathbb{R}^3$. Suppose that each of the coordinate functions $x(t)$, $y(t)$ and $z(t)$ is differentiable at $t = t_0$ in the familiar single-variable sense. Then we have, for $t \approx t_0$,

$$\begin{aligned} x(t) &\approx x(t_0) + (t - t_0)x'(t) \\ y(t) &\approx y(t_0) + (t - t_0)y'(t) \\ z(t) &\approx z(t_0) + (t - t_0)z'(t) \ . \end{aligned} \tag{2.3}$$

Define $\mathbf{x}_0 = \mathbf{x}(t_0)$ and $\mathbf{v} = (x'(t_0), y'(t_0), z'(t_0))$. We can express these approximations in vector notation as

$$\mathbf{x}(t) \approx \mathbf{x}_0 + (t - t_0)\mathbf{v} \ ,$$

telling us *which* straight line segment the graph of $\mathbf{x}(t)$ should look like near $\mathbf{x}_0$. This line, parameterized by $\mathbf{x}_0 + t\mathbf{v}$, is called the *tangent line* to the graph of $\mathbf{x}(t)$ at $t = t_0$.

Example 24 (A tangent line)**.** *Consider the vector valued function*

$$\mathbf{x}(t) = (x(t), y(t), z(t)) = (t \ , \ 2^{3/2}t^{3/2}/3 \ , \ t^2/2) \ .$$

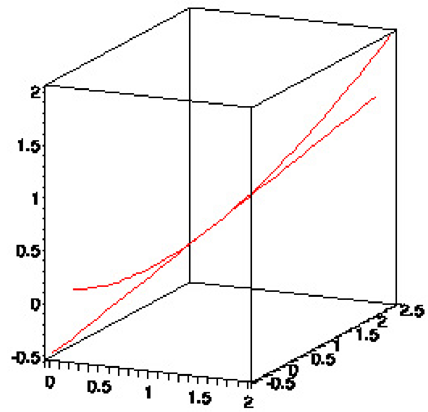
Then one computes, using single variable calculus,

$$x'(t) = 1 \ , \qquad y'(t) = 2^{1/2}t^{1/2} \quad \text{and} \quad z'(t)t \ .$$

Taking $t_0 = 1$ we have from (2.3) that for $t \approx t_0$,

$$\mathbf{x}(t) \approx \mathbf{x}_0 + (t - t_0)\mathbf{v} = (1, 2^{3/2}/3, 1/2) + (t - 1)(1, 2^{1/2}, 1) \ . \tag{2.4}$$

Here is a graph showing both the curve $\mathbf{x}(t)$ and the tangent line $\mathbf{x}_0 + t\mathbf{v}$ for $0 \leq t \leq 2$:



As you can see, the straight line is a very *close match to the curve for* $t \approx t_0 = 1$*: Both curves pass through* $\mathbf{x}(t_0)$ *at* $t = t_0$*, and they do so moving in the same direction. What you cannot see in this static picture is that they also move through this point at the same speed. That is, the linear motion and the curved motion "track each other" very well.*

Had we "zoomed in more", and shown the two graphs only for $0.9 \le t \le 1.1$*, the two graphs would have been pretty much indistinguishable. If we keep "zooming in" the two curves, and the motions along them, will look more and more "equivalent".*

To make precise mathematical sense of the intuitive notion of two "motions" along two parameterized curves being "equivalent" to some degree of accuracy at some time $t = t_0$, we make the following fundamental definition.

Definition 25 (Equivalence of order m at t_0)**.** *Let* $\mathbf{x}(t)$ *and* $\mathbf{y}(t)$ *be* $\mathbb{R}^n$ *valued functions defined on an interval* (a, b) *that includes* t_0*. Let* m *be any non-negative integer. Then the functions* $\mathbf{x}(t)$ *and* $\mathbf{y}(t)$ *are* equivalent of order m at t_0 *in case for all* $\epsilon > 0$*, there is a* $\delta_\epsilon > 0$ *so that*

$$|t - t_0| \le \delta_\epsilon \qquad \Rightarrow \qquad \|\mathbf{x}(t) - \mathbf{y}(t)\| \le \epsilon |t - t_0|^m \ . \tag{2.5}$$

In this case we write

$$\mathbf{x}(t) \sim_m \mathbf{y}(t) \text{ at } t_0 \ .$$

Notice that if $\mathbf{x}(t)$, $\mathbf{y}(t)$ and $\mathbf{z}(t)$ are three functions defined on an interval (a, b) about t_0 with values in $\mathbb{R}^n$, and $\mathbf{x}(t) \sim_m \mathbf{y}(t)$ at t_0, and $\mathbf{y}(t) \sim_m \mathbf{z}(t)$ at t_0,, then $\mathbf{x}(t) \sim_m \mathbf{z}(t)$ at t_0.

To see this note that by the triangle inequality,

$$\|\mathbf{x}(t) - \mathbf{z}(t)\| \le \|\mathbf{x}(t) - \mathbf{y}(t)\| + \|\mathbf{y}(t) - \mathbf{z}(t)\| \ .$$

Fix any $\epsilon > 0$, Since $\mathbf{x}(t) \sim_m \mathbf{y}(t)$ at t_0, there is a $\delta > 0$ so that the first term on the right is no more than $(\epsilon/2)|t - t_0|^m$ for $|t - t_0| < \delta$. Since $\mathbf{y}(t) \sim_m \mathbf{z}(t)$ at t_0, there is a $\tilde{\delta} > 0$ so that the second term on the right is no more than $(\epsilon/2)|t - t_0|^m$ for $|t - t_0| < \delta$. Then taking $\delta_\epsilon = \min\{\delta, \tilde{\delta}\}$, we have $\delta_\epsilon > 0$, and

$$|t - t_0| < \delta_\epsilon \qquad \Rightarrow \qquad \|\mathbf{x}(t) - \mathbf{z}(t)\| \le \epsilon |t - t_0|^m \ .$$

That is, our notion of equivalence is *transitive*.

Also, our definition is such that any function on (a, b) with values in $\mathbb{R}^n$ is equivalent to itself of every order m at every $t_0 \in (a, b)$. This makes it a proper notion of equivalence in the the usual mathematical sense.

Since for t close to t_0, so that $|t - t_0|$ is (much) less than 1, $|t - t_0|^m$ goes toward zero (rapidly) as n increases, the higher m is, the more stringent the condition of being equivalent of order m at t_0 is. But what is the condition telling us? Let us start with $m = 0$.

Theorem 16 (Equivalence of order zero and continuity)**.** *Let $\mathbf{x}(t)$ be a function defined on some interval (a, b) about t_0 with values in $\mathbb{R}^n$. Let $\mathbf{y}(t)$ be the constant function defined by $\mathbf{y}(t) = \mathbf{a}$ for all $t \in (a, b)$, and some $\mathbf{a} \in \mathbb{R}^n$. Then $\mathbf{x}(t) \sim_0 \mathbf{y}(t)$ at t_0 if and only if $\mathbf{x}(t)$ is continuous at t_0, and $\mathbf{x}(t_0) = \mathbf{a}$.*

Moreover, if $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are two continuous functions defined on an interval (a, b) about t_0 with values in $\mathbb{R}^n$, then $\mathbf{x}(t) \sim_0 \mathbf{y}(t)$ at t_0 if and only if

$$\mathbf{x}(t_0) = \mathbf{y}(t_0) \ .$$

Proof. For the first part, note that since $\|\mathbf{x}(t) - \mathbf{y}(t)\| = \|\mathbf{x}(t) - \mathbf{a}\|$, we have

$$|t - t_0| < \delta \quad \Rightarrow \quad \|\mathbf{x}(t) - \mathbf{y}(t)\| < \epsilon$$

is the same statement as

$$|t - t_0| < \delta \quad \Rightarrow \quad \|\mathbf{x}(t) - \mathbf{a}\| < \epsilon \ .$$

Whenever the latter is true, taking $t = t_0$ we see $\|\mathbf{x}(t_0) - \mathbf{a}\| < \epsilon$, and this is true for all $\epsilon > 0$ if and only if $\mathbf{a} = \mathbf{x}(t_0)$. Replacing $\mathbf{a}$ in the second line by $\mathbf{x}(t_0)$, we have the expression occurring in the definition of continuity at t_0. This proves the first part.

For the second part, since $\mathbf{x}(t)$ is continuous at t_0, $\mathbf{x}(t) \sim_0 \mathbf{x}(t_0)$ at t_0, and since since $\mathbf{y}(t)$ is continuous at t_0, $\mathbf{y}(t) \sim_0 \mathbf{y}(t_0)$ at t_0, where we have used the first part twice.

Then by transitivity of our equivalence relation, it follows that $\mathbf{x}(t) \sim_0 \mathbf{y}(t)$ at t_0 if and only if the two *constant* curves $\mathbf{x}(t_0)$ and $\mathbf{y}(t_0)$ are equivalent of order zero at t_0. This is the case if and only if $\|\mathbf{x}(t_0) - \mathbf{y}(t_0)\| = 0$. □

Next, let us consider equivalence to a more interesting class of "curves", namely parameterized lines.

Definition 26 (Parameterized line)**.** *A* parameterized line *is a function from $\mathbb{R}$ to $\mathbb{R}^n$ of the form*

$$\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{v}$$

for fixed vectors $\mathbf{x}_0$ and $\mathbf{v}$ in $\mathbb{R}^n$. Note that for any $t_0 \in \mathbb{R}$, we have

$$\mathbf{x}(t) = \mathbf{x}(t_0) + (t - t_0)\mathbf{v}$$

for all $t \in \mathbb{R}$.

Remark 1. *Let* $\mathbf{x}_0$ *and* $\mathbf{v}$ *be vectors in* $\mathbb{R}^n$. *The sets traced out by*

$$\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{v} \qquad \text{and} \qquad \widetilde{\mathbf{x}}(t) = \mathbf{x}_0 + t^3\mathbf{v}$$

as t *varies over* $\mathbb{R}$ *are the same line in* $\mathbb{R}^n$, *as a subset of* $\mathbb{R}^n$, *but they are different parameterizations of the same line. By a parameterized line, we mean a parameteriization of a line in which each coordinate is a linear function of the parameter.*

Theorem 17 (Equivalence of order one to a line)**.** *Let* $\mathbf{x}(t)$ *be a parameterized curve in* $\mathbb{R}^n$ *defined on some interval* (a, b) *about* t_0. *Then there is at most one parameterized line* $\mathbf{y}(t)$ *in* $\mathbb{R}^n$ *such that* $\mathbf{x}(t)$ *and* $\mathbf{y}(t)$ *are equivalent of order one at* t_0

Proof. Suppose $\mathbf{y}(t)$ and $\mathbf{z}(t)$ are two parameterized lines in $\mathbb{R}^n$, and both are equivalent to $\mathbf{x}(t)$ of order one at $t = t_0$. By the transitivity of our equivalence relation, then $\mathbf{y}(t) \sim_1 \mathbf{z}(t)$ at t_0. Since all parameterized lines are continuous, this means that $\mathbf{y}(t_0) = \mathbf{z}(t_0)$ by Theorem 16. Let us call this common point $\mathbf{x}_0$.

Thus, for some vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$, we have

$$\mathbf{y}(t) = \mathbf{x}_0 + (t - t_0)\mathbf{v} \qquad \text{and} \qquad \mathbf{z}(t) = \mathbf{x}_0 + (t - t_0)\mathbf{w} \ .$$

But then

$$\|\mathbf{y}(t) - \mathbf{z}(t)\| = |t - t_0|\|\mathbf{v} - \mathbf{w}\|$$

and so $\|\mathbf{y}(t) - \mathbf{z}(t)\| \leq \epsilon|t - t_0|$ on some interval about t_0 if and only if $\|\mathbf{v} - \mathbf{w}\| \leq \epsilon$. But this is true for arbitrary ϵ is and only if $\mathbf{v} = \mathbf{w}$. That is, the two lines are in fact the same. ☐

We now see clearly the added strength in being equivalent of order one at t_0 over being equivalent of order zero at t_0: Two parameterized lines are equivalent of order zero at t_0 if and only if they pass through the same point at t_0. But if they are equivalent of order one, they must be the exact same line.

We now come to a central definition:

Definition 27 (Derivatives of vector valued functions)**.** *Let* $\mathbf{x}(t)$ *be an* $\mathbb{R}^n$ *valued function of the variable* t. *We say that* $\mathbf{x}(t)$ *is differentiable at* $t = t_0$ *in case it is equivalent of order one at* t_0 *to a parameterized line*

Let us now "unpack" the meaning of this definition. Suppose $\mathbf{x}(t)$ is differentialbe at t_0. Let $\mathbf{y}(t)$ be a parameterized line such that $\mathbf{x}(t) \sim_1 \mathbf{y}(t)$ at t_0. By Theorem 17, this line is unique.

First, since equivalence of order one is stronger than equivalence of order zero, $\mathbf{x}(t) \sim_0 \mathbf{y}(t)$ at t_0. Since parameterized lines are continuous, Theorem 16 tells us that $\mathbf{y}(t)$ is equivalent of order zero at t_0 to the constant curve $\mathbf{y}(t_0)$. By the transitivity of equivalence, $\mathbf{x}(t)$ is equivalent of order zero at t_0 to the constant curve $\mathbf{y}(t_0)$. But then by Theorem 16 once more, $\mathbf{x}(t)$ is continuous at $t = t_0$, and $\mathbf{x}(t_0) = \mathbf{y}(t_0)$. In particular, differentiability at t_0 implies continuity at t_0.

We have just seen that the unique parameterized line $\mathbf{y}(t)$ such that $\mathbf{y}(t) \sim_1 \mathbf{x}(t)$ at t_0 satisfies $\mathbf{y}(t_0) = \mathbf{x}(t_0)$, and so this line has the form

$$\mathbf{y}(t) = \mathbf{x}(t_0) + (t - t_0)\mathbf{v}$$

for some $\mathbf{v} \in \mathbb{R}^n$. Therefore,

$$\|\mathbf{x}(t) - \mathbf{y}(t)\| = \|\mathbf{x}(t) - \mathbf{x}(t_0) - (t - t_0)\mathbf{v}\| \ ,$$

and the differentiability of $\mathbf{x}(t)$ at t_0 means that for all $\epsilon > 0$, there is a $\delta_\epsilon > 0$ so that

$$|t - t_0| < \delta_\epsilon \quad \Rightarrow \quad \|\mathbf{x}(t) - \mathbf{x}(t_0) - (t - t_0)\mathbf{v}\| < \epsilon|t - t_0| \ . \tag{2.6}$$

Now define $h = t - t_0$, and note that we can rewrite (2.6) as

$$|h| \leq \delta_\epsilon \qquad \Rightarrow \qquad \left\| \frac{1}{h}(\mathbf{x}(t + h) - \mathbf{x}(t_0)) - \mathbf{v} \right\| \leq \epsilon \ . \tag{2.7}$$

By what we have explained about convergence and limits in $\mathbb{R}^n$, this is equivalent to

$$\lim_{h \to 0} \frac{1}{h} \left(\mathbf{x}(t_0 + h) - \mathbf{x}(t_0)\right) = \mathbf{v} \ . \tag{2.8}$$

Moreover, when it comes to actually computing this vector $\mathbf{v}$, we can make ready use of what we know about single variable calculus: By what we have explained about convergence and limits in $\mathbb{R}^n$, (2.8) is equivalent to

$$\lim_{h \to 0} \frac{1}{h} \left(x_j(t_0 + h) - x_j(t_0)\right) = v_j \qquad \text{for} \qquad j = 1, \ldots, n \ .$$

Definition 28 (Derivative of a vector valued function)**.** *Let* $\mathbf{x}(t)$ *be a a vector valued function that is differentiable at* t_0*, Then its* derivative *at* t_0 *is the vector*

$$\mathbf{x}'(t_0) := \lim_{h \to 0} \frac{1}{h} \left(\mathbf{x}(t_0 + h) - \mathbf{x}(t_0)\right) \ ,$$

so that $\mathbf{y}(t) = \mathbf{x}(t_0) + (t - t_0)\mathbf{x}'(t_0)$ *is the unique parameterized line that is equivalent of order one to* $\mathbf{x}(t)$ *at* t_0*. This line is the* tangent line *to* $\mathbf{x}(t)$ *at* t_0*.*

As far as doing the computations necessary to evaluate the derivative of a vector valued function, we only need to differentiate the entries of $\mathbf{x}(t)$ one at a time.

- *To compute the derivative of* $\mathbf{x}(t)$*, you simply differentiate it entry by entry.*

Example 25 (Computing the derivative of a vector valued function of t))**.** *Let* $\mathbf{x}(t)$ *be given by (2.1). Then for any* $t \neq 0$*,*

$$\mathbf{x}'(t) = (-\sin(t), \cos(t), -1/t^2) \ .$$

Because we differentiate vectors entry by entry without mixing the entries up in any way, familiar rules for differentiating scalar valued functions hold for vector valued functions as well. In particular, the derivative of a sum is still the sum of the derivatives, etc.:

$$(\mathbf{x}(t) + \mathbf{y}(t))' = \mathbf{x}'(t) + \mathbf{y}'(t) \ . \tag{2.9}$$

We now turn to product rules. There are now several types of products to consider: product rules for scalar-vector multiplication and product rules for both the dot and cross products.

Theorem 18 (Differentiating dot and cross products)**.** *Suppose that* $\mathbf{v}(t)$ *and* $\mathbf{w}(t)$ *are differentiable vector valued functions for* t *in* (a,b) *with values in* $\mathbb{R}^n$*, and that both of these functions are differentiable at* $t_0 \in (a,b)$*. Then* $\mathbf{v}(t)\cdot\mathbf{w}(t)$ *is differentiable at* t_0 *and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{v}(t)\cdot\mathbf{w}(t)\right)\bigg|_{t=t_0} = \mathbf{v}'(t_0)\cdot\mathbf{w}(t_0) + \mathbf{v}(t_0)\cdot\mathbf{w}'(t_0)\ . \tag{2.10}$$

Also, if $n = 3$ *so that the cross product is defined,* $\mathbf{v}(t)\times\mathbf{w}(t)$ *is differentiable at* t_0 *and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{v}(t)\times\mathbf{w}(t)\right)\bigg|_{t=t_0} = \mathbf{v}'(t_0)\times\mathbf{w}(t_0) + \mathbf{v}(t_0)\times\mathbf{w}'(t_0)\ . \tag{2.11}$$

Proof: By definition we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{v}(t)\cdot\mathbf{w}(t)\right)\bigg|_{t=t_0} = \lim_{h\to 0}\frac{1}{h}\left(\mathbf{v}(t_0+h)\cdot\mathbf{w}(t_0+h) - \mathbf{v}(t_0)\cdot\mathbf{w}(t_0)\right)\ . \tag{2.12}$$

Now we use the device of "adding and subtracting" that is used to prove the single variable product rule to write

$$\begin{aligned}\mathbf{v}(t_0+h)\cdot\mathbf{w}(t_0+h) - \mathbf{v}(t_0)\cdot\mathbf{w}(t_0) &= [\mathbf{v}(t_0+h) - \mathbf{v}(t_0)]\cdot\mathbf{w}(t_0+h)\\ &= \mathbf{v}(t_0)\cdot[\mathbf{w}(t_0+h) - \mathbf{w}(t_0)]\end{aligned} \tag{2.13}$$

Note that this identity is true because we have simply added $\mathbf{v}(t_0)\cdot\mathbf{w}(t_0+h)$ in the first line on the right, and subtracted it back out in the second. The advantage is that now in each term, only one of $\mathbf{v}$ and $\mathbf{w}$ is changing.

Combining (2.12) and (2.13), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{v}(t)\cdot\mathbf{w}(t)\right)\bigg|_{t=t_0} &= \lim_{h\to 0}\left(\frac{\mathbf{v}(t_0+h)-\mathbf{v}(t_0)}{h}\cdot\mathbf{w}(t_0+h)\right)\\ &+ \lim_{h\to 0}\left(\mathbf{v}(t_0)\cdot\frac{\mathbf{w}(t_0+h)-\mathbf{w}(t_0)}{h}\right)\\ &= \mathbf{v}'(t_0)\cdot\mathbf{w}(t_0) + \mathbf{v}(t_0)\cdot\mathbf{w}'(t_0)\end{aligned}$$

The proof for cross products is exactly the same; simply replace each dot product with a cross product in the lines above. □

Finally there is the case of the product rule for scalar vector multiplication. If $g(t)$ is a real valued function defined on (a,b), and $\mathbf{x}(t)$ is an $\mathbb{R}^n$ valued function defined on (a,b), and if both are differentiable at $t_0 \in (a,b)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(g(t)\mathbf{x}(t)\right)\bigg|_{t=t_0} = g'(t_0)\mathbf{x}(t_0) + g(t_0)\mathbf{x}'(t_0)\ . \tag{2.14}$$

We leave the proof of this to the reader - treat the components one at a time.

We next present a simple consequence of Theroem 18 that we shall frequently use.

Theorem 19 (Othogonality for constant magnitude curves)**.** *Let* $\mathbf{w}(t)$ *be a differentiable curve in* $\mathbb{R}^n$ *defined on* (a,b) *such that for some* $\varrho > 0$*,* $\|bw(t)\| = \varrho$ *for all* $t \in (a,b)$*. That is, suppose the vector* $\mathbf{w}(t)$ *has constant magnitude. Then for all* $t \in (a,b)$*,*

$$\mathbf{w}(t)\cdot\mathbf{w}'(t) = 0\ .$$

Proof.

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\varrho^2 = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{w}(t)\cdot\mathbf{w}(t) = 2\mathbf{w}(t)\cdot\mathbf{w}'(t)\ .$$

□

2.1.3 Velocity and acceleration

Let $\mathbf{x}(t)$ be a function defined on (a,b) with values in $\mathbb{R}^n$. If $n = 3$, we can think of $\mathbf{x}(t)$ as representing the *position* of a point particle in physical space at time t. In this case it is natural to call $\mathbf{x}'(t)$ *velocity*, and we shall do for all values of n. The velocity gives the rate of change of the position, or more generally the *configuration* of some physical system more complicated than a point particle.

If the function $\mathbf{v}(t) = \mathbf{x}'(t)$ is differentiable, then $\mathbf{v}'(t)$ is called the *acceleration*, and is often denoted by $\mathbf{a}(t)$, so that $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{x}''(t)$. In this case we say that $\mathbf{x}(t)$ is *twice differentiable*, and *twice continuously differentiable* in case $\mathbf{a}(t) = \mathbf{x}''(t)$ is continuous. Thus, the acceleration is the second time derivative of the position (if it is twice differentiable) and gives the rate of change of the velocity vector.

For a parameterized line $\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{b}$, we have

$$\mathbf{v}(t) = \mathbf{x}'(t) = \mathbf{v}$$

and so the velocity is constant. Therefore,

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{0}$$

for all t. That is, parameterized lines have zero acceleration. For parameterized circles, matters are different.

Example 26 (Parameterized circle in $\mathbb{R}^3$)**.** *Let* $\mathbf{c}$ *and* $\mathbf{a}$ *be vectors on* $\mathbb{R}^3$ *with* $\mathbf{a} \neq \mathbf{0}$*. Let* $\varrho > 0$*. Consider the system of equations*

$$\begin{aligned} \|\mathbf{x}-\mathbf{c}\|^2 &= \varrho^2 \\ \mathbf{a}\cdot(\mathbf{x}-\mathbf{c}) &= 0\ . \end{aligned} \tag{2.15}$$

The first equation in this system is the equation for the sphere of radius ϱ *centered at* $\mathbf{c}$*. The second is the equation of the plane passing through* $\mathbf{c}$ *with normal direction along* $\mathbf{a}$*. The intersection of the plane and the sphere is a circle of radius* ϱ *in* $\mathbb{R}^3$*. In fact, if you "slice" a sphere be a plane through the center of the sphere, you get a so-called* great circle *on the sphere. Segments of great circles have a "geodesic property" that we shall study later in this chapter.*

In the mean time, let us parameterize the solutions set to (2.15). Let $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ *be an orthonormal basis of* $\mathbb{R}^3$ *such that* $\mathbf{u}_3 = \|\mathbf{a}\|^{-1}\mathbf{a}$*. We have seen how to construct such an orthonormal basis.*

Note that $\mathbf{a}\cdot(\mathbf{x}-\mathbf{c}) = 0$ *if and only if* $\mathbf{u}_3\cdot(\mathbf{x}-\mathbf{c}) = 0$*, and so* $\mathbf{x}$ *satisfies the second equation in (2.15) if and only if*

$$\mathbf{x}-\mathbf{c} = ((\mathbf{x}-\mathbf{c})\cdot\mathbf{u}_1)\mathbf{u}_1 + ((\mathbf{x}-\mathbf{c})\cdot\mathbf{u}_2)\mathbf{u}_2\ .$$

Then $\mathbf{x}$ *also satisfies the first equation in (2.15) if and only if*

$$((\mathbf{x}-\mathbf{c})\cdot\mathbf{u}_1)^2 + ((\mathbf{x}-\mathbf{c})\cdot\mathbf{u}_2)^2 = \varrho^2 \ .$$

Since all of the solutions of $a^2+b^2=\varrho^2$ *are given by* $(a,b)=\varrho(\cos\theta,\sin\theta)$ *for some* $0\le\theta<2\pi$*, we must have that*

$$(\mathbf{x}-\mathbf{c})\cdot\mathbf{u}_1 = \cos\theta \qquad \text{and} \qquad (\mathbf{x}-\mathbf{c})\cdot\mathbf{u}_2 = \sin\theta$$

for some $0\le\theta<2\pi$*,*

Thus,

$$\mathbf{x}(\theta) := \mathbf{c} + \varrho\cos\theta\mathbf{u}_1 + \varrho\sin\theta\mathbf{u}_2$$

for $0\le\theta<2\pi$ *is a parameterization of the solution set of (2.15).*

Now suppose that the angle θ *is increasing at a constant rate; i.e., that for some* $\omega>0$*, the angle* $\theta(t)$ *at time* t *is given by*

$$\theta(t) = \omega(t-t_0)$$

for some $t_0\in\mathbb{R}$*. Then writing* $\mathbf{x}(t)$ *to denote* $\mathbf{x}(\theta(t))$*, we have*

$$\mathbf{x}(t) = \mathbf{c} + \varrho[\cos(\omega(t-t_0))\mathbf{u}_1 + \sin(\omega(t-t_0))\mathbf{u}_2] \ .$$

With this parameterization $\mathbf{x}(t_0)=\mathbf{c}+\varrho\mathbf{u}_1$*,* $\mathbf{x}(t_0+\pi/2\omega)=\mathbf{c}+\varrho\mathbf{u}_2$*, and the period of the motion is* $2\pi/\omega$*.*

Now. let us compute the velocity and acceleration of $\mathbf{x}(t)$*. We compute:*

$$\mathbf{v}(t) = \mathbf{x}'(t) = \varrho\omega[-\sin(\omega(t-t_0))\mathbf{u}_1 + \cos(\omega(t-t_0))\mathbf{u}_2] \ ,$$

and

$$\mathbf{a}(t) = \mathbf{v}'(t) = -\varrho\omega^2[\cos(\omega(t-t_0))\mathbf{u}_1 + \sin(\omega(t-t_0))\mathbf{u}_2] \ .$$

Note that

$$\|\mathbf{v}(t)\| = \varrho\omega \qquad \text{and} \qquad \|\mathbf{a}(t)\| = \varrho\omega^2 \ . \tag{2.16}$$

Since $\|\mathbf{v}(t)\|$ *is constant, it follows from Theorem 19 that* $\mathbf{v}(t)$ *and* $\mathbf{a}(t)$ *are orthogonal for each* t*, as you can readily check.*

We have seen that a curve $\mathbf{x}(t)$ in $\mathbb{R}^n$ is differentiable at some t_0 provided it is well approximated by its tangent line $\mathbf{x}(t_0)+(t-t_0)\mathbf{x}'(t_0)$ at t_0 in the sense that the curve an the lines are equivalent of order one at t_0, and the tangent line is the only line with this property.

As we now explain, when a curve $\mathbf{x}(t)$ is twice differentiable on an interval (a,b); i.e., it can be differentiated twice at each t, and $\mathbf{x}''(t)$ is continuous, then at each $t_0\in(a,b)$ there is a unique *purely quadratic* curve $\mathbf{y}(t)$ that is equivalent of order two to $\mathbf{x}(t)$ at t_0.

Definition 29 (Pureley quadratic curve)**.** *A parameterized curve* $\mathbf{y}(t)$ *in* $\mathbb{R}^n$ *is* quadratic *if and inly if for fixed* $\mathbf{y}_0$*,* $\mathbf{v}$ *and* $\mathbf{a}$ *in* $\mathbb{R}^n$*,*

$$\mathbf{y}(t) = \mathbf{y}_0 + \mathbf{v} + \frac{1}{2}t^2\mathbf{a} \ ,$$

which is the case if and only if each entry of $\mathbf{y}(t)$ *is at most quadratic in* t*.*

Example 27 (Velocity and acceleration for a quadratic curve)**.** *Consider a quadratic curve* $\mathbf{y}(t) = \mathbf{y}_0 + \mathbf{v} + \frac{1}{2}t^2\mathbf{a}$*. Then for any* t_0 *we find*

$$\mathbf{y}(t_0) = \mathbf{y}_0 + t_0\mathbf{v} + \frac{1}{2}t_0^2\mathbf{a}\ , \qquad \mathbf{y}'(t_0) = \mathbf{v} + t_0\mathbf{a} \quad \text{and} \quad \mathbf{y}''(t_0) = \mathbf{a}\ .$$

You can then check directly that for all t*,*

$$\mathbf{y}(t) = \mathbf{y}(t_0) + (t-t_0)\mathbf{y}'(t_0) + \frac{1}{2}(t-t_0)^2\mathbf{y}''(t_0)\ ,$$

and so we can always write a quadratic curve in this form which puts the spotlight on how it behaves for t *near* t_0

Lemma 6 (Euivalence of order two for quadratic curves)**.** *Let* $\mathbf{z}(t)$ *and* $\mathbf{y}(t)$ *be two quadratic curves, and suppose that for some* t_0*,* $\mathbf{y}(t) \sim_2 \mathbf{z}(t)$ *at* t_0*. Then* $\mathbf{y}(t) = \mathbf{z}(t)$ *for all* t*.*

Proof. Quadratic curves are differentiable, and since equivalence of order two implies equivalence of order one, we have that $\mathbf{y}(t)$ is equivalent of order one to its tangent line at t_0, and so

$$\mathbf{y}(t) \sim_1 [\mathbf{y}(t_0) + (t-t_0)\mathbf{y}'(t_0)]\ .$$

Likewise, we have

$$\mathbf{z}(t) \sim_1 [\mathbf{z}(t_0) + (t-t_0)\mathbf{z}'(t_0)]\ .$$

Then, by the transitivity of equivalence, again using the fact that equivalence of order two implies equivalence of order one,

$$\mathbf{y}(t) \sim_1 [\mathbf{z}(t_0) + (t-t_0)\mathbf{z}'(t_0)]\ .$$

By Theorem 17, the two tangent lines are the same, and so

$$\mathbf{y}(t_0) = \mathbf{z}(t_0) \quad \text{and} \quad \mathbf{y}'(t_0) = \mathbf{z}'(t_0)\ . \tag{2.17}$$

Therefore, since $\mathbf{y}(t_0) + (t-t_0)\mathbf{y}'(t_0) - \mathbf{z}(t_0) - (t-t_0)\mathbf{z}'(t_0) = \mathbf{0}$,

$$\begin{aligned}
\mathbf{y}(t) - \mathbf{z}(t) &= \mathbf{y}(t) - \mathbf{z}(t) - [\mathbf{y}(t_0) + (t-t_0)\mathbf{y}'(t_0) - \mathbf{z}(t_0) + (t-t_0)\mathbf{z}'(t_0)] \\
&= [\mathbf{y}(t) - \mathbf{y}(t_0) - (t-t_0)\mathbf{y}'(t_0)] - [\mathbf{z}(t) - \mathbf{z}(t_0) - (t-t_0)\mathbf{z}'(t_0)] \\
&= \frac{1}{2}(t-t_0)^2[\mathbf{y}''(t_0) - \mathbf{z}''(t_0)]\ .
\end{aligned}$$

Hence, for any $\epsilon > 0$,

$$\|\mathbf{y}(t) - \mathbf{z}(t)\| \le \epsilon(t-t_0)^2 \quad \Rightarrow \|\mathbf{y}''(t_0) = \mathbf{z}''(t_0)\| \le 2\epsilon\ .$$

Because $\mathbf{y}(t) \sim_2 \mathbf{z}(t)$ at $t - 0$, the inequality on the left must be true for all t sufficiently close to t_0 no matter how small ϵ is. This means $\|\mathbf{y}''(t_0) = \mathbf{z}''(t_0)\| \le 2\epsilon$ for all $\epsilon > 0$, which means $\|\mathbf{y}''(t_0) = \mathbf{z}''(t_0)\| = 0$. That is $\mathbf{y}''(t_0) = \mathbf{z}(t_0)$. Combining this with (2.17) we see that all coefficients are equal and so $\mathbf{y}(t) = \mathbf{z}(t)$ for all t. □

Theorem 20 (Quadratic approximation)**.** *Let* $\mathbf{x}(t)$ *be a twice continuously differentiable curve in* $\mathbb{R}^n$ *defined on* (a, b)*, For each* $t_0 \in (a, b)$*, define the quadratic curve*

$$\mathbf{y}(t) = \mathbf{x}(t_0) + (t - t_0)\mathbf{x}'(t_0) + \frac{1}{2}(t - t_0)^2\mathbf{x}''(t_0) \ . \tag{2.18}$$

Then $\mathbf{x}(t) \sim_2 \mathbf{y}(t)$ *at* t_0*, and moreover* $\mathbf{y}(t)$ *is the unique quadratic curve that is equivalent of order two to* $\mathbf{x}(t)$ *at* t_0*.*

Proof. Each $x_i(t)$ is a twice continuously differentiable function form $\mathbb{R}$ to $\mathbb{R}$. Recall that by Taylors's Theorem with remainder, we have that

$$x_i(i) = x_i(t_0) + (t - t_0)x_i'(t_0) + \frac{1}{2}(t - t_0)^2 x_i''(t_0) + \int_{t_0}^{t} [x_i''(s) - x_i''(t_0)](t - s)\mathrm{d}s \ .$$

Since each $x_i''(t)$ is continuous on $(a < b)$, for any $\epsilon > 0$, there is a $\delta_{\epsilon,i} > 0$ so that

$$|s - t_0| \le \delta_{\epsilon,i} \quad \Rightarrow \quad |x_i''(s) - x_i''(t_0)| \le \epsilon \ .$$

Then, for —s $t - -t_0| \le \delta_{\epsilon,i}$,

$$\left| \int_{t_0}^{t} [x_i''(s) - x_i''(t_0)](t - s)\mathrm{d}s \right| \le \int_{\min\{t,t_0\}}^{\max\{t,t_0\}} |x_i''(s) - x_i''(t_0)||t - s|\mathrm{d}s \le \epsilon\frac{1}{2}(t - t_0)^2 \ .$$

Now define $\delta = \min\{\delta_{\epsilon,1}, \ldots, \delta_{\epsilon,n}\}$ which is strictly positive. We then have

$$\begin{aligned} |t - t_0| \le \delta \qquad \Rightarrow \qquad & \left\| \mathbf{x}(t) - \mathbf{x}(0) - (t - t_0)\mathbf{x}'(t_0) - \frac{1}{2}(t - t_0)^2\mathbf{x}''(t_0) \right\| \\ \le \quad & \left(\sum_{j=1}^{n} \left| \int_{t_0}^{t} [x_i''(s) - x_i''(t_0)]\mathrm{d}s \right|^2 \right)^{1/2} \le \sqrt{n}\epsilon(t - t_0)^2 \ . \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we can replace it with $\epsilon/\sqrt{n}$, decreasing $\delta > 0$ accordingly, and thus we see that $\mathbf{x}(t)$ is equivalent of order two at t_0 to the quadratic function given in (2.18).

Finally, if $\mathbf{z}(t)$ is any quadratic function such that $\mathbf{x}(t) \sim_2 \mathbf{z}(t)$ at t_0, then by the transitivity of equivalence, $\mathbf{y}(t) \sim_2 \mathbf{z}(t)$ at t_0, and then by the previous lemma, $\mathbf{z}(t) = \mathbf{y}(t)$ for all t. Thus, at each t_0, $\mathbf{x}(t)$ has a unique approximation by a quadratic function that is equivalent of order two at t_0. □

The quadratic approximation to a twice continuously differentiable curve $\mathbf{x}(t)$ is a much better approximation than the tangent line approximation. In particular, it contains information about *the curvature of the curve*, as we now prepare to explain. The first step is to write the velocity as the product of its magnitude and its direction vector.

Definition 30 (Speed and the unit tangent vector)**.** *The magnitude of the velocity vector is called the* speed*. We denote it by* $v(t)$*. That is,*

$$v(t) = |\mathbf{v}(t)| \ .$$

Provided that $v(t) \neq 0$*, we can define a unit vector valued function* $\mathbf{T}(t)$ *by*

$$\mathbf{T}(t) = \frac{1}{v(t)}\mathbf{v}(t) \qquad \text{so that} \qquad \mathbf{v}(t) = v(t)\mathbf{T}(t) \ .$$

The vector $\mathbf{T}(t)$ *is called the* unit tangent vector *at time* t*. It specifies the instantaneous direction of motion.*

Example 28 (Speed and the unit tangent vector)**.** *Let* $\mathbf{x}(t) = (t, 2^{3/2}t^{3/2}/3, t^2/2)$ *for* $t > 0$ *as in Example 25. There we found that* $\mathbf{v}(t) = (1, 2^{1/2}t^{1/2}, t)$*, and so the speed* $v(t)$ *is given by*

$$v(t) = \sqrt{1 + 2t + t^2} = 1 + t \ .$$

which is strictly positive for all $t > 0$*, and then we have*

$$\mathbf{T}(t) = \frac{1}{1+t}(1, 2^{1/2}t^{1/2}, t) \ .$$

Theorem 21. *Let* $\mathbf{x}(t)$ *be a twice differentiable curve, and suppose that the speed* $v(t)$ *is nonzero on some open interval* (b, c) *so that* $\mathbf{T}(t)$ *is defined for all* t *in this interval. Let* $\mathbf{a}(t) = \mathbf{a}_{\|}(t) + \mathbf{a}_{\perp}(t)$ *where we decompose* $\mathbf{a}(t)$ *using the direction* $\mathbf{T}(t)$*. Then*

$$\mathbf{a}_{\|}(t) = v'(t)\mathbf{T}(t) \qquad \text{and} \qquad \mathbf{a}_{\perp}(t) = v(t)\mathbf{T}'(t) \ . \tag{2.19}$$

Proof. Since $\mathbf{v}(t) = v(t)\mathbf{T}(t)$, we have from (2.14) that.

$$\mathbf{a}(t) = (v(t)\mathbf{T}(t))' = v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t) \ .$$

By Theorem 19, $\mathbf{T}(t)$ and $\mathbf{T}'(t)$ are orthogonal, and so $v'(t)\mathbf{T}(t)+$ $v(t)\mathbf{T}'(t)$ are orthogonal, and clearly the first of these vectors is a multiple of $\mathbf{T}(t)$. This proves (2.19). □

We refer to $\mathbf{a}_{\|}$ as the *tangential component* of the acceleration, and to $\mathbf{a}_{\perp}$ as the *normal component* of the acceleration. We see from (2.19) that the tangential component of the acceleration has to do with the rate of change of the speed, while the normal component has to do with the rate of change of the direction of the velocity vector, $\mathbf{T}(t)$.

In particular, when the speed is constant, so that $v'(t) = 0$, $\mathbf{a}_{\|}(t) = 0$, and then only the normal component of the acceleration can be non-zero. We encountered this already in Example 26.

Example 29 (Constant speed circular motion)**.** *Let* $\mathbf{x}(t)$ *be the curve in* $\mathbb{R}^3$

$$\mathbf{x}(t) = \mathbf{c} + \varrho[\cos(\omega(t - t_0))\mathbf{u}_1 + \sin(\omega(t - t_0))\mathbf{u}_2] \ . \tag{2.20}$$

that we considered in Example 26. Recall that $\varrho, \omega > 0$*. As we saw there, the speed is constant, and so there is no tangential component of the acceleration. By our computations there, we have*

$$v(t) = \varrho\omega \qquad \text{and} \qquad \|\mathbf{a}_{\perp}\| = \|\mathbf{a}\| = \varrho\omega^2 \ .$$

Therefore, for motion on a circle of radius ϱ *at constant speed* v*, we have*

$$\|\mathbf{a}\| = \|\mathbf{a}_{\perp}\| = \frac{v^2}{\varrho} \ .$$

Note that the smaller the radius of the circle, the more "tightly curved" the circle is, and the greater the magnitude of the acceleration at any given speed $v > 0$*.*

The previous example motivates the following definition.

Definition 31 (Curvature and the unit normal vector)**.** *Let* $\mathbf{x}(t)$ *be a twice differentiable curve in* $\mathbb{R}^n$*, and suppose that the speed* $v(t)$ *is nonzero on some open interval* (a, b) *so that* $\mathbf{T}(t)$ *is defined for all* t *in this interval.*

The curvature $\kappa(t)$ *at time* t *is defined by*

$$\kappa(t) = \frac{\|\mathbf{a}_\perp\|}{v^2(t)} \ , \tag{2.21}$$

and the radius of curvature $\varrho(t)$ *at time* t *is defined by*

$$\varrho(t) = \frac{1}{\kappa(t)} \ .$$

Furthermore, if $\|\mathbf{a}_\perp\| \neq 0$*, we define the* unit normal vector $\mathbf{N}(t)$ *by*

$$\mathbf{N}(t) = \frac{1}{\|\mathbf{a}_\perp\|}\mathbf{a}_\perp \ , \tag{2.22}$$

Comparing (2.19) and (2.22), we see that $\mathbf{N}(t)$ points in the same direction as $\mathbf{T}'(t)$. Thus, it points in the direction in which the curve is turning. Moreover, since whenever $\|\mathbf{a}_\perp\| \neq 0$,

$$\mathbf{a}_\perp = \|\mathbf{a}_\perp\|\frac{1}{\|\mathbf{a}_\perp\|}\mathbf{a}_\perp = \|\mathbf{a}_\perp\|\mathbf{N} \ ,$$

it follows from the definition that $\mathbf{a}_\perp = v^2\kappa\mathbf{N}$. Combining this with Theorem 21 yields:

Theorem 22. *Let* $\mathbf{x}(t)$ *be a twice differentiable curve in* $\mathbb{R}^n$*. Then*

$$\mathbf{a}(t) = v'(t)\mathbf{T}(t) + v^2(t)\kappa(t)\mathbf{N}(t) \ , \tag{2.23}$$

and

$$\mathbf{T}'(t) = v(t)\kappa(t)\mathbf{N}(t) \ . \tag{2.24}$$

Example 30 (Normal and tangential acceleration)**.** *Let* $\mathbf{x}(t) = \big(t, (2t)^{3/2}/3, t^2/2\big)$ *for* $t > 0$*. We have computed in Example 28 that*

$$v(t) = 1 + t \qquad \text{and} \qquad \mathbf{T}(t) = \frac{1}{1+t}\big(1, (2t)^{1/2}, t\big) \ .$$

Therefore, $v'(t) = 1$*, and so* $\mathbf{a}_\parallel(t) = \mathbf{T}(t)$*. Thus,*

$$\mathbf{a}_\parallel(t) = \frac{1}{1+t}\big(1, (2t)^{1/2}, t\big) \ .$$

This is the tangential component of the acceleration.

We next compute

$$\mathbf{a}(t) = \mathbf{x}''(t) = \big(0, (2t)^{-1/2}, 1\big) \ ,$$

the normal component is

$$\big(0, (2t)^{-1/2}, 1\big) - \frac{1}{1+t}\big(1, (2t)^{1/2}, t\big) = \frac{1}{1+t}\big(1, (1-t)(2t)^{-1/2}, 1\big) \ .$$

From here we compute

$$\|\mathbf{a}_\perp(t)\| = \frac{1}{\sqrt{2t}} \ .$$

Hence

$$\mathbf{N}(t) = \frac{\sqrt{2t}}{1+t}\big(-1, (1-t)(2t)^{-1/2}, 1\big)$$

and

$$\kappa(t) = \frac{\sqrt{2t}}{(1+t)^2} \qquad \text{and} \qquad \varrho(t) = \frac{(1+t)^2}{\sqrt{2t}}\ .$$

in $\mathbb{R}^3$, there is another way to compute curvature:

Theorem 23 (Computing curvature in $\mathbb{R}^3$)**.** *Let* $\mathbf{x}(t)$ *be a twice differentiable curve in* $\mathbb{R}^3$ *such that* $v(t_0) > 0$. *Let* $\mathbf{v}$ *and* $\mathbf{a}$ *denote the velocity and acceleration at time* t_0. *Let* v *and* κ *denote the speed and curvature at time* t_0. *Then*

$$\kappa = \frac{\|\mathbf{v}\times\mathbf{a}\|}{v^3}\ . \tag{2.25}$$

Proof: We compute:

$$\begin{aligned}\mathbf{v}\times\mathbf{a} &= (v\mathbf{T})\times(v'\mathbf{T} + v^2\kappa\mathbf{N})\\ &= vv'\mathbf{T}\times\mathbf{T} + v^3\kappa\mathbf{T}\times\mathbf{N}\\ &= v^3\kappa\mathbf{T}\times\mathbf{N}\end{aligned}$$

Since $\mathbf{T}$ and $\mathbf{N}$ are orthogonal unit vectors, $\mathbf{T}\times\mathbf{N}$ is a unit vector, and hence $|\mathbf{v}\times\mathbf{a}| = v^3\kappa$. □

The aspect of the formula (2.25) that is three dimensional is that it involves the cross product. Nonetheless, it can also be applied in two dimensions: We can consider any planar curve in $\mathbb{R}^2$ as a curve in $\mathbb{R}^3$ for which $x_3(t) = 0$ for all t. Thus everything we have deduced about curves in $\mathbb{R}^3$ applies to curves in $\mathbb{R}^2$ as a special case.

2.1.4 Torsion and the Frenet–Seret formulae for a curve in $\mathbb{R}^3$

Curves in $\mathbb{R}^3$ are especially important since we live in a three dimensional world. In this case, we can compute a unit normal to the plane through $\mathbf{x}(t_0)$ that contains both $\mathbf{T}(t_0)$ and $\mathbf{N}(t_0)$ by taking the cross product of $\mathbf{T}(t_0)$ and $\mathbf{N}(t_0)$.

Definition 32 (Binormal vector and osculating plane)**.** *Let* $\mathbf{x}(t)$ *be a twice differentiable curve in* $\mathbb{R}^3$. *Then at each* t_0 *for which* $v(t_0)\neq 0$ *and* $\kappa(t_0)\neq 0$, *so that* $\mathbf{T}(t_0)$ *and* $\mathbf{N}(t_0)$ *are well defined, the* binormal vector $\mathbf{B}(t_0)$ *is defined by*

$$\mathbf{B}(t_0) = \mathbf{T}(t_0)\times\mathbf{N}(t_0)\ , \tag{2.26}$$

and the osculating plane *at* t_0 *is the plane specified by the equation*

$$\mathbf{B}(t_0)\cdot(\mathbf{x}-\mathbf{x}(t_0)) = 0\ . \tag{2.27}$$

Since $\mathbf{B}(t_0)$ is orthogonal to $\mathbf{T}(t_0)$ and $\mathbf{N}(t_0)$, (2.27) is the equation of the plane through $\mathbf{x}(t_0)$ that contains both $\mathbf{T}(t_0)$ and $\mathbf{N}(t_0)$. Since $\mathbf{v} = v\mathbf{T}$ and $\mathbf{a} = v'\mathbf{T} + v^2\kappa\mathbf{N}$, $\mathbf{v}\times\mathbf{a} = v^3\kappa\mathbf{B}$, which yields the useful formulas

$$\mathbf{B}(t_0) = \frac{1}{v^3(t_0)\kappa(t_0)}\mathbf{v}(t_0)\times\mathbf{a}(t_0) = \frac{1}{\|\mathbf{v}(t_0)\times\mathbf{a}(t_0)\|}\mathbf{v}(t_0)\times\mathbf{a}(t_0)\ . \tag{2.28}$$

It follows that the direction of $\mathbf{B}$ is the same as that of $\mathbf{v} \times \mathbf{a}$. Therefore, the osculating plane at time t_0 is the plane through $\mathbf{x}(t_0)$ that contains $\mathbf{v}(t_0)$ and $\mathbf{a}(t_0)$. For this reason, the osculating plane is sometimes called the *instantaneous plane of motion*, and another equation for the osculating plane at $t = t_0$ is

$$(\mathbf{v}(t_0) \times \mathbf{a}(t_0)) \cdot (\mathbf{x} - \mathbf{x}(t_0)) = 0 \ .$$

In particular, it is not necessary to go through all the work of computing $\mathbf{T}$, $\mathbf{N}$ and then $\mathbf{B}$ if all you wanted to find was an equation for the osculating plane. You can find the equation directly from a computation of $\mathbf{v}$, $\mathbf{a}$ and $\mathbf{v} \times \mathbf{a}$.

We emphasize that we are assuming throughout these paragraphs, as in Definition 32, that $v(t_0) \neq 0$ and $\kappa(t_0) \neq 0$, so that $\mathbf{T}(t_0)$ and $\mathbf{N}(t_0)$ are well defined. Otherwise, it does not make sense to refer to "the" plane through $\mathbf{x}(t_0)$ containing $\mathbf{v}(t_0)$ and $\mathbf{a}(t_0)$.

Example 31 (An osculating plane). *Let* $\mathbf{x}(t) = (t, 2^{3/2}t^{3/2}/3, t^2/2)$ *for* $t > 0$. *We have computed that*

$$\mathbf{x}(1) = (1, 2^{3/2}/3, 1/2) \qquad \mathbf{v}(1) = (1, 2^{1/2}, 1) \qquad \text{and} \qquad \mathbf{a}(1) = (0, 2^{-1/2}, 1) \ .$$

We now compute

$$\mathbf{v}(1) \times \mathbf{a}(1) = (2^{-1/2}, -1, 2^{-1/2}) \ .$$

The equation for the osculating plane then is

$$(2^{-1/2}, -1, 2^{-1/2}) \cdot (x - 1, y - 2^{3/2}/3, z - 1/2) = 0$$

which reduces to

$$x - 2^{1/2}y + z = 6 \ .$$

Definition 33 (Planar curve in $\mathbb{R}^3$). *A curve* $\mathbf{x}(t)$ *in* $\mathbb{R}^3$, $a < t < b$, *is* planar *in case there is some plane in* $\mathbb{R}^3$ *that contains* $\mathbf{x}(t)$ *for all* $a < t < b$. *In other words,* $\mathbf{x}(t)$ *is planar in case there exists a non-zero vector* $\mathbf{n}$ *and a constant* d *such that* $\mathbf{n} \cdot \mathbf{x}(t) = d$ *for all* $a < t < b$.

Planar curves are easy to recognize when the plane is one of the coordinate planes: For example $\mathbf{x}(t) := (t, t^2, 0)$ is clearly a planar curve – a parabola in the x, y plane - for which we may take $\mathbf{n} = \mathbf{e}_3$ and $d = 0$. But planar curves are not always so easy to recognize. Consider the curve

$$\mathbf{x}(t) := (-1 + 2t - t^3, t + 3t^2 - 2t^3, 1 + 2t - 6t^2 + 2t^3) \ . \tag{2.29}$$

As we shall see, this is in fact a planar curve. How can we recognize that, and what is the plane that contains the curve?

Theorem 24 (The binormal vector and planar curves). *Let* $\mathbf{x}(t)$ *be a twice differentiable curve in* $\mathbb{R}^3$ *defined on* (a, b) *such that* $v(t)$ *and* $\kappa(t)$ *are non-zero for all* $a < t < b$. *Then* $\mathbf{x}(t)$ *is planar if and only if* $\mathbf{B}(t)$ *is a constant vector on* (a, b). *In this case, there is exactly one plane containing the curve, and for any* $t_0 \in (a, b)$, *if we define* $\mathbf{n} = \mathbf{B}(t_0)$ *and* $d = \mathbf{x}(t_0) \cdot \mathbf{B}(t_0)$, *then* $\mathbf{n} \cdot \mathbf{x} = d$ *is an equation for the unique plane containing the curve.*

Proof: Suppose that $\mathbf{B}(t)$ is constant on (a,b). Then for all $a < b < t$, and any $t_0 \in (a,b)$

$$(\mathbf{x}(t) \cdot \mathbf{B}(t_0))' = \mathbf{x}'(t) \cdot \mathbf{B}(t_0) = \mathbf{v}(t) \cdot \mathbf{B}(t) = \frac{1}{\|\mathbf{v}(t) \times \mathbf{a}(t)\|}\mathbf{v}(t) \cdot (\mathbf{v}(t) \times \mathbf{a}(t)) = 0$$

by the triple product identity. Hence for all $t \in (a,b)$

$$\mathbf{x}(t) \cdot \mathbf{B}(t_0) = \mathbf{x}(t_0) \cdot \mathbf{B}(t_0) \ .$$

This shows that with $\mathbf{n} := \mathbf{B}(t_0)$ and $d := \mathbf{B}(t_0) \cdot \mathbf{x}(t_0)$, the plane specified by $\mathbf{n} \cdot \mathbf{x} = d$ contains $\mathbf{x}(t)$ for all $t \in (a,b)$. There can be no other plane containing the curve since the intersection of two distinct planes in $\mathbb{R}^3$ is either empty or is a line. Since by hypothesis $\mathbf{x}(t)$ has non-zero curvature, it is not contained in any line. Hence the plane is unique.

On the other hand, suppose that $\mathbf{x}(t)$ is planar, and therefore satisfies $\mathbf{n} \cdot \mathbf{x}(t) = d$ for some non-zero $\mathbf{n}$ and some d. Differentiating twice we obtain

$$0 = (\mathbf{n} \cdot \mathbf{x}(t))' = \mathbf{n} \cdot \mathbf{v}(t) \qquad \text{and then} \qquad 0 = (\mathbf{n} \cdot \mathbf{v}(t))' = \mathbf{n} \cdot \mathbf{a}(t) \ .$$

Hence for all $t \in (a,b)$, $\mathbf{n}$ is orthogonal to both $\mathbf{v}(t)$ and $\mathbf{a}(t)$, Since by hypothesis the curvature is non-zero, $\mathbf{v}(t)$ and $\mathbf{a}(t)$ are not multiples of one another, and so

$$\mathbf{B}(t) = \frac{1}{\|\mathbf{v}(t) \times \mathbf{a}(t)\|}\mathbf{v}(t) \times \mathbf{a}(t) = \pm\frac{1}{\|\mathbf{n}\|}\mathbf{n} \ .$$

Since $\mathbf{B}(t)$ is continuous, the sign cannot change anywhere in (a,b), and so $\mathbf{B}(t)$ is constant whenever the curve is planar. □

Example 32 (Identifying a planar curve)**.** *Let $\mathbf{x}(t)$ be given by (2.29). Let us compute $\mathbf{B}(t)$ for this curve, and see whether it is constant or not. Before beginning the computation, it will pay to regroup the entries in $\mathbf{x}(t)$. Note that*

$$\mathbf{x}(t) = \mathbf{w}_0 + t\mathbf{w}_1 + t^2\mathbf{w}_2 + t^3\mathbf{w}_3$$

where

$$\mathbf{w}_0 := (-1,0,1) \ , \quad \mathbf{w}_1 := (2,1,2) \ , \quad \mathbf{w}_2 := (0,3,6) \ , \qquad \text{and} \quad \mathbf{w}_3 := (-1,-1,2) \ .$$

Then we have

$$\mathbf{v}(t) = \mathbf{w}_1 + 2t\mathbf{w}_2 + 3t^2\mathbf{w}_3 \qquad \text{and} \qquad \mathbf{a}(t) = 2\mathbf{w}_2 + 6t\mathbf{w}_3 \ .$$

Therefore

$$\begin{aligned} \mathbf{v}(t) \times \mathbf{a}(t) &= (\mathbf{w}_1 + 2t\mathbf{w}_2 + 3t^2\mathbf{w}_3) \times (2\mathbf{w}_2 + 6t\mathbf{w}_3) \\ &= 2(\mathbf{w}_1 + 3t^2\mathbf{w}_3) \times \mathbf{w}_2 + 6t(\mathbf{w}_1 + 2t\mathbf{w}_2) \times \mathbf{w}_3 \\ &= 2\mathbf{w}_1 \times \mathbf{w}_2 + 6t\mathbf{w}_1 \times \mathbf{w}_3 + 6t^2\mathbf{w}_2 \times \mathbf{w}_3 \ . \end{aligned}$$

We then compute

$$\mathbf{w}_1 \times \mathbf{w}_2 = 6(-2,2,1) \ , \quad \mathbf{w}_1 \times \mathbf{w}_3 = -3(-2,2,1) \ , \quad \text{and} \quad \mathbf{w}_2 \times \mathbf{w}_3 = 3(-2,2,1) \ .$$

Altogether then

$$\mathbf{v}(t) \times \mathbf{a}(t) = (12 - 18t + 18t^2)\big(-2, 2, 1\big) \qquad \text{and hence} \qquad \mathbf{B}(t) = \frac{1}{3}\big(-2, 2, 1\big) \ .$$

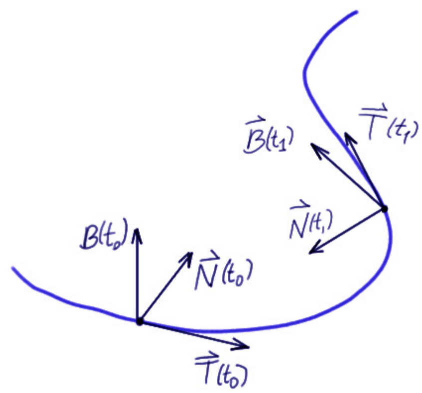
Since $\mathbf{x}(0) = \big(= 1, 0, 1\big)$*,* $\mathbf{B}(0) \cdot \mathbf{x}(0) = 1$*. Thus, the plane containing the curve satisfies the equation*

$$-2x + 2y + z = 1 \ .$$

As we have seen in our examples so far, the rate of change of the basis $\mathbf{T}(t)$ and $\mathbf{B}(t)$ tell us important information about the shape of a curve: The curvature $\kappa(t)$ is related to $\mathbf{T}'(t)$ through $\mathbf{T}'(t) = v(t)\kappa(t)\mathbf{N}(t)$, and the curve is planar if and only if $\mathbf{B}'(t) = 0$ for all t. But this only scratches the surface. There is much more to be learned by considering the rates of change of the vectors in the right handed orthonormal basis

$$\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\}$$

that is carried along by any twice differentiable curve in $\mathbb{R}^3$ with non-zero speed and curvature.



First, let us consider $\mathbf{B}'(t)$.

Lemma 7. *Let* $\mathbf{x}(t)$ *be a twice differentiable curve in* $\mathbb{R}^3$ *with non-zero speed and curvature. Then for each* t*,* $\mathbf{B}'(t)$ *is a multiple of* $\mathbf{N}(t)$*.*

Proof. $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ and so by Theorem 18

$$\mathbf{B}' = \mathbf{T}' \times \mathbf{N} + \mathbf{T} \times \mathbf{N}' = \mathbf{T} \times \mathbf{N}'$$

since $\mathbf{T}'$ is a multiple of $\mathbf{N}$. But $\mathbf{T} \times \mathbf{N}'$ is orthogonal to $\mathbf{T}$, and so $\mathbf{B}'$ is orthogonal to $\mathbf{T}$. Since $\mathbf{B}$ has constant magnitude, $\mathbf{B}'$ is orthogonal to $\mathbf{B}$ by Theorem 19. Since $\mathbf{B}'$ is orthogonal to both $\mathbf{T}$ and $\mathbf{B}$, and since $\{\mathbf{T}, \mathbf{N}, \mathbf{T}\}$ is an orthonormal basis, $\mathbf{B}'$ must be a multiple of $\mathbf{N}$. □

We now define *torsion*, $\tau(t)$, which quantifies the rate of change of the binormal vector $\mathbf{B}(t)$, and therefore quantifies the extent to which the curve is "twisting out of its osculating plane":

Definition 34 (Torsion)**.** *Let* $\mathbf{x}(t)$ *be a twice differentiable curve in* $\mathbb{R}^3$ *with non-zero speed and curvature for all* $t \in (a, b)$*. Then the torsion at* $t \in (a, b)$ *is the quantity* $\tau(t)$ *defined by*

$$\mathbf{B}'(t) = -v(t)\tau(t)\mathbf{N}(t) \ . \tag{2.30}$$

We have already seen that

$$\mathbf{T}'(t) = v(t)\kappa(t)\mathbf{N}(t)\ . \tag{2.31}$$

Notice the similarity between (2.30) and (2.31). The resaon for including the minus sign in (2.30) will become evident soon.

Theorem 25 (Computing torsion in $\mathbb{R}^3$)**.** *Let $\mathbf{x}(t)$ be a thrice differentiable curve in $\mathbb{R}^3$ such that $v(t_0) > 0$ and $\kappa(t_0) > 0$. Let $\mathbf{v}$ and $\mathbf{a}$ denote the velocity and acceleration at time t_0. Let v and κ denote the speed and curvature at time t_0. Then*

$$\tau = \frac{\mathbf{a}' \cdot \mathbf{v} \times \mathbf{a}}{v^6\kappa^2} = -\frac{\mathbf{x}''' \cdot \mathbf{x}'' \times \mathbf{x}'}{v^6\kappa^2}\ . \tag{2.32}$$

Proof: We start from the formula $\mathbf{v} \times \mathbf{a} = v^3\kappa\mathbf{B}$ that was derived in (2.26). Differentiating both sides we find

$$\begin{aligned}(\mathbf{v} \times \mathbf{a})' &= (v^3\kappa)'\mathbf{B} + (v^3\kappa)\mathbf{B}' \\ &= (v^3\kappa)'\mathbf{B} - \tau\kappa v^4\mathbf{N}\ .\end{aligned}$$

Taking the dot product of both sides with $\mathbf{a} = v'\mathbf{T} + v^2\kappa\mathbf{N}$ yields

$$\mathbf{a} \cdot (\mathbf{v} \times \mathbf{a})' = -\tau\kappa^2 v^6\ .$$

Solving for τ, we obtain the first formula in (2.32). Next, since $\mathbf{a} \cdot (\mathbf{v} \times \mathbf{a}) = \mathbf{0}$ for all t, $0 = (\mathbf{a} \cdot (\mathbf{v} \times \mathbf{a}))' = \mathbf{a}' \cdot (\mathbf{v} \times \mathbf{a}) + \mathbf{a} \cdot (\mathbf{v} \times \mathbf{a})'$ and so $\mathbf{a}' \cdot (\mathbf{v} \times \mathbf{a}) = -\mathbf{a} \cdot (\mathbf{v} \times \mathbf{a})'$. □

Finally, let us derive a formula for $\mathbf{N}'(t)$:

Lemma 8. *Let $\mathbf{x}(t)$ be a thrice differentiable curve in $\mathbb{R}^3$ with non-zero speed and curvature for all $t \in (a, b)$. Then for all $t \in (a, b)$,*

$$\mathbf{N}'(t) = -v(t)\kappa(t)\mathbf{T}(t) + v(t)\tau(t)\mathbf{B}(t)\ . \tag{2.33}$$

Proof. Since $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a right handed orthonormal basis, $\mathbf{N} = \mathbf{B} \times \mathbf{T}$. Therefore, by Theorem 18

$$\begin{aligned}\mathbf{N}' = (\mathbf{B} \times \mathbf{T})' &= \mathbf{B}' \times \mathbf{T} + \mathbf{B} \times \mathbf{T}' \\ &= -v\tau\mathbf{N} \times \mathbf{T} + \mathbf{B} \times (v\kappa\mathbf{N}) \\ &= v\tau\mathbf{B} - v\kappa\mathbf{T}\ ,\end{aligned}$$

where the last equality again uses the fact that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a right handed orthonormal basis of $\mathbb{R}^3$, and Theorem 9. □

Summarizing the results, we have proved the following:

Theorem 26 (Frenet–Seret formulae)**.** *Let $\mathbf{x}(t)$ be a thrice differentiable curve in $\mathbb{R}^3$ with non-zero speed and curvature at each t in some open interval so that $\mathbf{T}(t)$, $\mathbf{N}(t)$ and $\mathbf{B}(t)$ are all defined and differentiable on this interval. Then for all t in this interval,*

$$\begin{aligned}\mathbf{T}'(t) &= \quad v(t)\kappa(t)\mathbf{N}(t) \\ \mathbf{N}'(t) &= -v(t)\kappa(t)\mathbf{T}(t) + v(t)\tau(t)\mathbf{B}(t) \\ \mathbf{B}'(t) &= -v(t)\tau(t)\mathbf{N}(t)\ .\end{aligned}$$

There is a more useful way to express these three formulae.

Definition 35 (Darboux vector)**.** *Let* $\mathbf{x}(t)$ *be a twice differentiable curve with non-zero speed and curvature at each* t *in some open interval so that* $\mathbf{T}(t)$*,* $\mathbf{N}(t)$ *and* $\mathbf{B}(t)$ *are all defined on this interval. The* Darboux vector $\boldsymbol{\omega}$ *is defined on this interval by by*

$$\boldsymbol{\omega}(t) = \tau(t)\mathbf{T}(t) + \kappa(t)\mathbf{B}(t) \ .$$

The point of the definition is that since $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is constructed to be a right-handed orthonormal basis of $\mathbb{R}^3$, Theorem 9 says that

$$\mathbf{T} \times \mathbf{N} = \mathbf{B} \qquad \mathbf{N} \times \mathbf{B} = \mathbf{T} \qquad \text{and} \qquad \mathbf{B} \times \mathbf{T} = \mathbf{N} \ ,$$

and thus,

$$\begin{aligned}
\boldsymbol{\omega} \times \mathbf{T} &= (\tau\mathbf{T} + \kappa\mathbf{B}) \times \mathbf{T} = \quad \kappa\mathbf{N} \\
\boldsymbol{\omega} \times \mathbf{N} &= (\tau\mathbf{T} + \kappa\mathbf{B}) \times \mathbf{N} = -\kappa\mathbf{T} + \tau\mathbf{B} \\
\boldsymbol{\omega} \times \mathbf{B} &= (\tau\mathbf{T} + \kappa\mathbf{B}) \times \mathbf{B} = -\tau\mathbf{N} \ .
\end{aligned}$$

Comparing with Theorem 2.34, we see that

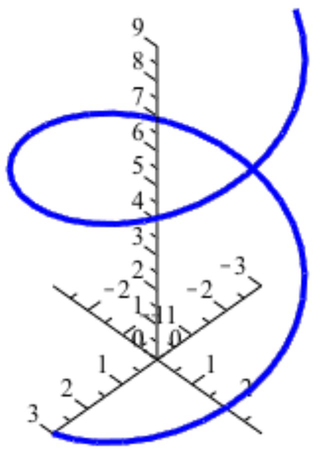
$$\begin{aligned}
\mathbf{T}'(t) &= v(t)\boldsymbol{\omega}(t) \times \mathbf{T}(t) \\
\mathbf{N}'(t) &= v(t)\boldsymbol{\omega}(t) \times \mathbf{N}(t) \\
\mathbf{B}'(t) &= v(t)\boldsymbol{\omega}(t) \times \mathbf{B}(t) \ .
\end{aligned} \tag{2.34}$$

As we shall see later in this chapter, this means that for small $h > 0$, the orthonormal basis $\{\mathbf{T}(t+h), \mathbf{B}(t+h), \mathbf{B}(t+h)\}$ is, up to errors of size h^2, what one would get by applying a rotation of angle $v(t)\|\boldsymbol{\omega}(t)\|$ about the axis of rotation in the direction of $\boldsymbol{\omega}(t)$. That is, the Darboux vector describes the instantaneous rate and direction of rotation of the orthonormal basis $\{\mathbf{T}(t), \mathbf{B}(t), \mathbf{B}(t)\}$.

Example 33 (Curvature and torsion for helices)**.** *Consider the curve* $\mathbf{x}(t)$ *given by*

$$\mathbf{x}(t) := (r\cos t, r\sin t, bt)$$

for some $r > 0$ *and* $b \neq 0$*. This curve is a helix: There is circular motion in the* x, y *variables, and linear motion in the* z *variable. A plot of the curve will look something like:*



The plot was made using the values $r = 3$ and $b = 1$ for $0 \le t \le 9$.

Let us compute the curvature, torsion, the orthonormal basis $\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\}$ and the Darboux vector $\boldsymbol{\omega}(t)$. To begin, we compute

$$\mathbf{v}(t) = \big(-r\sin t, r\cos t, b\big)$$

from which it follows that

$$v(t) = \sqrt{r^2 + b^2} \qquad \text{and} \qquad \mathbf{T}(t) = \frac{r}{\sqrt{r^2+b^2}}\big(-\sin t, \cos t, b/r\big)\ .$$

We next compute

$$\mathbf{a}(t) = \big(-r\cos t - r\sin t, 0\big)\ .$$

Then since $\mathbf{v}(t) \cdot \mathbf{a}(t) = 0$, the parallel component of the acceleration is zero, and so $\mathbf{a}_\perp(t) = \mathbf{a}(t)$. Since $\mathbf{N}(t)$ is the normalization of $\mathbf{a}_\perp(t)$, and hence in this case of $\mathbf{a}(t)$, it follows that

$$\|\mathbf{a}_\perp(t)\| = \|\mathbf{a}(t)\| = r \qquad \text{and} \qquad \mathbf{N}(t) = \big(-\cos t, -\sin t, 0\big)\ .$$

The curvature is

$$\kappa(t) := \frac{\|\mathbf{a}_\perp(t)\|}{v^2(t)} = \frac{r}{r^2+b^2}\ .$$

Let us pause to note that this is reasonable: If $b = 0$, the helix is simply a circle of radius r in the x, y plane, and so as b approaches zero, we must have that the curvature approaches $1/r$. On the other hand, if b is very large, the motion is essentially vertical, and the curvature is very small. This is in agreement with the formula we have found.

We next compute

$$\mathbf{v}(t) \times \mathbf{a}(t) = rb\big(\sin t, -\cos t, r/b\big)\ .$$

Hence

$$\mathbf{B}(t) = \frac{1}{\|\mathbf{v}(t) \times \mathbf{a}(t)\|}\mathbf{v}(t) \times \mathbf{a}(t) = \frac{b}{\sqrt{r^2+b^2}}\big(\sin t, -\cos t, r/b\big)\ .$$

Since $\mathbf{B}(t)$ *and* $\mathbf{N}(t)$ *are so simple in this case, the easiest way to compute the torsion* $\tau(t)$ *is directly from the defining relation* $\mathbf{B}'(t) = -v(t)\tau(t)\mathbf{N}(t)$. *We compute*

$$\mathbf{B}'(t) = -\frac{b}{\sqrt{r^2+b^2}}\big(\sin t, \sin t, 0\big) = -\frac{b}{\sqrt{r^2+b^2}}\mathbf{N}(t) \ ,$$

Thus

$$-\frac{b}{\sqrt{r^2+b^2}} = -v(t)\tau(t) \qquad \text{so that} \qquad \tau(t) = \frac{b}{r^2+b^2} \ .$$

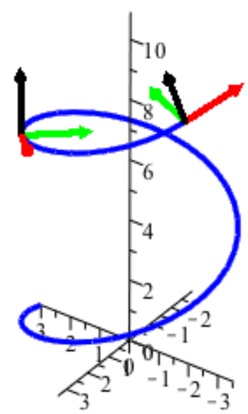
Of course, you could also have computed the curvature and torsion using the formulas from Theorems 23 and 25 respectively. However, those formulae do not always provide the simplest approach.

Notice that both the curvature and the torsion turn out to be constant. Let us now compute the Darboux vector $\boldsymbol{\omega}(t)$*:*

$$\boldsymbol{\omega}(t) = \tau(t)\mathbf{T}(t) + \kappa(t)\mathbf{B}(t) = \frac{rb}{r^2+b^2}(b/r + r/b)\big(0,0,1\big) \ .$$

Notice that this, too, is constant, despite the fact that neither $\mathbf{T}(t)$ *nor* $\mathbf{B}(t)$ *are constant.*

Here is a plot, once more for $r = 3$ *and* $b = 1$ *for* $0 \le t \le 9$, *but this time showing* $\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\}$ *for* $t = 7$ *and* $t = 9$*:*



2.1.5 Curvature and torsion are independent of parameterization.

The same path can be parameterized many ways. For example, consider

$$\mathbf{x}(t) = \big(\cos(t), \sin(t)\big) \qquad \text{and} \qquad \mathbf{y}(u) = \big(\cos(-u^3), \sin(-u^3)\big) \ .$$

As t and u vary over $\mathbb{R}$, both of these curves trace out the unit circle in $\mathbb{R}^2$, but they trace it out in different speeds, and one traces it out counterclockwise, and the other clockwise.

Definition 36 (Reparameterization)**.** *Let* $\mathbf{x}(t)$ *be a curve in* $\mathbb{R}^n$ *defined on an open interval* $(a,b) \subset \mathbb{R}$, *and let* $\mathbf{y}(u)$ *be another curve in* $\mathbb{R}^n$ *defined on an open interval* $(c,d) \subset \mathbb{R}$. *Either* a *or* c *may be* $-\infty$, *and either* b *or* d *may be* $+\infty$. *Then* $\mathbf{y}(u)$ *is a* reparameterization of $\mathbf{x}(t)$ *in case there is a a continuous, strictly monotone increasing or decreasing function* $t(u)$ *from* (c,d) *onto* (a,b) *such that*

$$\mathbf{y}(u(t)) = \mathbf{x}(t) \qquad \text{for all} \quad t \in (a,b) \ .$$

Example 34. *Define $t(u) = -u^3$ and $u(t) = -t^{1/3}$. Then with $\mathbf{x}(t) = \big(\cos(t), \sin(t)\big)$ and $\mathbf{y}(u) = \big(\cos(-u^3), \sin(-u^3)\big)$, we have both*

$$\mathbf{y}(u) = \mathbf{x}(t(u)) \quad \text{for all} \quad u \in \mathbb{R}$$

and

$$\mathbf{x}(t) = \mathbf{y}(u(t)) \quad \text{for all} \quad u \in \mathbb{R} \ .$$

Thus the $\mathbf{x}(t)$ and $\mathbf{y}(u)$ are reparameterizations of each other, and they both parameterize the unit circle.

As in the example, whenever $\mathbf{y}(u)$ is a reparameterization of $\mathbf{x}(t)$, then $\mathbf{x}(t)$ is a reparameterization of $\mathbf{y}(u)$. Indeed, if $t(u)$ is any continuous, strictly monotone increasing function $t(u)$ from (c, d) onto (a, b), then it is both one-to-one and onto, and so it has an inverse function $u(t)$ from (c, d) to (a, b) which is also continuous and strictly monotone increasing.

• *It turns out that while any curve can be parameterized in inifinitely many ways, the curvature at a point on the path is a purely geometric property of the path traced out by the curve – it is independent of the parameterization. Not only that, so is the unit normal vector, and, up to a sign, so is the unit tangent vector.*

To see this suppose that $\mathbf{x}(t)$ and $\mathbf{y}(u)$ are two parameterizations of the same path in $\mathbb{R}^n$. Suppose that

$$\mathbf{x}(t_0) = \mathbf{y}(u_0)$$

so that when $t = t_0$ and $u = u_0$, both curves pass through the same point. Let us suppose also that the two parameterizations are related in a smooth way, so that $t(u)$ is twice continuously differentiable in u.

Then, by the chain rule,

$$\mathbf{y}'(u) = \frac{\mathrm{d}}{\mathrm{d}u}\mathbf{y}(u) = \frac{\mathrm{d}}{\mathrm{d}u}\mathbf{x}(t(u)) = \left(\frac{\mathrm{d}t}{\mathrm{d}u}\right)\mathbf{x}'(t(u)) \ .$$

Evaluating at $u = u_0$, and recalling that $t_0 = t(u_0)$, we get the following relation between the speeds at which the two curve pass through the point in question:

$$\|\mathbf{y}'(u_0)\| = \left|\frac{\mathrm{d}t}{\mathrm{d}u}\right| \|\mathbf{x}'(t_0)\| \ .$$

Therefore,

$$\begin{aligned} \frac{1}{\|\mathbf{y}'(u_0)\|}\mathbf{y}'(u_0) &= \left(\left|\frac{\mathrm{d}t}{\mathrm{d}u}\right|^{-1}\frac{\mathrm{d}t}{\mathrm{d}u}\right)\frac{1}{\|\mathbf{x}(t_0)\|}\mathbf{x}'(t_0) \\ &= \pm\frac{1}{\|\mathbf{x}(t_0)\|}\mathbf{x}'(t_0) \ . \end{aligned}$$

The plus sign is correct is t is an increasing function of u, in which case the two parameterizations trace the path out in the same direction, and otherwise the minus sign is correct.

This shows that up to a sign, the unit tangent vector $\mathbf{T}$ at the point in question comes out the same for the two parameterizations.

Next, let us differentiate once more. We find

$$\begin{aligned}\mathbf{y}''(u) = \frac{\mathrm{d}}{\mathrm{d}u}\mathbf{y}'(u) &= \frac{\mathrm{d}}{\mathrm{d}u}\left(\left(\frac{\mathrm{d}t}{\mathrm{d}u}\right)\mathbf{x}'(t(u))\right)\\ &= \left(\frac{\mathrm{d}^2 t}{\mathrm{d}u^2}\right)\mathbf{x}'(t(u)) + \left(\frac{\mathrm{d}t}{\mathrm{d}u}\right)^2 \mathbf{x}''(t(u)) \ .\end{aligned}$$

Evaluating at $u = u_0$, and recalling that $t_0 = t(u_0)$, we find the following formula relating the acceleration along the two curves as they pass though the point in question:

$$\mathbf{y}''(u_0) = \left(\frac{\mathrm{d}^2 t}{\mathrm{d}u^2}\right)\mathbf{x}'(t_0) + \left(\frac{\mathrm{d}t}{\mathrm{d}u}\right)^2 \mathbf{x}''(t_0) \ .$$

Notice that the first term on the right is a multiple of $\mathbf{T}$, and hence when we decompose $\mathbf{y}''(u_0)$ into its tangential and orthogonal components, this piece contributes only to the tangential component. Hence

$$\mathbf{y}''_\perp(u_0) = \left(\frac{\mathrm{d}t}{\mathrm{d}u}\right)^2 \mathbf{x}''_\perp(t_0) \ .$$

Because of the square, $\mathbf{y}''_\perp(u_0)$ is a positive multiple of $\mathbf{x}''_\perp(t_0)$, and so these two vectors point in the exact same direction. That is,

$$\mathbf{N} = \frac{1}{\|\mathbf{y}''_\perp(u_0)\|}\mathbf{y}''_\perp(u_0) = \frac{1}{\|\mathbf{x}''_\perp(t_0)\|}\mathbf{x}''_\perp(t_0) \ ,$$

showing that the normal vector $\mathbf{N}$ is independent of the parameterization.

Next, we consider the curvature. Since

$$\begin{aligned}\frac{1}{\|\mathbf{y}'(u_0)\|^2}\|\mathbf{y}''_\perp(u_0))\| &= \left(\frac{\mathrm{d}t}{\mathrm{d}u}\right)^{-2}\frac{1}{|\mathbf{x}'(t_0)|^2}\left(\frac{\mathrm{d}t}{\mathrm{d}u}\right)^2\|\mathbf{x}''_\perp(t_0)\|\\ &= \frac{1}{\|\mathbf{x}'(t_0)\|^2}\|\mathbf{x}''_\perp(t_0)\| \ ,\end{aligned}$$

we get the exact same value for the curvature at the same point, using either parameterization. This shows that although in practice we use a particular parameterization to compute the curvature κ and the unit normal $\mathbf{N}$, the results do not depend on the choice of the parameterization, and are in fact an intrinsically geometric property of the path that the curve traces out.

So far what we have said about reparameterization is valid in $\mathbb{R}^n$ for all $n \geq 2$. In $\mathbb{R}^3$, there is more to say. In $\mathbb{R}^3$, we also have the binormal vector $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ and the torsion τ.

Since $\mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t)$, it follows that $\mathbf{B}(t)$ is well defined, independent of the parameterization, up to a sign. Then, consideration of the formula

$$\mathbf{B}'(t) = -v(t)\tau(t)\mathbf{N}(t)$$

under two parameterization shows that like the curvature, the torsion is independent of the parameterization, up to a sign. The calculations that show this are very similar to the calculuations we have just made for κ, $\mathbf{T}$ and $\mathbf{N}$, and are left to the reader. The conclusion is that the torsion also is determined by the geometry of the path itself, and not how fast or slow we move along it.

2.1.6 Speed and arc length

The speed $v(t)$ represents the rate of change of the distance traveled with time. Given some reference time t_0, define

$$s(t) = \int_{t_0}^{t} v(u)\mathrm{d}u \ . \tag{2.35}$$

Then by the Fundamental Theorem of Calculus,

$$\frac{\mathrm{d}}{\mathrm{d}t}s(t) = v(t)$$

and clearly $s(t_0) = 0$. Hence the rate of change of $s(t)$ is $v(t)$, which is the rate of change of the distance traveled with time, as one has moved along the path traced out by $\mathbf{x}(t)$.

Definition 37 (Arc length)**.** *The function $s(t)$ defined by (2.35) is called the arc length along the path traced out by $\mathbf{x}(t)$ since time t_0.*

Example 35 (Computation of arc length)**.** *Let $\mathbf{x}(t)$ be given by $\mathbf{x}(t) = (t, 2^{3/2}t^{3/2}/3, t^2/2)$ for $t > 0$ as in Example 9. Then, as we have seen, for all $t > 0$, $v(t) = 1 + t$. Therefore,*

$$s(t) = \int_0^t (1 + u)\mathrm{d}u = t + \frac{t^2}{2} \ .$$

If you took a piece of string, and cut it so it can be run along the path from the starting point to the position at time t, the length of the string would be $t + t^2/2$ units of distance.

By definition, $v(t) \geq 0$, and so $s(t)$ has a non-negative derivative. This means that it is an increasing function. As long as $v(t) > 0$; i.e., as long as the particle never comes to even an instantaneous rest, $s(t)$ is strictly monotone increasing. Let us suppose that for some $t_1 > t_0$, $v(t) > 0$ for all $t_0 < t < t_1$. Then $s(t)$ is strictly monotone increasing on the interval $[t_0, t_1]$.

Then for each $s \in [s(t_0), s(t_1)]$, there is exactly one value of $t \in [t_0, t_1]$ so that

$$s(t) = s \ . \tag{2.36}$$

This value of t, considered as a function of s, is the inverse function to the arc length function:

$$t(s) = t \ . \tag{2.37}$$

It answers a very simple question, namely: *How much time will have gone by when the distance travelled is s units of length?*

If you can compute an explicit expression for $s(t)$, such as the result $s(t) = t + t^2/2$ that we found in Example 9, what you then need to do to answer the question is to find the inverse function $t(s)$; i.e., to solve (2.36) to find t in terms of s:

Example 36 (Time as a function of arc length)**.** *Let $\mathbf{x}(t)$ be given by $\mathbf{x}(t) = (t, 2^{3/2}t^{3/2}/3, t^2/2)$ as in Example 35. Then, as we have seen, for all $t > 0$, $s(t) = t + (t^2/2)$. To find t as a function of s, write this as*

$$s = t + \frac{t^2}{2}$$

and solve for t *in terms of* s*. In this case,*

$$t + \frac{t^2}{2} = \frac{1}{2}((t+1)^2 - 1)$$

so $t = \sqrt{2s+1} - 1$*. That is,*

$$t(s) = \sqrt{2s+1} - 1 \ .$$

This function tells you how long it took to travel a given distance s *when moving along the curve.*

We can then get a new parameterization of our curve by defining $\mathbf{x}(s)$ by

$$\mathbf{x}(s) = \mathbf{x}(t(s)) \ .$$

This is called the *arc length parameterization.* We have changed our habits of notation somewhat: Now we use the sme letter $\mathbf{x}$ for both parameterizations to emphasize that they are two parameterizations of the same curve.

Example 37 (Arc length parameterization)**.** *Let* $\mathbf{x}(t) = (t, 2^{3/2}t^{3/2}/3, t^2/2)$ *as in Example 36. Then, as we have seen, for all* $t > 0$*,* $t(s) = \sqrt{2s+1} - 1$ *Therefore,*

$$\mathbf{x}(s) = \mathbf{x}(t(s)) = (\sqrt{2s+1} - 1, 2^{3/2}(\sqrt{2s+1} - 1)^{3/2}/3, (\sqrt{2s+1} - 1)^2/2) \ .$$

The arc length parameterization generally is complicated to work out explicitly. Even when you can work it out, it often looks a lot more complicated than whatever t parameterization you started with. So what is it good for?

The point about the arc length parameterization is that it is purely geometric, so that it helps us to understand the geometry of the path that a parameterized curve traces out. If we compute the rate of change of the unit tangent vector $\mathbf{T}$ as a function of s, we are computing the rate of turning per unit distance along the curve. This is an intrinsic property of the curve itself. If we compute rate of change of the unit tangent vector $\mathbf{T}$ as a function of t, we are computing something that depends on how fast we are moving on the curve, and not just on the curve itself. Indeed, if we use the arc length parameterization, $v(s) = 1$ for all s, and so the factors involving speed drop out of all of our formulas. For example,

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{x}(s) = \mathbf{T}(s)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{T}(s) = \kappa(s)\mathbf{N}(s) \ .$$

Often, this last formula is taken as the definition of the normal vector $\mathbf{N}$ and curvature κ. The advantage of this definition is that it is manifestly geometric, so that the normal vector $\mathbf{N}$ and curvature κ do not depend on the parameterization of the curve. The disadvantage is that it is generally very difficult to explicitly work out the arc length parameterization. In order to more quickly arrive at computational examples, we have chosen the form of the definition that is convenient for computation.

2.1.7 Geodesics in $\mathbb{R}^n$ and on the unit sphere

Let $\mathbf{u}$ and $\mathbf{w}$ be two unit vectors, where $\mathbf{w} \neq \pm\mathbf{u}$. The intersection of the unit sphere with the plane passing through $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{0}$ is a circle. Since the intersection is the solution set of the system of equations

$$\begin{aligned} \|\mathbf{x}\|^2 &= 1 \\ (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{x} &= 0 \end{aligned}$$

it is a circle of the sort we have parameterized in Example 26. Such a circle, produced by intersecting a plane through the origin and the unit sphere is called a *great circle* on the unit sphere.

As we shall see, when $\mathbf{w} \neq \pm\mathbf{u}$, the great circle passing through $\mathbf{u}$ and $\mathbf{w}$ consists of two circular arcs that may be prameterized using the menthod of Example 26. The one that passes from $\mathbf{u}$ to $\mathbf{w}$ without passing through $-\mathbf{u}$ will have the lesser arc length of the two. In fact, this cuve will have *less arc length than any other piecewise continuously differentiable curve on the unit sphere that runs from* $\mathbf{u}$ *to* $\mathbf{w}$. Such curves that minimize arclength are called *geodesics*.

In mathematical writing, it is usual to write S^2 to denote the unit sphere in $\mathbb{R}^3$, which is a "smooth" surface in $\mathbb{R}^3$, and as such is "two dimensional" in an obvious sort of way.

Here is the problem to be considered: Given two points in S^2; i.e., two unit vectors $\mathbf{u}$ and $\mathbf{w}$ in $\mathbb{R}^3$, we seek to find a continuous curve $\mathbf{u}(t)$, defined for $0 \le t \le T$, for some $T > 0$ that is piecewise continuously differentiable for $0 < t < T$, and such that:

(i) $\mathbf{u}(0) = \mathbf{u}$ and $\mathbf{u}(T) = \mathbf{w}$.

(ii) $\mathbf{u}(t) \in S^2$ for all $0 < t < T$.

(iii) The arc length of the curve as it runs from $\mathbf{u}$ to $\mathbf{w}$ is less than or equal to the arc length along any other curve of this same kind.

The requirement *(ii)* says that the curve $\mathbf{u}(t)$ must stay in the sphere S^2. If we dropped this requirement, it would be valid to consider the curve

$$\mathbf{u}(t) = (1-t)\mathbf{u} + t\mathbf{w}$$

for $T = 1$. This is the straight line segment joining $\mathbf{u}$ and $\mathbf{v}$, and since $\mathbf{u}'(t) = \mathbf{w} - \mathbf{u}$, the speed along this path is $v(t) = \|\mathbf{u}'(t)\| = \|\mathbf{w} - \mathbf{u}\|$. Thus, the arc length is

$$\int_0^1 v(t)\mathrm{d}t = \int_0^1 \|\mathbf{w} - \mathbf{u}\|\mathrm{d}t = \|\mathbf{w} - \mathbf{u}\| \ .$$

As you probably know, this straight line path from $\mathbf{u}$ to $\mathbf{w}$ has the least arc length among *all* piecewise continuously differentiable curves $\tilde{\mathbf{u}}(t)$ with $\tilde{\mathbf{u}}(0) = \mathbf{u}$ and $\tilde{\mathbf{u}}(T) = \mathbf{w}$, i.e., with the condition *(ii)* dropped:

Theorem 27 (Shortest paths in $\mathbb{R}^n$)**.** *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be any two distinct points in* $\mathbb{R}^n$*. Let* $\mathbf{x}(t)$ *be any curve in* $\mathbb{R}^n$ *that is continuous on* $[0, T]$ *for some* $T > 0$*, and piecewise continuously differentiable on* $(0, T)$ *with* $\mathbf{x}(0) = \mathbf{x}$ *and* $\mathbf{x}(T) = \mathbf{y}$*. Then the arc length of* $\mathbf{x}(t)$ *for* $0 \le t \le T$ *is at least* $\|\mathbf{y} - \mathbf{x}\|$*, and the arc length is exactly* $\|\mathbf{y} - \mathbf{x}\|$ *if and only if* $\mathbf{x}(t)$ *traverses the straight line segment from* $\mathbf{x}$ *to* $\mathbf{y}$ *without ever reversing the direction of travel.*

Proof. By the Fundamental Theorem of Calculus

$$\mathbf{y} - \mathbf{x} = \int_0^T \mathbf{x}'(t)\mathrm{d}t \qquad \text{and consequently} \qquad \|\mathbf{y} - \mathbf{x}\|^2 = \int_0^T (\mathbf{y} - \mathbf{x}) \cdot \mathbf{x}'(t)\mathrm{d}t \ .$$

By the Cauchy-Schwarz inequality

$$(\mathbf{y} - \mathbf{x}) \cdot \mathbf{x}'(t) \leq \|\mathbf{y} - \mathbf{x}\| \|\mathbf{x}'(t)\| \ , \tag{2.38}$$

and so

$$\|\mathbf{y} - \mathbf{x}\|^2 \leq \|\mathbf{y} - \mathbf{x}\| \left(\int_0^T \|\mathbf{x}'(t)\|\mathrm{d}t \right) \ .$$

Dividing through by $\|\mathbf{y} - \mathbf{x}\|$, we have

$$\|\mathbf{y} - \mathbf{x}\| \leq \int_0^T \|\mathbf{x}'(t)\|\mathrm{d}t \ , \tag{2.39}$$

and the quantity on the right is the arclength of the curve. There is equality in (2.39) if and only if there is equality in (2.38) for each t, and this means that the angle between $\mathbf{x}'(t)$ and $\mathbf{y} - \mathbf{x}$ is zero for each t. That is, for each t, $\mathbf{x}'(t)$ is a positive multiple of $\mathbf{y} - \mathbf{x}$, which means that $\mathbf{x}(t)$ lies on the straight line segment joining $\mathbf{x}$ and $\mathbf{y}$, and never reverses direction. □

Now we return to the sphere S^2, and let us consider only paths that stay on the sphere. This is a natural constraint: If you are looking for the shortest path from New York to Beijing, the straight line segment is not really relevant: You would have to dig an impressive tunnel to travel along it. So let us try to find a shortest path from $\mathbf{u}$ to $\mathbf{w}$ where $\mathbf{u}$ and $\mathbf{w}$ are on S^2, and where the path stays at all times on S^2.

For any fixed, distinct $\mathbf{u}, \mathbf{w} \in S^2$, we define $\mathcal{P}_{\mathbf{u},\mathbf{w}}$ to be the set of all continuous curves $\mathbf{u}(t)$ staying on S^2, that are defined on some interval $[0, T]$ for some $T > 0$, and that are piecewise continuously differentiable on $(0, T)$, and such that $\mathbf{u}(0) = \mathbf{u}$ and $\mathbf{u}(T) = \mathbf{w}$.

The arc length function, which assigns the value

$$\int_0^T \|\mathbf{u}'(t)\|\mathrm{d}t$$

to $\mathbf{u}(t) \in \mathcal{P}_{\mathbf{u},\mathbf{w}}$, is a real valued function on $\mathcal{P}_{\mathbf{u},\mathbf{w}}$. We seek that paths in $\mathcal{P}_{\mathbf{u},\mathbf{w}}$, if any, that *minimize* the arc length function on $\mathcal{P}_{\mathbf{u},\mathbf{w}}$. We shall initially suppose that $\mathbf{w} \neq -\mathbf{u}$, and come back to this special case later.

Theorem 28 (Geodesics on S^2)**.** *Let* $\mathbf{u}$ *and* $\mathbf{w}$ *be any two distinct points in* S^2 *with* $\mathbf{w} \neq -\mathbf{u}$*. Then the arc length of any path* $\mathbf{u}(t) \in \mathcal{P}_{\mathbf{u},\mathbf{w}}$ *is at least as large as*

$$\arccos(\mathbf{u} \cdot \mathbf{w}) \ ,$$

and the arc length is exactly $\arccos(\mathbf{u} \cdot \mathbf{w})$ *if and only* $\mathbf{u}(t)$ *traverses the arc of the great circle through* $\mathbf{u}$ *and* $\mathbf{w}$ *that does not pass through* $-\mathbf{u}$*, and without ever reversing the direction of travel.*

Proof. Decompose $\mathbf{w}$ into its components orthogonal and parallel to $\mathbf{u}$: $\mathbf{w} = \mathbf{w}_\perp + \mathbf{w}_\|$. Since $\mathbf{w} \neq \pm\mathbf{u}$, $\mathbf{w}_\perp \neq \mathbf{0}$, and so we may define a unit vector $\mathbf{z}$ by

$$\mathbf{z} = \frac{1}{\|\mathbf{w}_\perp\|}\mathbf{w}_\perp \ .$$

Then define an angle ϕ_1 by

$$\phi_1 = \arccos(\mathbf{w}\cdot\mathbf{u}) \ .$$

Becasue $\mathbf{w} \neq \pm\mathbf{u}$, $0 < \phi_1 < \pi$, and $\|\mathbf{w}_\|\|^2 = \cos^2\phi_1$, and $\|\mathbf{w}_\perp\|^2 = 1 - \cos^2\phi_1 = \sin^2\phi$. Since $0 < \phi_1 < \pi$, $\sin\phi_1 > 0$, and so $\mathbf{w} = \sin\phi_1\mathbf{z} + \cos\phi_1\mathbf{u}$. We now define the curve

$$\mathbf{u}(t) := \sin(t\phi_1)\mathbf{z} + \cos(t\phi_1)\mathbf{u} \ .$$

Evidently, $\mathbf{u}(0) = \mathbf{u}$, and by what we have seen just above, $\mathbf{u}(1) = \mathbf{w}$.

We compute

$$\mathbf{u}'(t) = \phi_1[-\cos(t\phi_1)\mathbf{z} + \sin(t\phi_1)\mathbf{u}] \ ,$$

and since $\mathbf{u}$ and $\mathbf{z}$ are orthonormal, $\|\mathbf{u}'(t)\| = \phi_1$. Therefore the arc length of this path is

$$\int_0^1 \|\mathbf{u}'(t)\|\mathrm{d}t = \phi_1 = \arccos(\mathbf{w}\cdot\mathbf{u}) \ .$$

Notice that every point on this path lies on the plane through $\mathbf{z}$, $\mathbf{u}$ and $\mathbf{0}$, and so it is an arc of a great circle, and is *the* arc of this great circle that does not pass through $-\mathbf{u}$ on the way to $\mathbf{w}$. Next we shall show that no other path does better.

Let us consider any path in $\mathcal{P}_{\mathbf{u},\mathbf{w}}$. Without loss of generality, we may assume that $\mathbf{u}(t) \neq \mathbf{u}$ and $\mathbf{u}(t) \neq \mathbf{w}$ for any $t \in (0, T)$, for if $\mathbf{u}(t) = \mathbf{u}$ for any $t > 0$, we may as well start over, and forget about the part of the path traveled so far, which was wasted travel. Likewise, if $\mathbf{u}(t) = \mathbf{w}$ for any $t < T$, we may as well stop the path already.

Next, define an angle $\phi(t)$ by

$$\phi(t) = \arccos(\mathbf{u}(t)\cdot\mathbf{u}) \ .$$

Since $\phi(0) = 0$ and $\phi(T) = \arccos(\mathbf{w}\cdot\mathbf{u})$, there is a least value of t for which $\phi(t) = \arccos(\mathbf{w}\cdot\mathbf{u})$, and $0 < T_* \leq T$. Since the function $\arccos(s)$ is continuously differentiable on $(0, 1)$ and since $\mathbf{u}(t)\cdot\mathbf{u} \in (0, 1)$ for $t \in (0, T_*)$, by the chain rule, $\phi(t) = \arccos(\mathbf{u}(t)\cdot\mathbf{u})$ is piecewise continuously differentiable on $(0, T_*)$, and $0 < \phi(t) < \pi$ on this interval.

Now decompose $\mathbf{u}(t)$ into its components parallel and orthogonal to $\mathbf{u}$: $\mathbf{u}(t) = \mathbf{u}_\|(t) + \mathbf{u}_\perp(t)$. We have

$$\mathbf{u}_\|(t) = (\mathbf{u}(t)\cdot\mathbf{u})\mathbf{u} = \cos\phi(t)\mathbf{u} \ .$$

Since $\|\mathbf{u}_\perp\|^2 = 1 - \|\mathbf{u}_\|\|^2 = 1 - \cos^2\phi(t) = \sin^2\phi(t)$ and since $0 < \phi(t) < \pi$ for $0 < t < T_*$, $\|\mathbf{u}_\perp(t) = \sin\phi(t) > 0$ for all $0 < t < T_*$. Thus we can define a time dependent unit vector $\mathbf{z}(t)$ by

$$\mathbf{z}(t) = \frac{1}{\|\mathbf{u}_\perp(t)\|}\mathbf{u}_\perp(t) \ .$$

It is left to the reader to check that that $\mathbf{z}(t)$ is piecewise continuously differentiable on $(0, T)$.

Then $\mathbf{u}_\perp(t) = \sin\phi(t)\mathbf{z}(t)$ and we have alread noted that $\mathbf{u}_\|(t) = \cos\phi(t)\mathbf{u}$. Therefore,

$$\mathbf{u}(t) = \sin\phi(t)\mathbf{z}(t) + \cos\phi(t)\mathbf{u} \ .$$

We compute

$$\mathbf{u}'(t) = \phi'(t)[\cos\phi(t)\mathbf{z}(t) - \sin\phi(t)\mathbf{u}] + \sin\phi(t)\mathbf{z}'(t) \ .$$

Since $\|\mathbf{z}(t)\| = 1$ for all $0 < t < T^*$, $\mathbf{z}'(t)\cdot\mathbf{z}(t) = 0$ for all such t. Likewise, since $\mathbf{z}(t)\cdot\mathbf{u} = 0$, and $\mathbf{u}$ is constant, differentiaiating yields $\mathbf{z}'(t)\cdot\mathbf{u} = 0$ for all t. Thus $\mathbf{z}'(t)$ is orthogonal to both $\mathbf{z}(t)$ and $\mathbf{u}(t)$. Therefore,

$$\begin{aligned}
\|\mathbf{u}'(t)\|^2 &= (\phi'(t))^2\|\cos\phi(t)\mathbf{z}(t) - \sin\phi(t)\mathbf{u}\|^2 + \sin^2\phi(t)\|\mathbf{z}'(t)\|^2 \\
&= (\phi'(t))^2[\cos^2\phi(t) + \sin^2\phi(t)] + \sin^2\phi(t)\|\mathbf{z}'(t)\|^2 \\
&= (\phi'(t))^2 + \sin^2\phi(t)\|\mathbf{z}'(t)\|^2 \\
&\geq (\phi'(t))^2 \ .
\end{aligned}$$

Hence, the arc length along the curve for $0 < t < T_*$ is

$$\int_0^{T_*} \|\mathbf{u}'(t)\|\mathrm{d}t \geq \phi(t_*) = \int_0^{T_*} |\phi'(t)|\mathrm{d}t \geq \int_0^{T_*} \phi'(t)\mathrm{d}t = \arccos(\mathbf{u}\cdot\mathbf{w}) \ ,$$

and there is equality if and only if $\phi'(t) \geq 0$ for all t $\mathbf{z}'(t) = 0$ for all t, meaning that $\mathbf{z}(t)$ is a constant unit vector $\mathbf{z}$ orthogonal to $\mathbf{u}$. In this case,

$$\mathbf{u}(t) = \sin\phi(t)\mathbf{z} + \cos\phi(t)\mathbf{u}$$

for $0 < t < T_*$ with $\phi(t)$ monotone increasing from 0 to ϕ_1. Notice that for each such t, $\mathbf{u}(t)$ lies in the plane through $\mathbf{z}$, $\mathbf{u}$ and $\mathbf{0}$, and so is on the great circle through which this plane slices the sphere.

Next, the arc length traversed between $0 < t < T^*$ is less than the arc length traversed between $0 < t < T$ unless $T_* = T$, so that if the arc length of our path is ϕ_1, then $T_* = T$ and

$$\mathbf{u}(T_*) = \mathbf{w} = \sin\phi(T_*)\mathbf{z} + \cos\phi(T_*)\mathbf{u} \ ,$$

Then the plane through $\mathbf{z}$, $\mathbf{u}$ and $\mathbf{0}$ is also the plane through $\mathbf{w}$, $\mathbf{u}$ and $\mathbf{0}$. Thus, for the arc length of the path to equal ϕ_1, it must traverse the arc of the great cicrcle $\mathbf{z}$, $\mathbf{u}$ and $\mathbf{0}$ that does not pass through $-\mathbf{u}$, and the angle between $\mathbf{u}(t)$ and $\mathbf{u}$ must be monotone increasing. □

There was nothing particularly three dimensional about the proof of Theorem 28. Indeed, it can be extended to arbitrary dimensions. Define S^n to be the set of all unit vectors in $\mathbb{R}^{n+1}$. The geometry of these higher dimensional spheres turns out to be important in many questions concerning physics and engineering. Indeed, the three dimensional sphere S^3 in four dimensional space $\mathbb{R}^4$ has a direct connection with rotations in the three dimensional space $\mathbb{R}^3$ that is important in many applications. This line of thought is developed in the exercises.

Finally, we come to the case $\mathbf{w} = -\mathbf{u}$. To rach $-\mathbf{u}$ starting from $\mathbf{u}$, one must first arrive at some point $\tilde{\mathbf{w}}$ that is very close, but not equal to $-\mathbf{u}$. By what we have seen above, the length of this part of the path is at least $\arccos(\tilde{\mathbf{w}}\cdot\mathbf{u})$, hence the length of the whole path to $\mathbf{u}$ is at least this large.

Taking $\tilde{\mathbf{w}}$ closer and closer to $-\mathbf{u}$, we see that the arclength is at least ϕ for any $\phi < \pi$, and hence it is at least π. There are infinitely many planes through $-\mathbf{u}$, $\mathbf{u}$ and $\mathbf{0}$, which are colinear, and so there are infinitely many great circles connection $\mathbf{u}$ to $-\mathbf{u}$. The arc length along any of them is π.

Definition 38 (The geodesic distance function on S^2)**.** *Define the function d_{S^2} on the Cartesian product $S^2 \times S^2$ by*

$$d_{S^2}(\mathbf{u}, \mathbf{w}) = \arccos(\mathbf{u} \cdot \mathbf{w})$$

for $\mathbf{u}, \mathbf{w}$ in S^2. This is the geodesic distance function on S^2

The function $d_{S^2}(\mathbf{u}, \mathbf{w})$ is a metric on S^2. That is,

(1) For all $\mathbf{u}, \mathbf{w} \in S^2$, $d_{S^2}(\mathbf{u}, \mathbf{w}) \geq 0$ and $d_{S^2}(\mathbf{u}, \mathbf{w}) = 0$ if and only if $\mathbf{u} = \mathbf{w}$.

(2) For all $\mathbf{u}, \mathbf{w} \in S^2$, $d_{S^2}(\mathbf{u}, \mathbf{w}) = d_{S^2}(\mathbf{w}, \mathbf{u})$.

(3) For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in S^2$, $d_{S^2}(\mathbf{u}, \mathbf{w}) \leq d_{S^2}(\mathbf{u}, \mathbf{v}) + d_{S^2}(\mathbf{v}, \mathbf{w})$.

Property *(1)* follows form the fact that $\mathbf{u} \cdot \mathbf{w} < 1$ for $\mathbf{u} \neq \mathbf{w}$ and with $\mathbf{u}, \mathbf{w} \in S^2$. Likewise, *(2)* is a consequence of the fact that $\mathbf{u} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{u}$.

The inequality *(3)* is the *triangle inequality* on S^2. Here is one way to see this using Theorem 28.

Build path from $\mathbf{u}$ to $\mathbf{w}$ as follows: Let $\mathbf{u}_1(t)$, $t \in [0, 1]$, be a shortest path from $\mathbf{u}$ to $\mathbf{v}$. Let $\mathbf{u}_2(t)$, $t \in [0, 1]$, be a shortest path from $\mathbf{v}$ to $\mathbf{w}$. Define a path $\mathbf{u}(t)$, $t \in [0, 2]$, from $\mathbf{u}$ to $\mathbf{w}$ by

$$\mathbf{u}(t) = \begin{cases} \mathbf{u}_1(t) & 0 \leq t \leq 1 \\ \mathbf{u}_2(t-1) & 1 \leq t \leq 2 \end{cases}$$

This path is continuous and piecewise continuously differentiable. Therefore, by Theorem 28, $d_{S^2}(\mathbf{u}, \mathbf{w}) = \arccos(\mathbf{u}, \mathbf{w})$ is less than or equal to the length of this composite path. But by construction, the length of the composite paths is the sum of the two lengths, namely $d_{S^2}(\mathbf{u}, \mathbf{v}) + d_{S^2}(\mathbf{v}, \mathbf{w})$

The proof we have given of the triangle inequality for the geodesic distance funciton on S^2 uses the rather sophisticated Theorem 28. But the triangle inequality simply says that for any three unit vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$,

$$\arccos(\mathbf{u} \cdot \mathbf{w}) \leq \arccos(\mathbf{u} \cdot \mathbf{v}) + \arccos(\mathbf{v} \cdot \mathbf{w}) \ . \tag{2.40}$$

In fact, it is possible to prove this directly, without considering paths.

Given three such unit vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, define

$$\theta = \arccos(\mathbf{u} \cdot \mathbf{v}) \qquad \text{and} \qquad \phi = \arccos(\mathbf{v} \cdot \mathbf{w}) \ .$$

Let us write $\mathbf{u} = \mathbf{u}_\| + \mathbf{u}_\perp$ and $\mathbf{w} = \mathbf{w}_\| + \mathbf{w}_\perp$ where parallel and perpendicular components are taken with respect to $\mathbf{v}$. Then

$$\mathbf{u} \cdot \mathbf{w} = \mathbf{u}_\| \cdot \mathbf{w}_\| + \mathbf{u}_\perp \cdot \mathbf{w}_\perp \ .$$

and

$$\mathbf{u}_\| \cdot \mathbf{w}_\| = \cos\theta\cos\phi \qquad \text{and} \qquad \mathbf{u}_\perp \cdot \mathbf{w}_\perp \geq -\sin\theta\sin\phi \ ,$$

where the Cauchy-Schwarz inequality implies the last part. Conclude that

$$\mathbf{u} \cdot \mathbf{w} \geq \cos(\theta + \phi) \ .$$

Now taking the arccos, we obtain an elmentary proof of the inequality (2.40).

Again, in this proof, we did not use any cross products or anything specific to $\mathbb{R}^3$. Thus, this proof shows that (2.40) is valid for any here unit vectors in $\mathbb{R}^n$, for any n, and thus we can define a metric; i.e., a distance function, on the n dimensional sphere in $\mathbb{R}^{n+1}$, which is the set of all unit vectors in $\mathbb{R}^{n+1}$, by

$$d_{S^n}(\mathbf{u}, \mathbf{w}) = \arccos(\mathbf{u} \cdot \mathbf{w}) \ .$$

This line of thought is developed in the exercises.

2.2 The prediction of motion

2.2.1 Newton's Second Law

Newton's Second Law states that if $\mathbf{x}(t)$ denotes the position at time t of a particle of mass m, and at each time t a force $\mathbf{F}(t)$ is acting on the particle, the acceleration of the particle, $\mathbf{a}(t)$, satisfies

$$\mathbf{a}(t) = \frac{1}{m}\mathbf{F}(t) \ . \tag{2.41}$$

For example, a particle of mass m in $\mathbb{R}^3$ that is subject to a constant gravitational field (pointing downwards by convention) is acted upon by the force

$$\mathbf{F} = -mg(0, 0, 1) \ ,$$

where g is the gravitational constant, and is about 9.8 meters/second^2 on the Earth.

Since $\mathbf{a}(t)$ is the derivative of the velocity $\mathbf{v}(t) = (v_1(t), v_2(t), v_3(t))$, this means that

$$\mathbf{v}'(t) = -g(0, 0, 1) \ . \tag{2.42}$$

This is a *differential equation* for $\mathbf{v}(t)$, it tells us what the derivative of $\mathbf{v}(t)$ is. (The term "derivative equation might be more descriptive, but it is not what is used.) The kind of motion described by(2.42) is called *ballistic motion*.

The differential equation is very simple; the right hand side is constant. If we are also given the initial value $\mathbf{v}(0) = \mathbf{v}_0$, we can solve for $\mathbf{v}(t)$ using the Fundamental Theorem of Calculus, entry by entry.

Given a vector a continuous vector valued function $\mathbf{w}(t)$ in $\mathbb{R}^n$, define $\int_a^b \mathbf{w}(t)\mathrm{d}t$ to be the vector in $\mathbb{R}^n$ whose jth component is

$$\int_0^b w_j(t)\mathrm{d}t \ .$$

That is, we simply integrate $\mathbf{w}(t)$ entry by entry. Then since we also differentiate entry by entry, the fundamental Theorem of Calculs, applied entry by entry, gives us that

- *Whenever $\mathbf{z}(t)$ is a continuously differentiable function in $\mathbb{R}^n$ for $a \le t \le b$,*

$$\mathbf{z}(b) = \mathbf{z}(a) + \int_a^b \mathbf{z}'(t)\mathrm{d}t \ . \tag{2.43}$$

Applying this to ballistic motion, if the initial velocity is $\mathbf{v}_0$, then

$$\mathbf{v}(t) = \mathbf{v}_0 + \int_0^1 -g(0,0,1)\mathrm{d}r = \mathbf{v}_0 - tg(0,0,1) \ .$$

Since this gives an explicit expression for $\mathbf{x}'(t) = \mathbf{v}(t)$, we can apply (2.43) once more to obtain

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_0^t [\mathbf{v}_0 - rg(0,0,1)]\mathrm{d}r = \mathbf{x}_0 + t\mathbf{v}_0 - \frac{gt^2}{2}(0,0,1) \ ,$$

where $\mathbf{x}_0$ is the initial position.

That is, we have solved Newton's Equation to deduce that the particle's trajectory is given by

$$\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{v}_0 - \frac{1}{2}gt^2\mathbf{e}_3 \ . \tag{2.44}$$

The term *ballistic motion* is used to describe the motion along such a trajectory where some "launching process" imparts the initial velocity, and after that the only force acting is gravity. Once you know the initial position and velocity, you know the whole trajectory, and can determine (or "predict") any aspect of the motion. For example, suppose that the initial position is at the origin; i.e., $\mathbf{x}(0) = \mathbf{0}$, and that $\mathbf{v}(0) = (a,0,b)$ with $a, b > 0$. This corresponds to "launching" the particle with this initial velocity, upwards and outwards along the $\mathbf{e}_1$ axis. What is the maximum height of the particle along the trajectory? What is the distance from the origin when it "hits the ground"?

Example 38 (Making predictions for ballistic motion)**.** *Let us answer the questions raised just above concerning ballistic motion. To do this, plug our initial position* $\mathbf{0}$ *and initial velocity* $(a,0,b)$ *into (2.44). We find that the height of the particle at time* t*,* $x_3(t)$*, is given by*

$$x_3(t) = tb - \frac{1}{2}gt^2 \ .$$

Completing the square,

$$x_3(t) = -\frac{g}{2}\left(\frac{b}{g} - t\right)^2 + \frac{1}{2}\frac{b^2}{g} \ . \tag{2.45}$$

The maximum height is achieved when the first term is zero, i.e., at $t = b/g$*, at which time the height is* $b^2/(2g)$*.*

The particle "hits the ground" when $x_3(t) = 0$ *for the second time. From (2.45), we see that* $x_3(t) = 0$ *if and only if*

$$\left(\frac{b}{g} - t\right)^2 = \frac{b^2}{g^2} \ .$$

The two solutions are $t = t_0 = 0$ *and* $t = t_1 = 2b/g$*. Since* $t_0 = 0$ *is the launch time,* $t_1 = 2b/g$ *is the time when the particle hits the ground. Evaluating* $\mathbf{x}(t_1)$ *at this time* t*, we find*

$$\mathbf{x}(t_1) = \left(\frac{2ab}{g}, 0, 0\right) \ .$$

Thus, the distance from the origin to where the particle hit the ground is $2ab/g$*.*

Let us go a little further with this, suppose that however you are launching the particle, the maximum launch speed you can achieve is 10 *meters/second. Then, to have the particle hit the* *ground as far away along the $\mathbf{e}_1$ axis as possible, you should use an initial velocity vector $(a, 0, b)$ with $a^2 + b^2 = 100$, but which one?*

Since $a^2 + b^2 - 2ab = (a - b)^2$, we have $2ab \leq a^2 + b^2$, with equality if and only if $a = b$. Thus to maximize $2ab$ under the constraint $a^2 + b^2 = 100$, you should use $a = b = \sqrt{50}$ meters/second. This corresponds to the particle at an initial angle of $\pi/4$ with the horizontal.

2.2.2 Ballistic motion with friction

The equation (2.42) only provides an accurate prediction for the motion of a projection only when the effects of friction are negligible. Frictional forces, for moderate velocities, can taken into account with a useful degree of accuracy by postulating a frictional force of the form

$$\mathbf{F}_{\text{fric}} := -\alpha \mathbf{v} \ , \tag{2.46}$$

where $\alpha > 0$ is a proportionality constant, and $\mathbf{v}$ is the velocity. The minus sign says that the frictional force opposes the motion of direction, whatever it is, and that the magnitude of the frictional force is proportional to the magnitude of the velocity; i.e., the speed. Adding the frictive force to the gravitational force, and applying Newton's Second Law, we obtain the differential equation

$$\mathbf{v}'(t) = -g(0, 0, 1) - \frac{\alpha}{m}\mathbf{v}(t) \ . \tag{2.47}$$

This is more complicated than (2.42) since in that equation, the right hand side was an explicit constant vector, and here, the right hand side depends on the unknown vector $\mathbf{v}(t)$. However, there is a standard device for reducing (2.47) to the "constant right hand side case". It is a very simple device, but very important: We shall use it again and again. Here it is: Regroup the terms in (2.47) and multiply through by $e^{t(\alpha/m)}$ to obtain

$$e^{t(\alpha/m)} \left[\mathbf{v}'(t) + \frac{\alpha}{m}\mathbf{v}(t)\right] = -e^{t(\alpha/m)} g(0, 0, 1) \ . \tag{2.48}$$

Defining the new dependent variable

$$\mathbf{z}(t) := e^{t(\alpha/m)}\mathbf{v}(t) \ ,$$

we see that (2.49) is equivalent to

$$\mathbf{z}'(t) = -e^{t(\alpha/m)} g(0, 0, 1) \ . \tag{2.49}$$

Now, the right hand side is not constant, but it is an explicit function of t that is easily integrated, so that, once again, we may apply the Fundamental Theorem of Calculus:

$$\begin{aligned} \mathbf{z}(t) - \mathbf{z}(0) &= -\int_0^t e^{t(\alpha/m)} g(0, 0, 1) \mathrm{d}r \\ &= -\frac{mg}{\alpha}(0, 0, e^{t(\alpha/m)} - 1) \ . \end{aligned}$$

Since $\mathbf{v}(0) = \mathbf{z}(0)$, and more generally, $\mathbf{v}(t) = e^{-t(\alpha/m)}\mathbf{z}(t)$, we have

$$\mathbf{v}(t) = e^{-t(\alpha/m)}\mathbf{v}(0) - \frac{mg}{\alpha}(0, 0, 1 - e^{-t(\alpha/m)}) \ .$$

Notice that

$$\lim_{t\to\infty} \mathbf{v}(t) = -\frac{mg}{\alpha}(0,0,1) \ .$$

This is the *terminal velocity* for ballistic motion with this frictive coefficient α.

Next, since $\mathbf{x}(t) = \mathbf{x}(0) + \int_0^t \mathbf{v}(r)\mathrm{d}r$, we have

$$\mathbf{x}(t) = \mathbf{x}(0) - \frac{m}{\alpha}\left[e^{-t\alpha/m} - 1\right]\mathbf{v}(0) - \frac{mg}{\alpha}\left(0,0,t + \frac{m}{\alpha}\left[e^{-t(\alpha/m)} - 1\right]\right) \ .$$

Now let consider the case in which $\mathbf{x}(0) = \mathbf{0}$, and that $\mathbf{v}(0) = (a,0,b)$ with $a, b > 0$. Then with $\mathbf{x}(t) = (x(t), y(t), z(t))$, we have

$$\begin{aligned} x(t) &= -\frac{m}{\alpha}\left[e^{-t\alpha/m} - 1\right]a \\ z(t) &= -\frac{m}{\alpha}\left[e^{-t\alpha/m} - 1\right]b - \frac{mg}{\alpha}\left(t + \frac{m}{\alpha}\left[e^{-t(\alpha/m)} - 1\right]\right) \ , \end{aligned}$$

and of course $y(t) = 0$ for all t. Using the Taylor approximation $e^s \approx 1 + s + s^2/2$ which is valid for small values of s, and the formulas derived above, we see that

$$\lim_{\alpha\to 0} x(t) = at \qquad \text{and} \qquad \lim_{\alpha\to 0} z(t) = bt - gt^2/2 \ ,$$

as we had found in the frictionless case.

Let us suppose once again that we are launching a particle of mass m on ballistic motion with friction coefficient α, and the maximum launch speed we can obtain is again 10 meters per second. If we want to particle to travel as far as possible before hitting the ground, and what angle should we launch it?

The particle hits the ground when $z(t) = 0$. However, this is now a transcendental equation, and we cannot simply solve for t to pug into $x(t)$. Instead, we write $(a,b) = 10(\cos\theta, \sin\theta)$, and define

$$\begin{aligned} f(t,\theta) &:= -10\frac{m}{\alpha}\left[e^{-t\alpha/m} - 1\right]\cos\theta \\ g(t,\theta) &:= -10\frac{m}{\alpha}\left[e^{-t\alpha/m} - 1\right]\sin\theta - \frac{mg}{\alpha}\left(t + \frac{m}{\alpha}\left[e^{-t(\alpha/m)} - 1\right]\right) \ . \end{aligned}$$

Since by the formulas derived above, $f(t,\theta)$ and $g(t,\theta)$ are the horizontal distance traveled and the height, respectively, at time t when the launch angle is θ, the problem is to maximize $f(t,\theta)$ subject to the *constraint* that $g(t,\theta) = 0$. That is, we seek to maximize $f(t,\theta)$ not over the set of all values of (t,θ), but only over those for which $g(t,\theta) = 0$. This is a classic problem in the theory of constrained optimization, and we will solve it later when we discuss the method of *Lagrange multipliers*. Our present emphasis, though, is on the derivation of the trajectory from Newton's Second Law.

2.2.3 Motion in a constant magnetic field and the Rotation Equation

Ballistic motion is particularly simple, but is fundamentally important. Another fundamentally important example concerns the motion of a charged particle in a constant magnetic field $\mathbf{H}$. If the particle has charge q and mass m, the Lorentz force law says that force $\mathbf{F}(t)$ due to the magnetic field that acts on the particle at time t is $\mathbf{F}(t) = q\mathbf{v}(t) \times \mathbf{H}$, and then Newton's Second Law yields

$$\mathbf{v}'(t) = \frac{q}{m}\mathbf{v}(t) \times \mathbf{H} \ . \tag{2.50}$$

Notice that the magnetic force is zero if the particle is not moving, or is moving parallel to $\mathbf{H}$. We shall solve (2.50) to find the particle's trajectory.

First, let us simplify our notation. Define a vector $\mathbf{b} \in \mathbb{R}^3$ by $\mathbf{b} = -\frac{q}{m}\mathbf{H}$ so that (2.50) becomes

$$\mathbf{v}'(t) = \mathbf{b} \times \mathbf{v}(t) \ , \tag{2.51}$$

and we assume $\mathbf{b} \neq \mathbf{0}$ to avoid trivialities.

In fact, this equation comes up all the time: We have already met it in different notation: Each of the three Frenet-Seret equations (2.34) written in terms of the Darboux vector is of this form.

We shall now find all solutions of (2.51) that satisfy the initial condition

$$\mathbf{v}(0) = \mathbf{v}_0 \tag{2.52}$$

where $\mathbf{v}_0$ is a given initial value of the velocity vector. Together, (2.51) and (2.52) specify an *initial value problem*. Once we have the solution in hand, we will see exactly why this equation comes up all the time.

The equation (2.51) is not so simple as the equation for ballistic motion with friction since due to the cross product, for each j, $v_j'(t)$ depends on each of $v_k(t)$ for $k \neq j$: The equation "mixes up" the entries of $\mathbf{v}(t)$. Thus, we cannot so easily "separate variables" as we did in the last subsection. However, it is not a complicated matter to solve (2.51) and obtain an explicit formula for $\mathbf{v}(t)$.

First of all, we show that if $\mathbf{v}(t)$ solves (2.51) then $\|\mathbf{v}(t)\| = \|\mathbf{v}(0)\|$ for all t; i.e., that that $v(t)$ is a *constant of the motion.* To see this, we differentiate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}(t)\|^2 = 2\mathbf{v}(t) \cdot \mathbf{v}'(t) = 2\mathbf{v}(t) \cdot (\mathbf{b} \times \mathbf{v}(t)) = 0$$

by the triple product identity.

This has an important consequence, namely that there is *at most* one solution to our initial value problem. Indeed if $\mathbf{v}(t)$ and $\mathbf{w}(t)$ are any two solutions of the initial value problem, define $\mathbf{z}(t) = \mathbf{v}(t) - \mathbf{w}(t)$, and note that $\mathbf{z}'(t) = \mathbf{b} \times \mathbf{z}(t)$. Indeed,

$$\mathbf{z}'(t) = \mathbf{v}'(t) - \mathbf{w}'(t) = \mathbf{b} \times (\mathbf{v}(t) - \mathbf{w}(t) = \mathbf{b} \times \mathbf{z}(t) \ .$$

By what we have seen just above, this means that $\|\mathbf{z}(t)\|$ is constant. But since $\mathbf{v}(t)$ and $\mathbf{w}(t)$ have the same initial value, namely $\mathbf{v}_0$, $\mathbf{z}(0) = \mathbf{0}$. Therefore, $\|\mathbf{z}(t)\| = \|\mathbf{z}(0)\| = 0$ for all t, and consequently, $\mathbf{v}(t) = \mathbf{w}(t)$ for all t: *The two solutions are in fact the same.* Hence, if we can find one solution of our initial value problem, it is the only solution there is, and the problem is completely solved.

There is a second constant of the motion, namely $\mathbf{b} \cdot \mathbf{v}(t)$. If $\mathbf{v}(t)$ solves $\mathbf{v}'(t) = \mathbf{b} \times \mathbf{v}(t)$, then

$$(\mathbf{b} \cdot \mathbf{v}(t))' = \mathbf{b} \cdot (\mathbf{b} \times \mathbf{v}(t)) = (\mathbf{b} \times \mathbf{b} \cdot \mathbf{v}(t) = 0 \ .$$

Then, since the component of $\mathbf{v}(t)$ parallel to $\mathbf{b}$, $\mathbf{v}_\parallel(t)$ is given by

$$\mathbf{v}_\parallel(t) = \frac{1}{\|\mathbf{b}\|^2}(\mathbf{b} \cdot \mathbf{v}(t))\mathbf{b} \ ,$$

it follows that $\mathbf{v}_\parallel(t) = \mathbf{v}_\parallel(t_0)$ for all t. That is, $\mathbf{v}_\parallel$ is a constant of the motion. This naturally means that $\|\mathbf{v}(t)_\parallel\|$ is constant, and since

$$v^2(t) = \|\mathbf{v}(t)\|^2 = \|\mathbf{v}_\parallel v(t)\|^2 + \|\mathbf{v}_\perp(t)\|^2 \ ,$$

$\|\mathbf{v}_\perp(t)\|$ is constant as well.

•*For any solution* $\mathbf{v}(t)$ *of (2.51), the vector* $\mathbf{v}_\|(t)$ *and the quantity* $\|\mathbf{v}_\perp(t)\|$ *are constant. As* t *varies, only the direction of* $\mathbf{v}_\perp(t)$ *changes.*

To take advantage of our constants of the motion, we first introduce an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ that is adapted to the problem at hand. We build this out of two orthogonal vectors that come along with the problem, namely $\mathbf{b}$ itself, and $(\mathbf{v}_0)_\perp$, the component of $\mathbf{v}_0$ that is orthogonal to $\mathbf{b}$. Notice that if $(\mathbf{v}_0)_\perp = 0$, then $\mathbf{b} \times \mathbf{v}_0 = 0$, and the initial value problem has the simple solution $\mathbf{v}(t) = \mathbf{v}_0$ for all t. By what we have said above, this is the only solution in this case.

Therefore, without loss of generality, we assume $(\mathbf{v}_0)_\perp \neq 0$, and define

$$\mathbf{u}_3 = \frac{1}{\|\mathbf{b}\|}\mathbf{b} \quad \text{and} \qquad \mathbf{u}_1 = \frac{1}{\|(\mathbf{v}_0)_\perp\|}(\mathbf{v}_0)_\perp \ .$$

Once $\mathbf{u}_3$ and $\mathbf{u}_1$ are defined, we have only one choice for $\mathbf{u}_2$ that makes $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ a right-handed orthonormal basis, namely $\mathbf{u}_2 = \mathbf{u}_3 \times \mathbf{u}_1$.

Now let $\big((y_1(t), y_2(t), y_3(t)\big)$ be the coordinate vector for $\mathbf{v}(t)$ with respect to the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. Then,

$$\mathbf{v}(t) = y_1(t)\mathbf{u}_1 + y_2(t)\mathbf{u}_2 + y_3(t)\mathbf{u}_3 \ .$$

The merit of this coordinate system based on $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is that

$$\mathbf{v}_\perp(t) = y_1(t)\mathbf{u}_1 + y_2(t)\mathbf{u}_2 \qquad \text{and} \qquad \mathbf{v}_\|(t) = y_3(t)\mathbf{u}_3 \ .$$

But since $\mathbf{v}_\|(t) = \mathbf{v}_\|(0) = (\mathbf{v}_0)_\|$, we have

$$\mathbf{v}(t) = y_1(t)\mathbf{u}_1 + y_2(t)\mathbf{u}_2 + (\mathbf{v}_0)_\| \ , \tag{2.53}$$

and only $y_1(t)$ and $y_2(t)$ remain to be determined. For this we return to the equation (2.51): Differentiating both sides of (2.53) and taking the cross product of both sides with $\mathbf{b}$, we deduce from (2.51) that

$$\begin{aligned} y_1'(t)\mathbf{u}_1 + y_2'(t)\mathbf{u}_2 &= y_1(t)\mathbf{b} \times \mathbf{u}_1 + y_2(t)\mathbf{b} \times \mathbf{u}_2 \\ &= \|\mathbf{b}\|(y_1(t)\mathbf{u}_2 - y_2(t)\mathbf{u}_1) \end{aligned}$$

where we have used $\mathbf{b} = \|\mathbf{b}\|\mathbf{u}_3$ several times. Equating the coefficients of $\mathbf{u}_1$ on both sides, and doing the same for $\mathbf{u}_2$, we conclude

$$y_1'(t) = -\|\mathbf{b}\|y_2(t) \qquad \text{and} \qquad y_2'(t) = \|\mathbf{b}\|y_1(t) \ . \tag{2.54}$$

Also from (2.53), we have that $y_1(0)\mathbf{u}_1 + y_2(0)\mathbf{u}_2 = (\mathbf{v}_0)_\perp = \|(\mathbf{v}_0)_\perp\|\mathbf{u}_1$, we have

$$y_1(0) = \|(\mathbf{v}_0)_\perp\| \qquad \text{and} \qquad y_2(0) = 0 \ . \tag{2.55}$$

This gives us what we need to determine $y_1(t)$ and $y_2(t)$: We know that

$$y_1^2(t) + y_2^2(t) = \|\mathbf{v}_\perp(t)\|^2 = \|(\mathbf{v}_0)_\perp\|^2$$

so that

• *For each t, $\big(y_1(t), y_2(t)\big)$ is a point on the centered circle of radius $\|(\mathbf{v}_0)_\perp\|^2$.*

Thus, for some function $\theta(t)$,

$$\big(y_1(t), y_2(t)\big) = \|(\mathbf{v}_0)_\perp\|\big(\cos\theta(t), \sin\theta(t)\big)\ . \tag{2.56}$$

Differentiating both sides, we find

$$y_1'(t) = -\theta'(t)y_2(t) \qquad \text{and} \qquad y_2'(t) = \theta'(t)y_1(t)\ .$$

Comparing with (2.54) we see that $\theta'(t) = \|\mathbf{b}\|$ for all t, and so $\theta(t) = \|\mathbf{b}\|t + \theta_0$ vfor some $\theta_0 \in [0, 2\pi)$. Using this in (2.56), and evaluating at $t = 0$, we find

$$\big(y_1(0), y_2(0)\big) = \|(\mathbf{v}_0)_\perp\|\big(\cos(\theta_0), \sin(\theta_0)\big)\ .$$

Comparing with (2.55) w see that $\theta_0 = 0$, so $\theta(t) = \|\mathbf{b}\|t$.

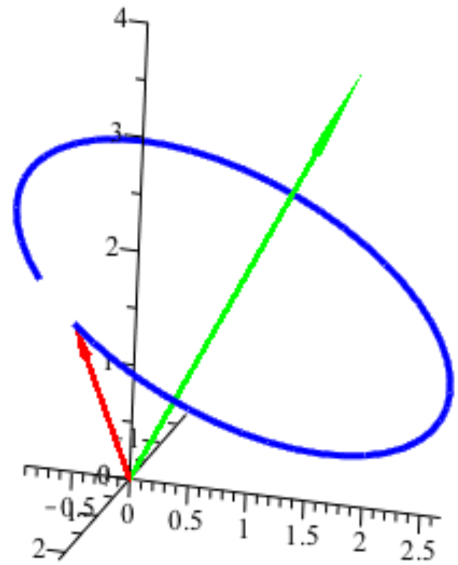
Now that we know $y_1(t)$ and $y_2(t)$, we know $\mathbf{v}(t)$. Altogether, the solution of (2.51) satisfying (2.52) is

$$\mathbf{v}(t) = \|(\mathbf{v}_0)_\perp\|(\cos(\|\mathbf{b}\|t)\mathbf{u}_1 + \sin(\|\mathbf{b}\|t)\mathbf{u}_2) + (\mathbf{v}_0)_\|\ . \tag{2.57}$$

Now let us look at our explicit solution (2.57), and see "what it is": The component $\mathbf{v}_\|(t)$ of $\mathbf{v}(t)$ parallel to $\mathbf{b}$ stays constant, while the component $(\mathbf{v}_0)_\perp(t)$ of $\mathbf{v}(t)$ orthogonal to $\mathbf{b}$ simply rotates in the plane through $\mathbf{0}$ that is orthogonal to $\mathbf{b}$, and does so at the constant *angular velocity* $\|\mathbf{b}\|$, by which we mean $\theta'(t) = \|\mathbf{b}\|$ for all t. We conclude from our formula (2.57):

• *The equation $\mathbf{v}'(t) = \mathbf{b}\times\mathbf{v}(t)$ describes rotation about an axis in the direction of $\mathbf{b}$ at constant angular velocity $\|\mathbf{b}\|$. If one observes the rotation in the plane orthogonal to $\mathbf{b}$ from a point along the positive axis in the direction of $\mathbf{b}$, the rotation is counter-clockwise.*

Here is a plot showing an axis of rotation $\mathbf{b}$, and initial vector $\mathbf{v}_0$, and the curve swept out by $\mathbf{v}(t)$ up the time it almost comes back to the starting point. As you see, when viewed from out along the axis of rotation, the direction of rotation is counter-clockwise.



Therefore, we call $\mathbf{v}'(t) = \mathbf{b} \times \mathbf{v}(t)$ the *Rotation Equation.* We have proved:

Theorem 29 (The Rotation Equation)**.** *For any given vectors* $\mathbf{b}$ *and* $\mathbf{v}_0$ *in* $\mathbb{R}^3$*, there is one and only one solution to the system of equations*

$$\mathbf{v}'(t) = \mathbf{b} \times \mathbf{v}(t) \qquad \text{and} \qquad \mathbf{v}(0) = \mathbf{v}_0 \ .$$

If $\mathbf{b} = \mathbf{0}$*, or if* $\mathbf{v}_0$ *is a multiple of* $\mathbf{b}$*, this solution is* $\mathbf{v}(t) = \mathbf{v}_0$ *for all* t*. Otherwise it is given by (2.57).*

This gives us another geometric way to think about what the cross product represents: Take a non-zero vector $\mathbf{v} \in \mathbb{R}^3$ and another non-zero vector $\mathbf{b}$ in $\mathbb{R}^3$, and produce a curve $\mathbf{v}(t)$ by rotating $\mathbf{v}$ through that angle $\|\mathbf{b}\|t$ about the axis $\mathbf{b}$ so that the rotation in the plane orthogonal to $\mathbf{b}$ is counter-clockwise when viewed from along the positive axis in the direction of $\mathbf{b}$. Then,

$$\mathbf{b} \times \mathbf{v} = \mathbf{v}'(0) \ .$$

That is,

$$\mathbf{v}(t) = \mathbf{v} + t\mathbf{b} \times \mathbf{v} + \mathcal{O}(t^2) \ .$$

One can loosely express this as saying that the cross product $\mathbf{b} \times \mathbf{v}$ specifies the increment in $\mathbf{v}$ under an infinitesimal rotation about the positive $\mathbf{b}$ axis.

Example 39 (Solving the Rotation Equation)**.** *Let* $\mathbf{b} := (1,4,8)$ *and* $\mathbf{v}_0 := (2,0,2)$*. Let us solve the Rotation Equation* $\mathbf{v}'(t) = \mathbf{b} \times \mathbf{v}(t)$ *with* $\mathbf{v}(0) = \mathbf{v}_0$*.*

First, we compute that $\|\mathbf{b}\| = 9$*. Hence*

$$\|\mathbf{b}\| = 9 \qquad \text{and} \qquad \mathbf{u}_3 = \frac{1}{9}(1,4,8) \ .$$

Next, we compute $\mathbf{v}_0 \cdot \mathbf{u}_3 = 2$*. Hence*

$$(\mathbf{v}_0)_\perp = (2,0,2) - \frac{2}{9}(1,4,8) = \frac{2}{9}(8,-4,1) \ .$$

Therefore,

$$\|(\mathbf{v}_0)_\perp\| = 2 \qquad \text{and} \qquad \mathbf{u}_1 = \frac{1}{9}(8, -4, 1) \ .$$

Finally,

$$\mathbf{u}_2 = \mathbf{u}_3 \times \mathbf{u}_1 = \frac{1}{9}(4, 7, -4) \ .$$

We now have all the information we need to evaluate the formula (2.57). We get

$$\mathbf{v}(t) = 2\cos(9t)\mathbf{u}_1 + 2\sin(9t)\mathbf{u}_2 + 2\mathbf{u}_3 \ .$$

More explicitly, using our evaluation of $\mathbf{u}_1$*,* $\mathbf{u}_2$ *and* $\mathbf{u}_3$*, this is*

$$\mathbf{v}(t) = \frac{1}{9}\big(16\cos(9t) + 8\sin(9t) + 2 \ , -8\cos(9t) + 14\sin(9t) + 8 \ , \ 2\cos(9t) - 8\sin(9t) + 16\big) \qquad (2.58)$$

The plot we have given just above Theorem 29 is of this solution for $0 \le t \le 2$*, except that the length of* $\mathbf{b}$ *is reduced by a factor of* 2 *in the plot; the plot is easier to understand when* $\|\mathbf{b}\|$ *and* $\|\mathbf{v}_0\|$ *are not too far apart.*

As we have explained in connection with motion in a magnetic field, it is natural to think of a solution $\mathbf{v}(t)$ of the Rotation Equation as a particle velocity. Since Theorem 29 gives us an explicit formula for $\mathbf{v}(t)$ in terms of $\mathbf{v}_0$ and $\mathbf{b}$, if we are also given the initial position $\mathbf{x}_0$, we can integrate $\mathbf{v}(t)$ to get

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_0^t \mathbf{v}(r)\mathrm{d}r \ .$$

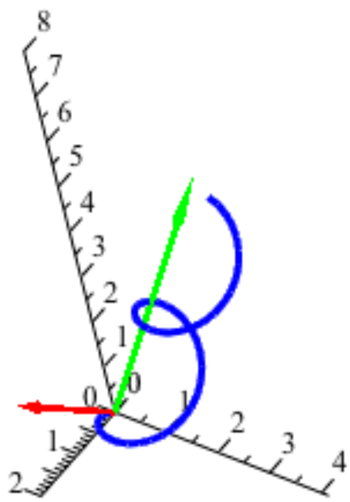
When $\mathbf{v}(t)$ is given by the formula (2.57), the time dependence is very simple, and it is easy to do the integrals explicitly.

The result is:

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{x}_0 - \frac{\|(\mathbf{v}_0)_\perp\|}{\|\mathbf{b}\|}\sin(\|\mathbf{b}\|t)\mathbf{u}_1 + \frac{\|(\mathbf{v}_0)_\perp\|}{\|\mathbf{b}\|}(1 - \cos(\|\mathbf{b}\|t))\mathbf{u}_2 + t(\mathbf{v}_0)_\| \\ &= \left[\mathbf{x}_0 + \frac{\|(\mathbf{v}_0)_\perp\|}{\|\mathbf{b}\|}\mathbf{u}_2\right] + \left[\frac{\|(\mathbf{v}_0)_\perp\|}{\|\mathbf{b}\|}(\sin(\|\mathbf{b}\|t)\mathbf{u}_1 - \cos(\|\mathbf{b}\|t)\mathbf{u}_2) + t(\mathbf{v}_0)_\|\right] \ , \qquad (2.59) \end{aligned}$$

which we recognize as the parameterization of a Helix. This shows that the trajectory of a charged particle in a constant magnetic field is a helix.

Here is a plot of the helix produced by using the solution of the rotation equation found in Example 39 in (2.59), using $\mathbf{x}_0 = \mathbf{0}$, and also showing the initial velocity vector $\mathbf{v}_0$ and $\mathbf{b}$:



There is another context in which this is natural: If $\mathbf{x}(s)$ is a curve in $\mathbb{R}^3$ parameterized by arclength, then $\mathbf{x}'(s) = \mathbf{T}(s)$, and by the Frenet-Seret formulae (2.34), we then have $\mathbf{T}'(s) = \boldsymbol{\omega}(s) \times \mathbf{T}(s)$.

Thus, whenever the Darboux vector $\boldsymbol{\omega}$ is constant along a curve $\mathbf{x}(s)$, the unit tangent vector $\mathbf{T}(s)$ satisfies

$$\mathbf{T}'(s) = \boldsymbol{\omega} \times \mathbf{T}(s)$$

where $\boldsymbol{\omega}$ is the constant value of the Darboux vector. Thus, Theorem 29 gives us an explicit formula for $\mathbf{T}(s)$ in terms of $\mathbf{T}(0)$ and $\boldsymbol{\omega}$, making appropriate adjustments to the notation only. Then since $\mathbf{x}'(s) = \mathbf{T}(s)$, we can use the fundamental Theorem of Calculus to find an explicit formula for $\mathbf{x}(s)$ itself: *As we have just seen,* $\mathbf{x}(s)$ *will be a helix.*

This raises the question: *When is the Darboux vector constant?*

Lemma 9 (Constant curvature and torsion)**.** *Let* $\mathbf{x}(t)$*be a thrice differentiable curve with non-zero speed and curvature for* $a < t < b$*. Then the Darboux vector* $\boldsymbol{\omega}(t)$ *is constant on the interval (a,b) if and only if the curvature* $\kappa(t)$ *and the torsion* $\tau(t)$ *are constant on* (a, b)*.*

Proof: Suppose first that the curvature κ and torsion τ are constant. Then $\boldsymbol{\omega}(t) = \tau\mathbf{T}(t) + \kappa\mathbf{B}(t)$. Differentiating, and using the Frenet-Seret formulae (2.34),

$$\begin{aligned}
\boldsymbol{\omega}'(t) &= \tau\mathbf{T}'(t) + \kappa\mathbf{B}'(t) \\
&= v(t)\tau\boldsymbol{\omega}(t) \times \mathbf{T}(t) + v(t)\kappa\boldsymbol{\omega}(t) \times \mathbf{B}(t) \\
&= v(t)\boldsymbol{\omega}(t) \times (\tau\mathbf{T}(t) + \kappa\mathbf{B}(t)) \\
&= v(t)\boldsymbol{\omega}(t) \times \boldsymbol{\omega}(t) = \mathbf{0}\ .
\end{aligned}$$

Thus, when the curvature and torsion are constant, so is the Darboux vector.

For the converse, suppose that the Darboux vector is constant. Then $\tau(t) = \boldsymbol{\omega} \cdot \mathbf{T}(t)$, and so

$$\tau'(t) = \boldsymbol{\omega} \cdot \mathbf{T}'(t) = v(t)\kappa(t)\boldsymbol{\omega} \cdot \mathbf{N}(t) = 0$$

since the Darboux vector is always orthogonal to $\mathbf{N}$. A similar calculation shows that $\kappa'(t) = 0$. □

Thus, whenever, the curvature and torsion are constant, the Darboux vector is constant, and thus the curve $\mathbf{x}(s)$ is a helix. Conversely, we have already seen that for any helix, the curvature and torsion are constant. Hence we have proved:

Theorem 30 (Curvature, torsion and helices)**.** *A thrice differentiable curve with non-zero speed and curvature is a helix if and only if it has constant curvature and torsion.*

2.2.4 Planetary motion

Consider a planet of mass m orbiting a star of mass M. According to Newton's Universal Theory of Gravitation, the force that attracts the planet to the star has magnitude

$$\frac{GMm}{r^2}$$

where G is the *gravitational constant* and r is the distance between the centers of the star and the planet.

The center of mass will stay fixed, and since the star is generally much, much more massive than the planet, it is an excellent approximation to regard the star as fixed. We take its position to be the origin of our coordinate system, and denote the position of the planet at time t by $\mathbf{x}(t)$. Since the gravitational force acting on the planet is directed towards the star, and hence towards $\mathbf{0}$, Newton's second law tells us that the motion of the planet satisfies

$$\mathbf{x}''(t) = -\frac{GM}{\|\mathbf{x}(t)\|^3}\mathbf{x}(t) \ . \tag{2.60}$$

We are going to determine the orbit of the planet, which is to say, the path traced out by $\mathbf{x}(t)$. The key to this, as in our study of motion in a constant magnetic field, is to find *constants of the motion*. The crucial initial observation for us there was that in a constant magnetic field $\mathbf{H}$, $v(t)$ and $\mathbf{H}\cdot\mathbf{v}(t)$ are both independent of time. This gave us the plane in which $\mathbf{v}(t)$ was then found to rotate.

The key to easily solving (2.60) is to find constants of the motion. There are two *vector-valued* constants of the motion for (2.60):

Definition 39 (Momentum, angular momentum and the Runge-Lenz vector)**.** *Let* $\mathbf{x}(t)$ *be a twice differentiable curve in* $\mathbb{R}^3$ *satisfying (2.60). Then the momentum* $\mathbf{p}(t)$*, the* angular momentum $\mathbf{L}(t)$ *and the* Runge-Lenz vector $\mathbf{A}(t)$ *of the planet are given by*

$$\mathbf{p}(t) = m\mathbf{x}'(t) = m\mathbf{v}(t), \tag{2.61}$$

$$\mathbf{L}(t) = \mathbf{x}(t) \times \mathbf{p}(t) \ . \tag{2.62}$$

and

$$\mathbf{A}(t) = \mathbf{p}(t) \times \mathbf{L}(t) - GMm^2\frac{\mathbf{x}(t)}{\|\mathbf{x}(t)\|} \ . \tag{2.63}$$

The momentum is not a constant of the motion for this system: computing the derivative, we find

$$\mathbf{p}'(t) = m\mathbf{x}''(t) = -\frac{GMm}{\|\mathbf{x}(t)\|^3}\mathbf{x}(t) \ . \tag{2.64}$$

But notice that $\mathbf{p}'(t)$ is a multiple of $\mathbf{x}(t)$. Because of this and the fact that $\mathbf{p}(t)$ is a multiple of $\mathbf{x}'(t)$,

$$\mathbf{L}'(t) = \mathbf{x}'(t) \times \mathbf{p}(t) + \mathbf{x}(t) \times \mathbf{p}'(t) = \mathbf{0} \ .$$

This shows that $\mathbf{L}$ is a constant of the motion, and in fact, since it is vector valued, it provides us three scalar constants of the motion.

Next, let us compute $\mathbf{A}'(t)$ in two steps. First, since $\mathbf{L}'(t) = \mathbf{0}$,

$$\begin{aligned} (\mathbf{p}(t) \times \mathbf{L}(t))' &= \mathbf{p}'(t) \times \mathbf{L}(t) \\ &= -\frac{GMm}{\|\mathbf{x}(t)\|^3}\mathbf{x}(t) \times (\mathbf{x}(t) \times \mathbf{p}(t)) \ . \end{aligned} \tag{2.65}$$

Using Laplace's identity $\mathbf{u} \times (\mathbf{v} \times \mathbf{z}) = (\mathbf{u} \cdot \mathbf{z})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{z}$ from Theorem 10, we find

$$\mathbf{x} \times (\mathbf{x} \times \mathbf{p}) = (\mathbf{x} \cdot \mathbf{p})\mathbf{x} - \|\mathbf{x}\|^2\mathbf{p} \ .$$

Therefore,

$$(\mathbf{p}(t) \times \mathbf{L}(t))' = -\frac{GMm}{\|\mathbf{x}(t)\|^3}(\mathbf{x}(t) \cdot \mathbf{p}(t))\mathbf{x}(t) + \frac{GMm}{\|\mathbf{x}(t)\|}\mathbf{p}(t) \ . \tag{2.66}$$

Second, we compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\|\mathbf{x}(t)\|}\mathbf{x}(t)\right) = -\frac{1}{\|\mathbf{x}(t)\|^2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{x}(t)\|\right)\mathbf{x}(t) + \frac{1}{\|\mathbf{x}(t)\|}\mathbf{x}'(t)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{x}(t)\| = \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\mathbf{x}(t) \cdot \mathbf{x}(t)} = \frac{1}{\sqrt{\mathbf{x}(t) \cdot \mathbf{x}(t)}}\mathbf{x}(t) \cdot \mathbf{x}'(t) = \frac{1}{\|\mathbf{x}(t)\|}\mathbf{x}(t) \cdot \mathbf{x}'(t) \ .$$

Thus, altogether,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{GMm^2}{\|\mathbf{x}(t)\|}\mathbf{x}(t)\right) = -\frac{GMm}{\|\mathbf{x}(t)\|^3}(\mathbf{x}(t) \cdot \mathbf{p}(t))\mathbf{x}(t) + \frac{GMm}{\|\mathbf{x}(t)\|}\mathbf{p}(t) \ . \tag{2.67}$$

Combining (2.66), (2.67) and the definition of the Runge-Lenze vector $\mathbf{A}$, we see that $\mathbf{A}'(t) = \mathbf{0}$. Summarizing our conclusions, we have proved:

Theorem 31 (Constants of the motion for planetary orbits)**.** *Let* $\mathbf{x}(t)$ *be a solution of (2.60). Then the angular momentum vector* $\mathbf{L}(t)$ *and the Runge-Lenz vector* $\mathbf{A}(t)$ *are both constants of the motion:*

$$\mathbf{L}(t) = \mathbf{L}(0) = \mathbf{L} \qquad \text{and} \qquad \mathbf{A}(t) = \mathbf{A}(0) = \mathbf{A}$$

for all t.

We are now ready to solve for the orbits. Consider any orbit with given $\mathbf{A}$ and $\mathbf{L}$. Notice that $\mathbf{L} = \mathbf{0}$ if and only if the motion of the planet is straight towards or away from the star, and of course this does not describe an orbit. Therefore, to avoid trivialities, let us suppose that $\mathbf{L} \neq \mathbf{0}$. Since by definition, the angular momentum is orthogonal to $\mathbf{x}(t)$, the orbit lies in the plane given by the equation

$$\mathbf{L} \cdot \mathbf{x} = 0 \ .$$

This plane is called the *orbital plane.* (It is, in fact, the osculating plane to the orbit at each time t.)

Next, suppose that $\mathbf{A} = 0$. Then $\mathbf{x}(t) \cdot \mathbf{A} = 0$ for all t, and by then the definition of $\mathbf{A}(t)$,

$$\mathbf{x}(t) \cdot \mathbf{A}(t) = \mathbf{x}(t) \cdot \mathbf{p}(t) \times \mathbf{L}(t) - GMm^2 \|\mathbf{x}(t)\| \ , \tag{2.68}$$

and so $\mathbf{x}(t) \cdot \mathbf{p}(t) \times \mathbf{L}(t) = GMm^2\|\mathbf{x}(t)\|$. Using the triple product identity

$$\mathbf{x}(t) \cdot \mathbf{p}(t) \times \mathbf{L}(t) = \mathbf{L}(t) \cdot (\mathbf{x}(t) \times \mathbf{p}(t)) = \mathbf{L}(t) \cdot \mathbf{L}(t) = \|\mathbf{L}\|^2 \ . \tag{2.69}$$

Thus, (2.68) reduces to

$$\|\mathbf{x}(t)\| = \frac{\|\mathbf{L}\|^2}{GMm^2} \ .$$

This means that $\mathbf{x}(t)$ traces out a circle around the star in the orbital plane, and with the radius

$$R = \frac{\|\mathbf{L}\|^2}{GMm^2} \ . \tag{2.70}$$

Since the orbit is circular, the velocity $\mathbf{x}'(t)$ is tangent to the circle and therefore orthogonal to $\mathbf{x}(t)$. It follows that, $\|\mathbf{L}\| = m\|\mathbf{x}(t)\|\|\mathbf{x}'(t)\| = mR\|\mathbf{x}'(t)\|$ and hence the speed $v = \|\mathbf{x}'(t)\|$ is constant and given by

$$v = \frac{\|\mathbf{L}\|}{mR} \ . \tag{2.71}$$

Thus, when $\mathbf{A} = 0$, the motion is constant speed circular motion in the orbital plane. We can get a relation between v and R by eliminating $\|\mathbf{L}\|$ between (2.70) and (2.71)

$$GMm^2R = \|\mathbf{L}\|^2 = v^2m^2R^2 \ ,$$

and so

$$v = \sqrt{\frac{GM}{R}} \ . \tag{2.72}$$

This is usually expressed in terms of the *period* of the orbit, since that is what one can most easily determine directly by astronomical observation. The period of the orbit, traditionally denoted T, is the time it takes the planet to complete one circular orbit. This is given by

$$T = \frac{2\pi R}{v} \ . \tag{2.73}$$

Then from (2.73) and (2.72) we obtain

$$T^2 = \left(\frac{4\pi^2}{GM}\right) R^3 \ , \tag{2.74}$$

which is the usual form of Kepler's Third Law in the case of circular orbits, with one very important contribution by Newton: Kepler's law stated that the square of the period was proportional to the cube of the radius, but did not provide a formula for the proportionality constant. Newton's derivation does, and moreover, the constant G, giving the strength of the gravitational force, can be measured on the Earth. Then, making astronomical observations from the Earth, one can determine the radii and the periods of the orbits of the other planets circling the Sun. Using this data in (2.74), we can calculate M, the mass of the Sun.

• *In this way, one can "weigh" the Sun with a telescope, since the orbits of the planets are nearly circular.*

The estimates of the value of the Sun computed in this way would vary depending on which planet is used in part because the orbits are not exactly circular. To get a better treatment, let us now consider the case $\mathbf{A} \neq \mathbf{0}$.

From (2.68) and (2.69), we have

$$\mathbf{x}(t) \cdot \mathbf{A} = \|\mathbf{L}\|^2 - GMm^2 \|\mathbf{x}(t)\| \ . \tag{2.75}$$

Identifying the orbital plane with $\mathbb{R}^2$ in such a way that $\mathbf{A}$ points in the $\mathbf{e}_1$ direction, and and writing $\mathbf{x}(t) = (x(t), y(t))$, we can rewrite (2.75) as

$$GMm^2 \sqrt{x^2(t) + y^2(t)} = \|\mathbf{L}\|^2 - \|\mathbf{A}\| x(t) \ .$$

Dividing through by GMm^2, we see that for all t, $(x, y) := (x(t), y(t))$ satisfies

$$\sqrt{x^2 + y^2} = b - ax \tag{2.76}$$

where

$$a = \frac{\|\mathbf{A}\|}{GMm^2} \qquad \text{and} \qquad b = \frac{\|\mathbf{L}\|^2}{GMm^2} \ ,$$

which, after some simple algebra, leads to the conclusion that $(x(t), y(t))$ satisfies

$$(G^2M^2m^4 - \|\mathbf{A}\|^2)x^2 + G^2M^2m^4y^2 + 2\|\mathbf{L}\|^2\|\mathbf{A}\|x = \|\mathbf{L}\|^2$$

for all t. This is the equation of a conic section. It gives a closed orbit if and only if the coefficient of x^2 is strictly positive; i.e., if and only if $\|\mathbf{A}\|^2 < G^2M^2m^4$. In this case, the orbit will be an ellipse.

In fact, we can say more: For $b > 0$ and $0 \leq a < 1$, the equation (2.76) is the equation of and ellipse in the x, y plane with one focus at the origin, and the other at the point $(-2f, 0)$, semi-major axis of length R_+ running along the x-axis, and semi-minor axis of length R_- r inning along the y-axis, where

$$f = \left(\frac{a}{1 - a^2}\right) b \qquad R_+ = \frac{1}{1 - a^2} b \quad \text{and} \quad R_- = \frac{1}{\sqrt{1 - a^2}} b \ . \tag{2.77}$$

To see this, we verify that

$$\|(x, y) - (-2f, 0)\| + \|(x, y) - (0, 0)\| = 2R_+ \tag{2.78}$$

if and only if x, y satisfies (2.76) with a and b related to f and R_+ by the first two equalities in (2.77). A standard geometric definition of an ellipse is that it is the set of points in the plane for which the sum of the distances from two fixed points in the plane – the two *foci* - is constant, and the constant value is necessarily the length of the major axis. It is left as an exercise to prove this, though it may well be familiar from high-school geometry. However, note that the set of points (x, y) satisfying (2.78) is exactly the set of points such that the sum of the distances from $(-2f, 0)$ and $(0, 0)$ is $2c$, in which necessarily $c \geq f$, and c is the length of the semi-major axis.

Thus, all closed orbits are ellipses with the sun at one focus. In this case the magnitude of the Runge-Lenz vector necessarily satisfies

$$\|\mathbf{A}\| \leq GMm^2$$

and the direction is from the Sun out to the point on the ellipse that is closest to the Sun, which is known as the *perihelion*. (The point at the other end of the major axis, which is the point farther from the sun, is the *aphelion*.

Kepler's First Law states that the obits of the planets are ellipses with the sun at one focus, and we have now explained how this may be derived from Newton's Universal Theory of Gravitation. q Kelper's Second Law states that as the planet orbits the Sun, the straight line segment connecting the Sun and the planets sweeps out equal areas in equal times. As we shall see later on when we discuss area, this is a direct consequence of something we have already observed, namely the fact that $\mathbf{L}$, and in particular $\|\mathbf{L}\|$ is constant.

2.2.5 The specification of curves through differential equations

We have now seen several examples in which curves $\mathbf{x}(t)$ are specified by giving some *initial data*, such as $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{x}'(0) = \mathbf{v}_0$ and then a formula for the acceleration $\mathbf{x}''(t)$ in terms of the position $\mathbf{x}(t)$ and velocity $\mathbf{x}'(t)$:

$$\mathbf{x}''(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{x}'(t)) \ . \tag{2.79}$$

Equation (2.79) is an example of a *differential equation*. (The terminology "derivative equation" might seem more apt, since the equation (2.79) relates the second derivative $\mathbf{x}''(t)$ to the first derivative $\mathbf{x}'(t)$, and to $\mathbf{x}(t)$. However, the terminology was coined when people thought more commonly in terms of differentials and infinitesimals than derivatives.)

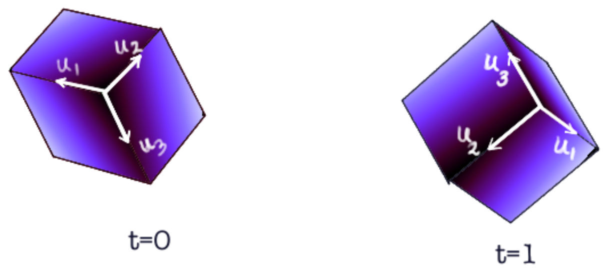
In ballistic motion without friction, the function $\mathbf{f}$ is simply constant: $\mathbf{f}(\mathbf{x}, vv) = \mathbf{f}_0$. In ballistic motion with friction, $\mathbf{f}(\mathbf{x}, \mathbf{v})$ is a linear function of $\mathbf{v}$, and is independent of $\mathbf{x}$: $\mathbf{f}(\mathbf{x}, \mathbf{v}) = \mathbf{f}_0 - \alpha\mathbf{v}$. In the Rotation Equation, $\mathbf{f}(\mathbf{x}, \mathbf{v})$ is a somewhat more complicated linear function of $\mathbf{v}$: $\mathbf{f}(\mathbf{x}, \mathbf{v}) = \mathbf{b} \times \mathbf{v}$. Finally, in the case of plenty motion, we have $\mathbf{f}(\mathbf{x}, \mathbf{v})$ independent of $\mathbf{v}$, but depending in a non-linear manner on $\mathbf{x}$:

$$\mathbf{f}(\mathbf{x}, \mathbf{v}) = -\frac{GM}{\|\mathbf{x}\|^3}\mathbf{x} \ .$$

As we have seen in each of these cases, for each given $\mathbf{x}_0$ and $\mathbf{v}_0$, there exists a unique curve $\mathbf{x}(t)$ that satisfies the equation (2.79) and also the initial conditions $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{x}'(0) = \mathbf{v}_0$. Later, we shall prove general existence and uniqueness theorems telling us when a differential equation and initial data are guaranteed to specify a unique curve, and we shall also develop methods for computing the solution. This more general investigaion of the subject will have to wait until we have developed more tools of multivariable calculus. However, the idea of specifying a curve through a differential equation together with initial data is far too important to be postponed so far into the future, and as we have seen, we can already explicitly solve a number of important cases with the tools presently at our disposal.

2.3 Rotations, continuity and the right hand rule

In this section, we return to rigid body motion, and apply some of what we have learned recently to this topic. As before, imagine a rigid cubical box moving in three dimensional space. Here is a picture showing the box shaped object at two times: $t = 0$ and $t = 1$:



As it moves, the box carries with it a "reference frame" of three unit vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$. Thus, as we have explained in Chapter One, rigid body motion involves a contiuous time dependent orthonormal frame $\{\mathbf{u}_1(t), \mathbf{u}_2(t), \mathbf{u}_3(t)\}$

The orthonormal basis $\{\mathbf{u}_1(t), \mathbf{u}_2(t), \mathbf{u}_3(t)\}$ is right handed in case $\mathbf{u}_1(t) \times \mathbf{u}_2(t) \cdot \mathbf{u}_3(t) = 1$, and is left handed in case $\mathbf{u}_1(t) \times \mathbf{u}_2(t) \cdot \mathbf{u}_3(t) = -1$, and ± 1 are the only possible values for this triple product.

Now, if $\mathbf{u}_j(t)$ is continuus for each $j = 1, 2, 3$, then $\mathbf{u}_1(t) \times \mathbf{u}_2(t) \cdot \mathbf{u}_3(t)$ is a continuouos function of t. Since it only has two possibles values, and canot jump from one to the other, it must be constant. That is, under our continuity assumption,

- *Let* $\{\mathbf{u}_1(t), \mathbf{u}_2(t), \mathbf{u}_3(t)\}$ *be a continuously time dependent orthonormal basis of* $\mathbb{R}^3$. *Then if* $\{\mathbf{u}_1(0), \mathbf{u}_2(0), \mathbf{u}_3(0)\}$ *is right-handed, so is* $\{\mathbf{u}_1(t), \mathbf{u}_2(t), \mathbf{u}_3(t)\}$ *for every* t.

In particular, it is impossible to "continuously interpolate" between a right-handed orthonormal basis and a left-handed orthonormal basis: If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a right-handed orthonormal basis and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a left-handed orthonormal basis, *there does not exist any* continuously time dependent orthonormal basis $\{\mathbf{u}_1(t), \mathbf{u}_2(t), \mathbf{u}_3(t)\}$, $0 \le t \le 1$ with

$$\mathbf{u}_j(0) = \mathbf{u}_j \qquad \text{and} \qquad \mathbf{u}_j(1) = \mathbf{v}_j$$

for $j = 1, 2, 3$.

However, as we shall now show, if $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are both right-handed (or both left-handed), then there is a continuous interpolation between them, and one such interpolation is through a "rotation about a fixed axis at constant angular velocity". The following lemma concerning Householder reflections is the key to our investigation.

Lemma 10 (Householder reflections and the cross product)**.** *Let* $\mathbf{u}$ *be any unit vector in* $\mathbb{R}^3$ *and let* $\mathbf{a}$ *and* $\mathbf{b}$ *be any two vectors in* $\mathbb{R}^3$. *Then*

$$\mathbf{h_u}(\mathbf{a} \times \mathbf{b}) = -\mathbf{h_u}(\mathbf{a}) \times \mathbf{h_u}(\mathbf{b}) \ ,$$

Proof: Direct computation shows that

$$\begin{aligned}
\mathbf{h_u}(\mathbf{a} \times \mathbf{b}) + \mathbf{h_u}(\mathbf{a}) \times \mathbf{h_u}(\mathbf{b}) &= 2[\mathbf{a} \times \mathbf{b} - (\mathbf{a} \times \mathbf{b} \cdot \mathbf{u})\mathbf{u} - (\mathbf{a} \cdot \mathbf{u})\mathbf{u} \times \mathbf{b} - (\mathbf{b} \cdot \mathbf{u})\mathbf{a} \times \mathbf{u}] \\
&= 2[(\mathbf{a} \times \mathbf{b})_\perp - (\mathbf{a} \cdot \mathbf{u})\mathbf{u} \times \mathbf{b} + (\mathbf{b} \cdot \mathbf{u})\mathbf{u} \times \mathbf{a}] \qquad (2.80)
\end{aligned}$$

where $(\mathbf{a}\times\mathbf{b})_\perp$ is the component of $\mathbf{a}\times\mathbf{b}$ orthogonal to $\mathbf{u}$.

However, since $\mathbf{u}$ is a unit vector, Lagrange's identity gives us:

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})_\perp &= -\mathbf{u}\times[\mathbf{u}\times(\mathbf{a}\times\mathbf{b})]\\ &= -\mathbf{u}\times[(\mathbf{u}\cdot\mathbf{b})\mathbf{a}-(\mathbf{u}\cdot\mathbf{a})\mathbf{b}]\\ &= -(\mathbf{u}\cdot\mathbf{b})\mathbf{u}\times\mathbf{a}+(\mathbf{u}\cdot\mathbf{a})\mathbf{u}\times\mathbf{b}\end{aligned}$$

Using this in (2.80), one obtains $\mathbf{h_u}(\mathbf{a}\times\mathbf{b})+\mathbf{h_u}(\mathbf{a})\times\mathbf{h_u}(\mathbf{b})=\mathbf{0}$. □

Since Householder reflections preserve dot products, and hence lengths and angles, we know that whenever $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ is a right-handed orthonormal basis, then

$$\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\} := \{\mathbf{h_u}(\mathbf{u}_1),\mathbf{h_u}(\mathbf{u}_2),\mathbf{h_u}(\mathbf{u}_3)\} \tag{2.81}$$

an orthonormal basis. By the lemma,

$$\mathbf{v}_1\times\mathbf{v}_2=\mathbf{h_u}(\mathbf{u}_1)\times\mathbf{h_u}(\mathbf{u}_2)=-\mathbf{h_u}(\mathbf{u}_1\times\mathbf{u}_3)=-\mathbf{h_u}(\mathbf{u}_3)=-\mathbf{v}_3\ ,$$

so that $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ is left-handed. Likewise if $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ is left-handed, (2.81) defines a right-handed orthonormal basis.

Now we are ready to draw some important conclusions.

Theorem 32 (Right handed orthonormal bases and reflection)**.** *Let* $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ *and* $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ *be two distinct right-handed orthonormal bases. Then there are unit vecturs* $\mathbf{u}$ *and* $\mathbf{v}$ *such that*

$$\mathbf{h_v}(\mathbf{h_v}(\mathbf{u}_j))=\mathbf{v}_j \qquad \text{for}\quad j=1,2,3\ . \tag{2.82}$$

Proof. Since the bases are distinct, we must have $\mathbf{u}_j\neq\mathbf{v}_j$ for some j. By cyclicly permuting the indices, we may suppose that $\mathbf{u}_1\neq\mathbf{v}_1$.

Let $\mathbf{u}=\|\mathbf{u}_1-\mathbf{v}_1||^{-1}(\mathbf{u}_1-\mathbf{v}_1)$. Then $\mathbf{h_u}(\mathbf{u}_1)=\mathbf{v}_1$, and we then define $\mathbf{w}_j:=\mathbf{h_u}(\mathbf{u}_j)$ for $j=1,2$, so that we have the left handed orthonormal basis

$$\{\mathbf{v}_1,\mathbf{w}_2,\mathbf{w}_3\}=\{\mathbf{h_u}(\mathbf{u}_1),\mathbf{h_u}(\mathbf{u}_2),\mathbf{h_u}(\mathbf{u}_3)\}\ .$$

Now suppose that $\mathbf{w}_2=\mathbf{v}_2$. Then we must have $\mathbf{w}_3=-\mathbf{v}_3$. In this case, we take $\mathbf{v}:=\mathbf{w}_3$. Then since this vector is orthogonal to both $\mathbf{v}_1$ and $\mathbf{w}_2=\mathbf{v}_2$, $\mathbf{h_v}(\mathbf{v}_1)=\mathbf{v}_1$ and $\mathbf{h_v}(\mathbf{w}_2)=\mathbf{w}_2=\mathbf{v}_2$. Finally, $\mathbf{h_v}(\mathbf{w}_3)=-\mathbf{w}_3=\mathbf{v}_3$. That is,

$$\{\mathbf{h_v}(\mathbf{v}_1),\mathbf{h_b}(\mathbf{w}_2),\mathbf{h_v}(\mathbf{w}_3)\}=\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}\ .$$

Thus, in this case, successiviely applying $\mathbf{h_u}$ and then $\mathbf{h_v}$ transforms $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ into $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$

On the other hand, if $\mathbf{w}_2\neq\mathbf{v}_2$, we define $\mathbf{v}=\|\mathbf{v}_2-\mathbf{w}_2||^{-1}(\mathbf{v}_2-\mathbf{w}_2)$, so that $\mathbf{h_v}(\mathbf{w}_2)=\mathbf{v}_2$. Note that $\mathbf{w}_2$ and $\mathbf{v}_2$ are both orthogonal to $\mathbf{v}_1$ since $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ and $\{\mathbf{v}_1,\mathbf{w}_2,\mathbf{w}_3\}$ are both orthonormal bases. But then $\mathbf{v}$ is orthogonal to $\mathbf{v}_1$, and so $\mathbf{h_v}(\mathbf{v}_1)=\mathbf{v}_1$.

Now since $\{\mathbf{v}_1,\mathbf{w}_2,\mathbf{w}_3\}$ is a left handed orthonormal basis,

$$\{\mathbf{h_v}(\mathbf{v}_1),\mathbf{h_b}(\mathbf{w}_2),\mathbf{h_v}(\mathbf{w}_3)\}=\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{h_v}(\mathbf{w}_3)\}$$

is a right handed orthonormal basis. Since any two vectors determine the third, and since $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ is right handed, it must be that $\mathbf{h_v}(\mathbf{w}_2)=\mathbf{v}_3$. Either way, we have proved (2.82). □

In what follows, let us fix two distinct right-handed orthonormal bases $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, and let us define $\mathbf{f}$ by

$$\mathbf{f}(\mathbf{x}) = \mathbf{h}_{\mathbf{v}}(\mathbf{h}_{\mathbf{u}}(\mathbf{x}))$$

where $\mathbf{h}_{\mathbf{v}}$ and $\mathbf{h}_{\mathbf{u}}$ are the Householder reflections provided by Theorem 32 so that $\mathbf{f}$ transforms $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ into $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

If it were the case that $\mathbf{v} = \pm\mathbf{u}$, then we would have $\mathbf{h}_{\mathbf{v}} = \mathbf{h}_{\mathbf{u}}$, and $\mathbf{f}$ would be the identity transformation. Since the two orthonormal bases are distinct, $\mathbf{v} \neq \pm\mathbf{u}$, and hence $\mathbf{v} \times \mathbf{u} \neq \mathbf{0}$. Therefore, we may define a unit vector $\mathbf{a}$ by $\mathbf{a} = \dfrac{1}{\|\mathbf{u} \times \mathbf{v}\|}\mathbf{u} \times \mathbf{v}$. Then $\mathbf{a}$ is orthognal to both $\mathbf{u}$ and $\mathbf{v}$ so that

$$\mathbf{f}(\mathbf{a}) = \mathbf{h}_{\mathbf{v}}(\mathbf{h}_{\mathbf{u}}(\mathbf{a})) = \mathbf{h}_{\mathbf{v}}(\mathbf{a}) = \mathbf{a} \ .$$

Define $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ by $\mathbf{a}_1 = \mathbf{a}$, $\mathbf{a}_2 = \mathbf{u}$ and $\mathbf{a}_3 = \mathbf{a}_1 \times \mathbf{a}_2$, and note that $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is an orthonormal basis. Define $\Theta \in [0, \pi]$ by

$$\Theta := \arccos(\mathbf{v} \cdot \mathbf{u}) \ . \tag{2.83}$$

Writing $\mathbf{v}$ out in this basis, and using the fact that $\mathbf{a}_1 = \mathbf{a}$ is orthogonal to $\mathbf{v}$, we have

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{a}_2)\mathbf{a}_2 + (\mathbf{v} \cdot \mathbf{a}_3)\mathbf{a}_3 \ .$$

By definition, $\mathbf{v} \cdot \mathbf{a}_2 = \mathbf{v} \cdot \mathbf{u} = \cos\Theta$. Next,

$$\mathbf{v} \cdot \mathbf{a}_3 = \mathbf{v} \cdot \mathbf{a} \times \mathbf{u} = -\mathbf{v} \cdot \mathbf{u} \times \mathbf{a} = -\mathbf{v} \times \mathbf{u} \cdot \mathbf{a} = \|\mathbf{u} \times \mathbf{v}\| = \sin\Theta \ .$$

Thus, $\mathbf{v} = \cos\Theta\mathbf{u} + \sin\Theta\mathbf{a}_3$, and if we define $\mathbf{v}(t)$ by

$$\mathbf{v}(t) := \cos(t\Theta)\mathbf{u} + \sin(t\Theta)\mathbf{a}_3 \ , \tag{2.84}$$

if follows that $\mathbf{v}(t)$ is a continuous function of t, with $\mathbf{v}(0) = \mathbf{u}$ and $\mathbf{v}(1) = \mathbf{v}$.

Given this interpolation between $\mathbf{u}$ and $\mathbf{v}$, define the t dependent orthogonal transformation $\mathbf{f}_t$ by

$$\mathbf{f}_t(\mathbf{x}) := \mathbf{h}_{\mathbf{v}(t)}(\mathbf{h}_{\mathbf{u}}(\mathbf{x})) \ . \tag{2.85}$$

Since $\mathbf{v}(0) = \mathbf{u}$, and since $\mathbf{h}_u \circ \mathbf{h}_u$ is the identity, $\mathbf{f}_0$ is the identity transformation, and by construction $\mathbf{f}_1$ transforms $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ into $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Consequently, if we define

$$\mathbf{u}_j(t) = \mathbf{f}_t(\mathbf{u}_j) \qquad j = 1, 2, 3 \quad \text{and} \quad 0 \le t \le 1 \ ,$$

$\{\mathbf{u}_1(t), \mathbf{u}_2(t), \mathbf{u}_3(t)\}$ interpolates between $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

We now claim that it does so not only continously, but also differentiably, and in fact, the interplation process is a rotation about the $\mathbf{a}_1$ axis 2Θ at steady angular speed 2Θ, with the inerpoltation accomplished in unit time. We shall show this by showing that for any $\mathbf{x}_0 \in \mathbb{R}^3$, if we define a curve $\mathbf{x}(t)$ by

$$\mathbf{x}(t) := \mathbf{f}_t(\mathbf{x}_0) \ ,$$

then is the solution of the rotation equation

$$\mathbf{x}'(t) = (2\Theta\mathbf{a}_1) \times \mathbf{x}(t) \tag{2.86}$$

with initial condition $\mathbf{x}(0) = \mathbf{x}_0$.

To verify this, we compute both sides of (2.86). It is simplest to first compute $\mathbf{f}_t(\mathbf{a}_j)$ for $j = 1, 2, 3$. It is left as an exercise to check that

$$\begin{aligned} \mathbf{f}_t(\mathbf{a}_1) &= \mathbf{a}_1 \\ \mathbf{f}_t(\mathbf{a}_2) &= \cos(t2\Theta)\mathbf{a}_2 + \sin(t2\Theta)\mathbf{a}_3 \\ \mathbf{f}_t(\mathbf{a}_3) &= -\sin(t2\Theta)\mathbf{a}_2 + \cos(t2\Theta)\mathbf{a}_3 \ . \end{aligned} \tag{2.87}$$

As we shall now explain, this means that the transformation $\mathbf{f}_t$ is *rotation through the angle* $t2\Theta$ *about the axis in the* $\mathbf{a}_1$ *direction.*

Now expand $\mathbf{x}_0$ in the orthonormal basis $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$:

$$\mathbf{x}_0 = (\mathbf{x}_0 \cdot \mathbf{a}_1)\mathbf{a}_1 + (\mathbf{x}_0 \cdot \mathbf{a}_2)\mathbf{a}_2 + (\mathbf{x}_0 \cdot \mathbf{a}_3)\mathbf{a}_3 \ .$$

Since reflections are linear transformations,

$$\mathbf{f}_t(\mathbf{x}_0) = (\mathbf{x}_0 \cdot \mathbf{a}_1)\mathbf{f}_t(\mathbf{a}_1) + (\mathbf{x}_0 \cdot \mathbf{a}_2)\mathbf{f}_t(\mathbf{a}_2) + (\mathbf{x}_0 \cdot \mathbf{a}_3)\mathbf{f}_t(\mathbf{a}_3) \ .$$

Using the computations just abiove, we compute that

$$\mathbf{f}_t(\mathbf{x}_0) = \tilde{x}_0\mathbf{a}_1 + [\cos(t2\Theta)\tilde{y}_0 - \sin(t2\Theta)\tilde{z}_0]\mathbf{a}_2 + [\sin(t2\Theta)\tilde{y}_0 + \cos(t2\Theta)\tilde{z}_0]\mathbf{a}_3 \ . \tag{2.88}$$

We now claim that the curve $\mathbf{x}(t)$ defined by $\mathbf{x}(t) = \mathbf{f}_t(\mathbf{x}_0)$ To see this, we differentiatiate and find

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{f}_t(\mathbf{x}_0) = 2\Theta[-\sin(t2\Theta)\tilde{y}_0 - \cos(t2\Theta)\tilde{z}_0]\mathbf{a}_2 + 2\Theta[\cos(t2\Theta)\tilde{y}_0 \sin(t2\Theta)\tilde{z}_0]\mathbf{a}_3 \ .$$

Likewise, using the fact that $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is a right handed orthonormal basis, we compute

$$(2\Theta\mathbf{a}_1) \times \mathbf{f}_t(\mathbf{x}_0) = 2\Theta[-\sin(t2\Theta)\tilde{y}_0 - \cos(t2\Theta)\tilde{z}_0]\mathbf{a}_2 + 2\Theta[\cos(t2\Theta)\tilde{y}_0 \sin(t2\Theta)\tilde{z}_0]\mathbf{a}_3 \ .$$

This proves that $\mathbf{x}(t) := \mathbf{f}_t(\mathbf{x}_0)$ satisfies (2.86), and it is clear that $\mathbf{x}(0) = \mathbf{x}_0$.

We now see that the interpolation process carrying $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ into $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is rotation about the axis $\mathbf{a}_1$ at steady angular veolcity 2Θ, with the interpolation complete in unit time. We summarize:

Theorem 33 (Rotations and reflections)**.** *Given two unit vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^3$ *with* $\mathbf{v} \neq \pm\mathbf{u}$*, define*

$$\theta := \frac{1}{2}\arccos(\mathbf{v} \cdot \mathbf{u})$$

and

$$\mathbf{a} := \frac{1}{\|\mathbf{u} \times \mathbf{v}\|}\mathbf{u} \times \mathbf{v} \ .$$

Then the transformation $\mathbf{f}$ *defined by*

$$\mathbf{f}(\mathbf{x}) := \mathbf{h}_{\mathbf{v}}(\mathbf{h}_{\mathbf{u}}(\mathbf{x})) \tag{2.89}$$

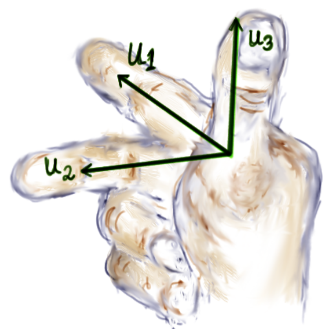
is rotation by an angle θ *about the axis along* $\mathbf{a}$*.*

Thus, every rotation in $\mathbb{R}^3$ *can be written as the composition product of two Householder reflections: Given a unit vector* $\mathbf{a}$ *and an angle* θ *in* $[0, 2\pi]$*, find any two unit vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in the plane orthogonal to* $\mathbf{a}$ *such that* $\mathbf{u} \cdot \mathbf{v} = \cos(\theta/2)$*. Then for any such choice of* $\mathbf{u}$ *and* $\mathbf{v}$*, (2.89) expresses the rotation through the angle* θ *about the axis along* $\mathbf{a}$ *as the product of two reflections.*

We can finally explain the terminology "right handed orthonormal basis". We begin by making an identification of $\mathbb{R}^3$ with the physical three dimension space around us. This requires us to identify the standard basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ in $\mathbb{R}^3$ with three orthogonal directions in physical space.

To do this, fix three orthogonal directions in physical space – for instance, East, North and "straight up" might be good choices for somebody standing anywhere on the Earth except the North or South Poles. Next, take your right hand, and arrange you thumb and fingers so that your thumb, index finger and middle finger each point in one of these three orthogonal directions, as in the picture below. *At this stage of the process*, we number the directions: Identify $\mathbf{e}_1$ with the direction in which your index finger points, identify $\mathbf{e}_2$ with the direction in which your middle finger points, and identify $\mathbf{e}_3$ with the direction in which your index thumb points.

Now let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be some other set of three orthogonal directions. Try to rigidly rotate your right hand (keeping the index finger, middle finger and thumb orthogonal) around so that your index finger points in the direction of $\mathbf{u}_1$, your middle finder points in the direction of $\mathbf{u}_2$, and your thumb points in the direction of $\mathbf{u}_3$, as in the picture:



- *If this is possible, then the basis* $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ *is right handed, and otherwise, it is not.*

Indeed, if this motion of your hand *is possible*, then the motion of your hand provides a continuous interpolation between the reference basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. By what we have seen at the beginning, this means that $(\mathbf{u}_1 \times \mathbf{u}_2) \cdot \mathbf{u}_3 = (\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3 = 1$ and hence $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is right handed.

Conversely, if $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ *is right-handed*, like $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, then there is a rotation process that carries $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ over to $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. Therefore, if you arrange your right hand so that your index finger points in the $\mathbf{e}_1$ direction, your middle finger in the $\mathbf{e}_2$ direction, and your thumb in the $\mathbf{e}_3$ direction, and you then rotate your right hand about the corresponding axis of rotation, through the corresponding angle of rotation, your right hand will indeed be oriented as in the picture.

We may now also explain the "*right-hand rule*": Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors in $\mathbb{R}^3$ such that neither is a multiple of the other. Let $\mathbf{b}_\perp$ be the component of $\mathbf{b}$ that is orthogonal to $\mathbf{a}$, and define a right-handed orthonormal basis by

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{a}\|}\mathbf{a}\ , \quad \mathbf{u}_2 = \frac{1}{\|(\mathbf{b})_\perp\|}(\mathbf{b})_\perp \quad \text{and} \quad \mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2\ .$$

Then

$$\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}_\perp = \|\mathbf{a}\|\|\mathbf{b}_\perp\|\mathbf{u}_1 \times \mathbf{u}_3 = \|\mathbf{a}\|\|\mathbf{b}_\perp\|\mathbf{u}_3\ .$$

That is, $\mathbf{a} \times \mathbf{b}$ is a positive multiple of $\mathbf{u}_3$.

This means that if you configure your right hand as in the picture, with your thumb pointing in the direction of $\mathbf{a}$, and your index finger in the plane containing $\mathbf{a}$ and $\mathbf{b}$, then your middle finger points in the direction of $\mathbf{u}_3$; i.e., in the direction of $\mathbf{a} \times \mathbf{b}$. This is commonly called the *right-hand rule* for the direction of $\mathbf{a} \times \mathbf{b}$.

2.4 Exercises

2.1 Let $\mathbf{x}(t) = (t + 1\ ,\ t^2)$. This is a parameterization of the parabola $y = (x - 1)^2$.

(a) Compute $\mathbf{v}(t) = \mathbf{x}'(t)$ and $\mathbf{a}(t) = \mathbf{x}''(t)$.

(b) Compute $v(t)$ and $\mathbf{T}(t)$.

(c) Find the tangent line to this curve at $t = 1$.

2.2 Let $\mathbf{x}(t) = \left(t^{-2}\ ,\ 4/\sqrt{t}\ ,\ t\right)$ for $t > 0$.

(a) Compute $\mathbf{v}(t) = \mathbf{x}'(t)$ and $\mathbf{a}(t) = \mathbf{x}''(t)$.

(b) Compute $v(t)$ and $\mathbf{T}(t)$.

(c) Find the tangent line to this curve at $t = 1$.

2.3 Let $\mathbf{x}(t)$ and $\mathbf{y}(t)$ be two continuous curves in $\mathbb{R}^n$. Show that $f(t) := \mathbf{x}(t) \cdot \mathbf{y}(t)$ is a continuous real valued function of t. Also for $n = 3$, show that

$$\mathbf{z}(t) := \mathbf{x}(t) \times \mathbf{y}(t)$$

is a continuous curve in $\mathbb{R}^3$.

2.4 Let $\mathbf{x}(t) = \left(\cos(t)\ , \sin(t)\ , t/\pi\right)$ where $r > 0$. The curve $\mathbf{x}(t)$ is a helix in $\mathbb{R}^3$.

(a) Compute $\mathbf{v}(t)$ and $\mathbf{a}(t)$.

(b) Compute $v(t)$ and $\mathbf{T}(t)$.

(c) Compute the curvature $\kappa(t)$ and the torsion $\tau(t)$, as well as $\mathbf{N}(t)$ and $\mathbf{B}(t)$.

(d) Compute the Darboux vector $\boldsymbol{\omega}(t)$.

(e) Find the tangent line to this curve at $t = \pi/4$, and the equation of the osculating plane to the curve at $t = \pi/2$. Find the intersection of this line and plane.

2.5: Let $\mathbf{x}(t)$ be the curve given by

$$\mathbf{x}(t) = (e^t \cos t, e^t \sin t, e^t)\ .$$

(a) Compute the arc length $s(t)$ as a function of t, measured from the starting point $\mathbf{x}(0)$, and find an arc-length parameterization of this curve

(b) Compute curvature $\kappa(t)$ and torsion $\tau(t)$ as a function of t.

(c) Find an equation for the osculating plane at time $t = 0$

2.6: Let $\mathbf{x}(t)$ be the curve given by

$$\mathbf{x}(t) = (t^{3/2}, 3t, 6t^{1/2})$$

for $t > 0$.

(a) what is the arc length along the curve between $\mathbf{x}(1)$ and $\mathbf{x}(4)$?

(b) Compute curvature $\kappa(t)$ and torsion $\tau(t)$ as a function of t.

(c) Find an equation for the osculating plane at $t = 1$, and find a parameterization of the tangent line to the curve at $t = 1$.

2.7: Let $\mathbf{x}(t)$ be the curve given by $\mathbf{x}(t) = (t, t^2/2, t^3/3)$.

(a) Find the equation of the osculating plane at $t = 1$.

(b) Compute the distance from the origin to the osculating plane at $t = 1$.

2.8: Let $\mathbf{x}(t)$ be the curve given by

$$\mathbf{x}(t) = (2t, t^2, t^3/3) \ .$$

(a) Compute the arc length $s(t)$ as a function of t, measured from the starting point $\mathbf{x}(0)$.

(b) Compute curvature $\kappa(t)$ and torsion $\tau(t)$ as a function of t.

(c) Find equations for the osculating planes at time $t = 0$ and $t = 1$, and find a parameterization of the line formed by the intersection of these planes.

2.9 Consider the ellipse in $\mathbb{R}^2$ given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a, b > 0$.

(a) Show that the path traced out by the parameterized curve $\mathbf{x}(t) = (a\cos(t), b\sin(t))$ is this ellipse. In other words, $\mathbf{x}(t) = (a\cos(t), b\sin(t))$ is a parameterization of this ellipse.

(b) Compute the curvature $\kappa(t)$, and find the minimum and maximum values of curvature on the ellipse, and the places where the curvature takes on these values.

2.10 Let $\mathbf{x}(t)$ be the curve given by $\mathbf{x}(t) = (t\ ,\ \sqrt{2}\ln(t)\ ,\ 1/t)$ for $t > 0$.

(a) Find the arc length along the curve from $\mathbf{x}(1)$ to $\mathbf{x}(3)$.

(b) Find the arc length along the curve from $\mathbf{x}(1)$ to $\mathbf{x}(t)$ as a function of t.

(c) Find the arc length parameterization $\mathbf{x}(s)$ of this curve.

2.11 Find the arc length along the parabola $y = (x-1)^2$ from the point $(0, 1)$ to the point $(1, 0)$. (See Excercise 2.1.)

2.12 Find the arc length parameterization of the curve given by $\mathbf{x}(t) = (t^{-2}\ ,\ 4/\sqrt{t}\ ,\ t)$ for $t > 0$. (See Excercise 2.2.) What is the arc length along the segment of the curve joining $\mathbf{x}(1)$ and $\mathbf{x}(4)$?

2.13 Let $\mathbf{b} = (2, 1, 2)$. Let $\mathbf{x}(t)$ be the curve given satisfying the initial value problem

$$\mathbf{x}'(t) = \mathbf{b} \times \mathbf{x}(t) \qquad \text{and} \qquad \mathbf{x}(0) = (1, 1, 1) \ .$$

(a) Compute $\mathbf{x}(\pi)$ and find the arc length along the curve from $\mathbf{x}(0)$ to $\mathbf{x}(\pi)$.

(b) Compute the curvature and torsion for this curve as a function of t.

2.14 Let $\mathbf{b} = (4,7,4)$. Let $\mathbf{x}(t)$ be the curve given satisfying the initial value problem

$$\mathbf{x}'(t) = \mathbf{b} \times \mathbf{x}(t) \qquad \text{and} \qquad \mathbf{x}(0) = (2,2,1)\ .$$

(a) Compute $\mathbf{x}(\pi)$ and find the arc length along the curve from $\mathbf{x}(0)$ to $\mathbf{x}(\pi)$.

(b) Compute the curvature and torsion for this curve as a function of t.

2.15 Show that for $b > 0$ and $0 \le a < 1$, the set of points (x, y) that satisfy (2.76) is an ellipse with one focus at the origin, and the other at $(-2f, 0)$, and semi-major axis R_+ where f and R_+ are given in terms of a and b by (2.77).

2.16 Let $\mathbf{x}(t)$ be the curve given by $\mathbf{x}(t) = (\cos t + 1, \cos t + \sin t, \sin t + 1)$.

(a) Compute curvature $\kappa(t)$ and torsion $\tau(t)$ as a function of t.

(b) Find an equation for the osculating plane at time $t = 0$

(c) Find the distance between the plane given by $x - y + z = 0$ to $\mathbf{x}(t)$ as a function of t.

2.17 Consider the helix whose Darboux vector is $(3,0,4)$, with $\mathbf{x}(0) = \mathbf{0}$, and $\{\mathbf{T}(0), \mathbf{N}(0), \mathbf{B}(0)\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Find a formula for $\mathbf{x}(s)$, the arc-length parameterization of the helix.

2.18 The latitude and longitude of Milan Italy is $45^\circ\ 27''\ N \quad 9^\circ\ 10''\ E$. The latitude and longitude of Cairo Egypt is $30^\circ\ 2''\ N \quad 31^\circ\ 21''\ E$. Using this information, and the value of 6371 kilometers for the radius of the Earth, and the assumption that the Earth is spherical, compute the length of the shortest route on the surface of the Earth from Milan to Cairo.

2.19 Consider the vectors

$$\mathbf{u} = \frac{1}{3}(2,1,0,2) \qquad \text{and} \qquad \mathbf{w} = \frac{1}{15}(10,-5,8,6)\ .$$

These vectros both belong to S^3, the unit sphere in $\mathbb{R}^4$. Find a continuous curve $\mathbf{u}(t)$ defined on some interval $[0, T]$, some $T > 0$, that is continuously differentiable on $(0, T)$, with each $\mathbf{u}(t) \in S^3$, $\mathbf{u}(0) = \mathbf{u}$, $\mathbf{u}(T) = \mathbf{w}$, and whose arc length is minimal among all such curves.

2.20 Let $\mathbf{a} = \frac{1}{3}(2,1,2)$. Let $\mathbf{u} = \frac{1}{3}(1,2,-2)$, and note that this is a unit vector orthogonal to $\mathbf{a}$. Find a unit vector $\mathbf{v}$ so that $\mathbf{f}(\mathbf{x}) := \mathbf{h}_{\mathbf{v}}(\mathbf{h}_{\mathbf{u}}(\mathbf{x}))$ is the rotation of $\mathbf{x}$ through the angle $\theta = \pi/3$ about the axis along $\mathbf{a}$, and then compute $\mathbf{f}((1,1,1))$.

2.21 Verify that the formulas in (2.87) are correct. You will need to use the double angle formulas $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ and $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$.