9.1. (a) We know that V is scalar-valued on $\mathbf{R}^n$, $\mathbf{F}$ is a vector field on $\mathbf{R}^n$. We are given that $\mathbf{F} = -\nabla V$ and since $\mathbf{x} = \mathbf{x}(t)$ is a flow curve of $\mathbf{F}$, $\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t))$ by definition of flow curve. So, by the chain rule,

$$\frac{d}{dt}V(\mathbf{x}(t)) = \nabla V(\mathbf{x}(t)) \bullet \mathbf{x}'(t) = -\mathbf{F}(\mathbf{x}(t)) \bullet \mathbf{F}(\mathbf{x}(t)) = -\|\mathbf{F}(\mathbf{x}(t))\|^2 \leq 0.$$

Therefore $V(\mathbf{x}(t))$ is non-increasing on any flow curve.

(b) Now we are given that $\mathbf{F} = A\nabla V$ and since $\mathbf{x} = \mathbf{x}(t)$ is a flow curve of $\mathbf{F}$, $\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t))$. By the chain rule, on the given flow curve,

$$\frac{d}{dt}V(\mathbf{x}(t)) = \nabla V(\mathbf{x}(t)) \bullet \mathbf{x}'(t) = \nabla V(\mathbf{x}(t)) \bullet \mathbf{F}(\mathbf{x}(t)) = \nabla V(\mathbf{x}(t)) A \nabla V = 0$$

because A is antisymmetric. (For any antisymmetric A and any vector $\mathbf{v}$, $\mathbf{v} \bullet A\mathbf{v} = 0$.) Therefore $V(\mathbf{x}(t))$ is constant on the flow curve.

9.3. Try a replacement trick. Let C' be the straight segment from $(0, 7)$ to $(0, 0)$, and let D be the region enclosed by the closed curve consisting of C followed by C'. Considering $\mathbf{F}$ as a vector field in $\mathbf{R}^3$, $\mathbf{F}(x, y, z) = (\sin x + y, 3x + y, 0)$, we have

$$\operatorname{curl} \mathbf{F} = (0, 0, 2), \quad \operatorname{div} \mathbf{F} = \cos x + 1.$$

The curl looks simple so we try using Stokes' Theorem. The surface integral, over D, is just a double integral

$$\int_C \mathbf{F} \bullet \mathbf{T}\, ds + \int_{C'} \mathbf{F} \bullet \mathbf{T}\, ds = \int_D \operatorname{curl} \mathbf{F} \bullet \mathbf{N}\, dS.$$

Since C and C' are counterclockwise, the preferred unit normal $\mathbf{N}$ is $\mathbf{k} = (0, 0, 1)$.

$$\int_D \operatorname{curl} \mathbf{F} \bullet \mathbf{N}\, dS = \int_D 2d^2\mathbf{x} = 2 \cdot \text{area of } D = 2 \cdot ((9/2) + 7 + 4) = 31$$

Along the path C' we can use $t = y$ as the parameter, with $x = 0$. Then $\mathbf{T}\, ds = (x', y')\, dt = (0, 1)\, dt$ and $\mathbf{F} \bullet \mathbf{T}\, ds = (0 + y)\, dt = t\, dt$, so

$$\int_{C'} \mathbf{F} \bullet \mathbf{T}\, ds = \int_7^0 t\, dt = -49/2.$$

Putting these together,

$$\int_C \mathbf{F} \bullet \mathbf{T}\, ds = \int_D \operatorname{curl} \mathbf{F} \bullet \mathbf{N}\, dS - \int_{C'} \mathbf{F} \bullet \mathbf{T}\, ds = 31 - (-49/2) = 111/2$$

Alternatively we can compute $\int_C \mathbf{F} \bullet \mathbf{T}\, ds$ from a direct parametrization of C in three pieces. We can use $t = x$ in all cases:

$C_1 : x = t, y = t,\ 0 \leq t \leq 3$; $C_2 : x = t, y = 2t-3,\ 3 \leq t \leq 4$; $C_3 : x = t, y = (1/2)(-t+14),\ 4 \geq t \geq 0.$

Then $\int_C \mathbf{F} \bullet \mathbf{T}\, ds = \int_{C_1} \mathbf{F} \bullet \mathbf{T}\, ds + \int_{C_2} \mathbf{F} \bullet \mathbf{T}\, ds + \int_{C_3} \mathbf{F} \bullet \mathbf{T}\, ds =$

$$\int_0^3 (\sin t+t+3t+t)\, dt+\int_3^4 (\sin t+2t-3+2(3t+2t-3))\, dt+\int_4^0 (\sin t+(1/2)(-t+14)-(1/2)(3t+(1/2)(-t+14)))\, dt$$

When the integrals are expanded, the three integrals of $\sin t$ add to $\int_0^0 \sin t\, dt = 0$, so we can disregard them. The rest total

$$\int_0^3 5t\, dt + \int_3^4 (12t-9)\, dt + \int_4^0 ((-7/4)t + (7/2))\, dt = 111/2.$$

9.5. Let C_1 be the line segment from $(0,1)$ to $(0,0)$, and C_2 the line segment from $(0,0)$ to $(0,1)$. Let D be the quarter-disk enclosed by C, C_1, and C_2. Then by Stokes' Theorem,

$$\int_C \mathbf{G}\bullet\mathbf{T}\, ds+\int_{C_1} \mathbf{G}\bullet\mathbf{T}\, ds+\int_{C_2} \mathbf{G}\bullet\mathbf{T}\, ds = \int_D \operatorname{curl} \mathbf{G}\bullet\mathbf{N}\, dS = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathbf{e}_3\bullet\mathbf{e}_3\, dA = \int_D dA = \frac{\pi}{4}.$$

Also C_1 is parametrized by $x=0$, $y=1-t$, from $t=0$ to $t=1$ so $\mathbf{T}\, ds = -\mathbf{e}_2\, dt$ and

$$\int_{C_1} \mathbf{G} \bullet \mathbf{T}\, ds = \int_0^1 (P(0,1-t), Q(0,1-t)) \bullet (-\mathbf{e}_2)\, dt = -\int_0^1 Q(0,1-t)\, dt = -\int_0^1 0\, dt = 0,$$

and similarly

$$\int_{C_2} \mathbf{G} \bullet \mathbf{T}\, ds = \int_0^1 (P(t,0), Q(t,0)) \bullet (\mathbf{e}_1)\, dt = \int_0^1 P(t,0)\, dt = \int_0^1 0\, dt = 0.$$

Combining what we have so far,

$$\int_C \mathbf{G} \bullet \mathbf{T}\, ds = \frac{\pi}{4}.$$

Also $\displaystyle\int_C \nabla\varphi \bullet \mathbf{T}\, ds = \varphi(x,y)\Big|_{(1,0)}^{(0,1)} = 2-1=1$. Finally

$$\int_C \mathbf{F} \bullet \mathbf{T}\, ds = \int_C \mathbf{G} \bullet \mathbf{T}\, ds + \int_C \nabla\varphi \bullet \mathbf{T}\, ds = \frac{\pi}{4}+1.$$

9.7. By Lemma 23 and Theorem 98, a vector field continuously differentiable in all of $\mathbf{R}^3$ is a curl if and only if its divergence is identically 0.

$\operatorname{div} \mathbf{F} = 1+x$, $\operatorname{div} \mathbf{G} = 0$, $\operatorname{div} \mathbf{H} = 2$ so only $\mathbf{G}$ is a curl.

9.9. The key here is to show that at each point (x,y,z) of the cone $z=\sqrt{x^2+y^2}$,

$$\operatorname{curl} \mathbf{F} \text{ is tangent to the cone,}$$

i.e., curl $\mathbf{F}$ is perpendicular to the normal vector $\mathbf{N}$ to the cone. For then by Stokes' Theorem, if we let $\mathcal{S}$ be the portion of the cone enclosed by C,

$$\oint_C \mathbf{F} \bullet \mathbf{T}\, ds = \int_{\mathcal{S}} \operatorname{curl} \mathbf{F} \bullet \mathbf{N}\, dS = \int_{\mathcal{S}} 0\, dS = 0.$$

To see that $\operatorname{curl} \mathbf{F} \bullet \mathbf{N} = 0$, parametrize the cone as $x = r\cos\theta$, $y = r\sin\theta$, $z = r$, and let $\mathbf{x} = \mathbf{x}(r,\theta) = (r\cos\theta, r\sin\theta, r)$. Then a normal vector is

$$\mathbf{n} = \frac{\partial \mathbf{x}}{\partial r} \times \frac{\partial \mathbf{x}}{\partial \theta} = (\cos\theta, \sin\theta, 1) \times (-r\sin\theta, r\cos\theta, 0) = (-r\cos\theta, -r\sin\theta, r) = (-x, -y, z), \text{ while}$$
$$\operatorname{curl} \mathbf{F} = (2yz, -2xz, 0), \text{ yielding } \operatorname{curl} \mathbf{F} \bullet \mathbf{n} = 0, \text{ as desired.}$$

9.11. Parametrize $\mathcal{S}$ by

$$\mathbf{x} = (r\cos\theta, r\sin\theta, 1 - r^2)$$

Since the plane in question has equation $z = 1 - x$, the region D of the parameter $(x, y-)$ plane corresponding to $\mathcal{S}$ is defined by the inequality

$$1 - x^2 - y^2 \geq 1 - x, \text{ i.e., } x^2 + y^2 \leq x, \text{ or in polar coordinates } r \leq \cos\theta, \ -\pi/2 \leq \theta \leq \pi/2.$$

(D is the circle $(x - (1/2))^2 + y^2 = (1/2)^2$ as can be seen by completing the square.)

We compute

$$\mathbf{N}\, dS = \pm \mathbf{T}_r \times \mathbf{T}_\theta\, dr\, d\theta = \pm(\cos\theta, \sin\theta, -2r) \times (-r\sin\theta, r\cos\theta, 0)\, dr\, d\theta = \pm(2r^2\cos\theta, 2r^2\sin\theta, r)\, dr\, d\theta$$

and since the downward normal is specified in the question, the minus sign is correct. Then

$$\begin{aligned}
\mathbf{F}(x,y,z) \bullet \mathbf{N}\, dS &= (r^2\cos\theta\sin\theta, r(1-r^2)\sin\theta, r(1-r^2)\cos\theta) \bullet (-2r^2\cos\theta, -2r^2\sin\theta, -r)\, dr\, d\theta \\
&= \int_{-\pi/2}^{\pi/2} \int_0^{\cos\theta} -2r^4\cos^2\theta\sin\theta - 2r^3\sin^2\theta + 2r^5\sin^2\theta - r^2\cos\theta + r^4\cos\theta\, dr\, d\theta \\
&= \int_{-\pi/2}^{\pi/2} -\frac{2}{5}\cos^7\theta\sin\theta - \frac{1}{2}\cos^4\theta\sin^2\theta + \frac{1}{3}\cos^6\theta\sin^2\theta - \frac{1}{3}\cos^4\theta + \frac{1}{5}\cos^5\theta\, d\theta
\end{aligned}$$

etc.

9.13. First calculate $\operatorname{div} \mathbf{F} = 1$. By the divergence theorem the flux out of $\mathcal{V}$ equals $\int_{\mathcal{V}} \operatorname{div} \mathbf{F}\, dV = \int_{\mathcal{V}} dV$.

We use cylindrical coordinates. The upper surface of $\mathcal{V}$ is $z = 4 - r^2$ and the lower surface is $z^2 = 6 - r^2$. The surfaces intersect where $z = z^2 - 2$, i.e., where $z = 2$, and thus where $r = \sqrt{2}$. (The root $z = -1$ is irrelevant to $\mathcal{V}$, since $z > 0$ everywhere in $\mathcal{V}$.)

Therefore $\mathcal{V}$ is described in cylindrical coordinates by

$$0 \le \theta \le 2\pi,\ 0 \le r \le \sqrt{2},\ \sqrt{6-r^2} \le z \le 4-r^2, \text{ and so the answer is}$$

$$\int_{\mathcal{V}} dV = \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{2}} \int_{z=\sqrt{6-r^2}}^{4-r^2} r\,dz\,dr\,d\theta = 2\pi \int_{r=0}^{\sqrt{2}} r(4-r^2) - r\sqrt{6-r^2}\,dr \quad \text{etc.}$$

9.15. (a) $\operatorname{curl}\mathbf{F} = \mathbf{0}$ and $\operatorname{curl}\mathbf{G} = (4y, 0, 0)$

(b) $\mathbf{F} = \nabla\varphi$ for some φ. By the method of Example 150, $\varphi(x,y,z) = x^2y - x^2z - y^2z + 2x - 3z$. The answer is not unique; any constant can be added to φ.

9.17. Since $\mathcal{S}$ is a closed surface we first calculate $\operatorname{div}\mathbf{F} = 2(x+y)$. We will use the divergence theorem, and first describe $\mathcal{V}$ in cylindrical coordinates. The sphere $z = \sqrt{4-r^2}$ and the surface $z = 1/r$ meet where

$$(1/r)^2 = 4 - r^2,\ r > 0,$$

which yields $r^2 = 2 \pm \sqrt{3}$. The two values of r are $r_+ = \sqrt{2+\sqrt{3}}$ and $r_- = \sqrt{2-\sqrt{3}}$.

By the divergence theorem the total flux in question equals

$$\begin{aligned}\int_{\mathcal{V}} \operatorname{div}\mathbf{F}\,dV &= \int_{\theta=0}^{2\pi} \int_{r=r_-}^{r_+} \int_{z=1/r}^{\sqrt{4-r^2}} 2r(\cos\theta + \sin\theta) r\,dz\,dr\,d\theta \\ &= \int_{\theta=0}^{2\pi} (\cos\theta + \sin\theta)\,d\theta \cdot \int_{r=r_-}^{r_+} \int_{z=1/r}^{\sqrt{4-r^2}} 2r^2\,dz\,dr = 0\end{aligned}$$

because the θ-integral equals 0.

9.19. (a) $\operatorname{div}\mathbf{F} = 2yz - 2x$, $\operatorname{div}\mathbf{G} = 0$, $\operatorname{curl}\mathbf{F} = \mathbf{0}$, $\operatorname{curl}\mathbf{G} = (0, 2y(1-x), 2xz - 2 - 2z + 2y)$.

(b) We take $\mathbf{N}$ to point up. Let $\mathcal{S}_1$ be the unit disk in the x,y-plane. Then $\mathcal{S}$ and $\mathcal{S}_1$ form the boundary of a solid $\mathcal{V}$. By the divergence theorem

$$\int_{\mathcal{S}} \mathbf{G} \bullet \mathbf{N}\,dS + \int_{\mathcal{S}_1} \mathbf{G} \bullet \mathbf{N}\,dS = \int_{\mathcal{V}} \operatorname{div}\mathbf{G}\,dV = 0$$

where $\mathbf{N}$ points outward from $\mathcal{V}$, i.e., upward on $\mathcal{S}$ and downward on $\mathcal{S}_1$. Thus for $\mathcal{S}_1$, $\mathbf{N}\,dS = -\mathbf{e}_3\,dx\,dy$, and

$$\int_{\mathcal{S}_1} \mathbf{G} \bullet \mathbf{N}\,dS = \int_{\mathcal{S}_1} \mathbf{G} \bullet \mathbf{e}_3\,dx\,dy = \int_{\mathcal{S}_1} x^2y\,dx\,dy = 0,$$

by symmetry with respect to the x-axis. Therefore

$$\int_{\mathcal{S}} \mathbf{G} \bullet \mathbf{N}\,dS = 0.$$

(c) Since $\operatorname{curl} \mathbf{F} = \mathbf{0}$, $\mathbf{F} = \nabla\varphi$ for some φ. By the method of Example 150, one such φ is $\varphi(x, y, z) = x^2yz - xy^2$.

(d) The endpoints of C are $(0, 0, 0)$ and $(-1, -2, 2)$. Then

$$\int_C \mathbf{F} \bullet \mathbf{T}\, ds = \int_C \nabla\varphi \bullet \mathbf{T}\, ds = \varphi(x, y, z)\Big|_{(0,0,0)}^{(-1,-2,2)} = -4 + 4 = 0.$$

9.21. As this is a circulation integral, before starting, compute

$$\operatorname{curl} \mathbf{G} = (1, 1, 1).$$

Because this is in the plane of the triangle C, the circulation will be 0.

In more detail, let $\mathcal{S}$ be the triangle bounded by C, so that $\mathcal{S}$ is a planar surface. Since that plane contains the vector from $(0, 0, 0)$ to $(2, 2, 2)$, $\operatorname{curl} \mathbf{G}$ is in the plane, so $\operatorname{curl} \mathbf{G} \bullet \mathbf{N} = 0$. By Stokes' Theorem

$$\int_C \mathbf{G} \bullet \mathbf{T}\, ds = \int_{\mathcal{S}} \operatorname{curl} \mathbf{G} \bullet \mathbf{N}\, dS = 0.$$

9.25. Since it's a flux integral to be calculated, first calculate

$$\operatorname{div} \mathbf{F} = 2yz - 2x.$$

Although this is not 0, when it is (triple) integrated over any solid with rotational symmetry about the z-axis, the result will be 0 by a symmetry argument. (Details below.) So we plan to change the surface $\mathcal{S}$ of integration to the unit disk $\mathcal{S}_1$ in the x, y-plane at zero cost.

Let $\mathcal{V}$ be the solid enclosed by $\mathcal{S}$ and $\mathcal{S}_1$. Then by the divergence theorem

$$\int_{\mathcal{S}} \mathbf{F} \bullet \mathbf{N}\, dS + \int_{\mathcal{S}_1} \mathbf{F} \bullet \mathbf{N}\, dS = \int_{\mathcal{V}} \operatorname{div} \mathbf{F}\, dV$$

where the normal $\mathbf{N}$ in the second integral points outward from $\mathcal{V}$, that is, it points down.

Let's calculate the right side first:

$$\int_{\mathcal{V}} \operatorname{div} \mathbf{F}\, dV = \int_{\mathcal{V}} (2yz + 2x)\, dV = 2 \int_{\mathcal{V}} yz\, dV + 2 \int_{\mathcal{V}} x\, dV.$$

Because $\mathcal{V}$ has rotational symmetry about the z-axis it is in particular symmetric with respect to the x-axis. But the transformation $y \to -y$ changes yz to $-yz$, so the first integral on the right equals 0. By a similar argument using symmetry with respect to the y-axis, the second integral is also 0. Thus

$$\int_{\mathcal{V}} \operatorname{div} \mathbf{F}\, dV = 0.$$

Next consider $\int_{\mathcal{S}_1} \mathbf{F} \bullet \mathbf{N}\, dS$. Since $\mathcal{S}_1$ is the unit disk and $\mathbf{N}$ points down, as we saw above, $\mathbf{N} = -\mathbf{e}_3$ and $dS = dx\,dy$. So,

$$\int_{\mathcal{S}_1} \mathbf{F} \bullet \mathbf{N}\, dS = \int_{\mathcal{S}_1} -x^2 y\, dx\, dy = 0$$

by another symmetry argument with respect to the x-axis.

Finally

$$\int_{\mathcal{S}} \mathbf{F} \bullet \mathbf{N}\, dS = \int_{\mathcal{V}} \operatorname{div} \mathbf{F}\, dV - \int_{\mathcal{S}_1} \mathbf{F} \bullet \mathbf{N}\, dS = 0 - 0 = 0.$$

9.27. The outside part $\mathcal{S}_1$ of $\mathcal{S}$ is described in terms of parameters r and θ by $\mathbf{x}_{out} = (r\cos\theta, r\sin\theta, \sqrt{4 - r^2})$. The inside part $\mathcal{S}_2$ is $\mathbf{x}_{in} = (r\cos\theta, r\sin\theta, 1/r)$. As computed in #9.17 above, the two pieces meet where $r = r_{\pm} = \sqrt{2 \pm \sqrt{3}}$, and there $z = 1/r = z_{\mp} = \sqrt{2 \mp \sqrt{3}}$, respectively. For each of $\mathcal{S}_1$ and $\mathcal{S}_2$, the underlying region of the r, θ- parameter plane is the annulus (ring)

$$0 \le \theta \le 2\pi, \quad \sqrt{2 - \sqrt{3}} \le r \le \sqrt{2 + \sqrt{3}}.$$

(a) The area is computed in an example in Chapter 7.

(b) $\operatorname{div} \mathbf{F} = 2$ so we can use the divergence theorem;

$$\begin{aligned}
\int_{\mathcal{S}} \mathbf{F} \bullet \mathbf{N}\, dS = \int_{\mathcal{V}} 2\, dV &= 2 \int_{\theta=0}^{2\pi} \int_{r=\sqrt{2-\sqrt{3}}}^{\sqrt{2+\sqrt{3}}} \int_{z=1/r}^{\sqrt{4-r^2}} r\, dz\, dr\, d\theta \\
&= 4\pi \int_{r=\sqrt{2-\sqrt{3}}}^{\sqrt{2+\sqrt{3}}} (r\sqrt{4 - r^2} - 1)\, dr = 4\pi \left(\sqrt{2} - \frac{2}{3} \right)
\end{aligned}$$