9.2. Given $\mathbf{x}_0 = (x_0, y_0)$ we have to find functions $x = x(t)$ and $y = y(t)$ such that the path $\mathbf{x}(t) = (x(t), y(t))$ satisfies $\mathbf{x}'(t) = (x'(t), y'(t)) = \mathbf{F}(\mathbf{x}(t)) = (y(t), x(t))$, and $\mathbf{x}(0) = \mathbf{x}_0$. The equations to be satisfied are

$$x'(t) = y(t), \quad y'(t) = x(t), \quad x(0) = x_0, \quad y(0) = y_0.$$

In particular, $x''(t) = x(t)$ and $y''(t) = y(t)$.

The hyperbolic functions $\cosh t = (e^t + e^{-t})/2$ and $\sinh t = (e^t - e^{-t})/2$ both equal their second derivatives. Also $\cosh 0 = 1$ and $\sinh 0 = 0$. As a first try, $x(t) = x_0 \cosh t$ and $y(t) = y_0 \cosh t$ satisfy our initial conditions, but they don't satisfy $x' = y$ and $y' = x$, so this is not a correct answer. This is corrected by taking

$$x(t) = x_0 \cosh t + y_0 \sinh t, \quad y(t) = y_0 \cosh t + x_0 \sinh t.$$

Then $\Phi_t(x_0, y_0) = (x(t), y(t)) = (x_0, y_0) \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}$.

9.4. (a) One computes that

$$\left(\frac{3t}{1+t^3}\right)^3 + \left(\frac{3t^2}{1+t^3}\right)^3 = 3\left(\frac{3t}{1+t^3}\right)\left(\frac{3t^2}{1+t^3}\right)$$

It is easy to check $\mathbf{x}(0) = (0,0) = \lim_{t\to\infty} \mathbf{x}(t)$.

(b) There seems to be a minus sign missing.

$d\mathbf{x}^\perp(t) = (-y'(t), x'(t)\,dt$. $\mathbf{x}(t) \bullet d\mathbf{x}^\perp(t) = (x, y) \bullet (-y', x')\,dt$ can be computed directly or by the observations $x/y = 1/t$, and

$$x'y - xy'\,dt = y^2 \frac{x'y - xy'}{y^2}\,dt = y^2\,d\left(\frac{x}{y}\right) = y^2\left(-\frac{1}{t^2}\right)\,dt = -\frac{9t^2}{(1+t^3)^2}.$$

To compute the area of the leaf, note that $\operatorname{div}(x, y) = 2$.

By the divergence theorem if $\mathbf{N}$ is the outward pointing unit normal, and D is the leaf-region enclosed by C,

$$\int_C \mathbf{x} \bullet \mathbf{N}\,ds = \int_D \operatorname{div} \mathbf{x}\,dA = 2\int_D dA = 2 \cdot \text{ area of } D.$$

Now $\mathbf{N}\,ds = \pm d\mathbf{x}^\perp$ so $\mathbf{x} \bullet \mathbf{N}\,ds = \pm\mathbf{x} \bullet d\mathbf{x}^\perp = \pm 9t^2/(1+t^3)^2\,dt$. So using the substitution $u = 1 + t^3$,

$$\text{area of } D \;= \frac{1}{2}\int_C \mathbf{x} \bullet \mathbf{N}\,ds = \pm\frac{1}{2}\int_0^\infty \frac{9t^2}{(1+t^3)^2}\,dt = \pm\frac{1}{2}\int_1^\infty \frac{3\,du}{u^2} = \pm\frac{3}{2}.$$

Obviously the + sign is the right choice.

(c) At the furthest out point of the loop, $\mathbf{N}$ should be $(1/\sqrt{2}, 1/\sqrt{2})$ and in particular should have positive coordinates. Now $\mathbf{N}\,ds = \pm(-y', x')dt$ and x is decreasing at that point of the loop, so x' is negative. The result is that the proper sign is given by $\mathbf{N}\,ds = (y', -x')$. Then

$$\int_C \mathbf{F} \bullet \mathbf{N}\,ds = \int_0^\infty (x^2 - y^2)y' - 2xyx'$$

9.6. Left side: The curve C is the intersection of $z = x^2 + y^2$ with the plane $z = 1$, so it's the unit circle in the plane $z = 1$. Since $\mathbf{N}$ points up, C is traversed counterclockwise when viewed from above. So C is parametrized by

$$\mathbf{x}(t) = (\cos t, \sin t, 1),\ 0 \le t \le 2\pi.$$

Then

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_0^{2\pi} (\sin^2 t, \cos t, 1) \bullet (-\sin t, \cos t, 0)\, dt = \int_0^{2\pi} -\sin^3 t + \cos^2 t\, dt = \pi.$$

Right side: parametrize $\mathcal{S}$ by x and y,

$$\mathbf{x} = (x, y, x^2 + y^2),\ \mathbf{T}_x = (1, 0, 2x),\ \mathbf{T}_y = (0, 1, 2y),\ \mathbf{T}_x \times \mathbf{T}_y = (-2x, -2y, 1)$$

Because the z coordinate of $\mathbf{T}_x \times \mathbf{T}_y$ is positive, $\mathbf{T}_x \times \mathbf{T}_y$ gives the preferred direction of the normal vector.

Next, $\operatorname{curl} \mathbf{F} = (0, 0, 1 - 2y)$. So letting D be the unit disk in the x, y-plane,

$$\int_{\mathcal{S}} \operatorname{curl} \mathbf{F} \bullet \mathbf{N}\, dS = \int_D (0, 0, 1 - 2y) \bullet (-2x, -2y, 1)\, dx\, dy = \int_D dx\, dy - \int_D 2y\, dx\, dy = \pi - 0 = \pi.$$

9.8. $\operatorname{curl} \mathbf{F} = \mathbf{0}$, $\operatorname{curl} \mathbf{G} = (0, xy, *)$, $\operatorname{curl} \mathbf{H} = (0, y - 1, *)$. Only $\mathbf{F}$ is a gradient. $\mathbf{F} = \nabla\varphi$ where $\varphi(x, y, z) = xy^3 + xyz^2$. φ can be found by a good guess and check, or systematically as follows.

In detail, (where $*$ means "it doesn't matter what it is"),

$$\begin{aligned}
\varphi(x, y, z) &= \int_0^x \mathbf{F}(t, 0, 0) \bullet \mathbf{e}_1\, dt + \int_0^y \mathbf{F}(x, t, 0) \bullet \mathbf{e}_2\, dt + \int_0^z \mathbf{F}(x, y, t) \bullet \mathbf{e}_3\, dt \\
&= \int_0^x (0, 0, 0) \bullet \mathbf{e}_1\, dt + \int_0^y (*, 3t^2 x, 0) \bullet \mathbf{e}_2\, dt + \int_0^z (*, *, 2xyt) \bullet \mathbf{e}_3\, dt = \int_0^x 0\, dt + \int_0^y 3t^2 x\, dt + \int_0^z 2xyt\, dt \\
&= xy^3 + xyz^2
\end{aligned}$$

9.10. By the Divergence Theorem, the outward flux equals

$$\begin{aligned}
\int_{\mathcal{V}} \operatorname{div} \mathbf{F}\, dV &= \int_{\mathcal{V}} (x^2 + y + 3z)\, dV = \int_{x=0}^1 \int_{y=2}^4 \int_{z=1}^5 (x^2 + y + 3z)\, dz\, dy\, dx \\
&= \int_{x=0}^1 \int_{y=2}^4 \int_{z=1}^5 x^2\, dz\, dy\, dx + \int_{x=0}^1 \int_{y=2}^4 \int_{z=1}^5 y\, dz\, dy\, dx + \int_{x=0}^1 \int_{y=2}^4 \int_{z=1}^5 3z\, dz\, dy\, dx \\
&= \frac{8}{3} + 24 + 72
\end{aligned}$$

9.12. $\operatorname{div} \mathbf{F} = 2$ so the divergence theorem may be useful.

Let $\mathcal{S}_1$ be the elliptical disk $4x^2 + 9y^2 = 27$ in the plane $z = 3$ and let $\mathcal{V}$ be the solid between $\mathcal{S}$ and $\mathcal{S}_1$. (This is the same $\mathcal{V}$ as in problem 14.) Using the upward pointing normal $\mathbf{N}$ on $\mathcal{S}_1$, so that both $\mathbf{N}$'s point into $\mathcal{V}$, the divergence theorem gives

$$\int_{\mathcal{S}} \mathbf{F} \bullet \mathbf{N}\, dS + \int_{\mathcal{S}_1} \mathbf{F} \bullet \mathbf{N}\, dS = -\int_{\mathcal{V}} 2\, dV = -2(\text{volume of } \mathcal{V}).$$

Next, parametrize $\mathcal{S}_1$ by x and y, so that $\mathbf{N}\, dS = \mathbf{e}_3\, dx\, dy$. On $\mathcal{S}_1$, $\mathbf{F} \bullet \mathbf{N}\, dS = (x, 0, z) \bullet \mathbf{e}_3\, dx\, dy = z\, dx\, dy = 3\, dx\, dy$ so

$$\int_{\mathcal{S}_1} \mathbf{F} \bullet \mathbf{N}\, dS = \int_{\mathcal{S}_1} 3\, dx\, dy = 3(\text{area of } \mathcal{S}_1), \text{ and therefore}$$

$$\int_{\mathcal{S}} \mathbf{F} \bullet \mathbf{N}\, dS = -3A - 2V$$

where A and V are the area of $\mathcal{S}_1$ and the volume of $\mathcal{V}$, respectively.

To calculate A, $\mathcal{S}_1$ is an ellipse with major and minor axes $a = \sqrt{27}/2$ and $b = \sqrt{27}/3$ so $A = \pi ab = \pi(27/6) = (9/2)\pi$.

To calculate V, let D be the ellipse $4x^2 + 9y^2 = 27$ in the x, y-plane. Then, making the change of variables $u = 2x$, $v = 3y$, so that $x = u/2$ and $y = v/3$, we have a Jacobian factor of $1/6$:

$$V = \int_D (\sqrt{36 - 4x^2 - 9y^2} - 3)\, dA = \int_{D'} (\sqrt{36 - u^2 - v^2} - 3) \cdot \frac{1}{6}\, du\, dv$$

where D' is the disk $u^2 + v^2 = 27$ in the (u, v)-plane.

Switching to polar coordinates via $u = r\cos\theta$, $v = r\sin\theta$ we have

$$V = \frac{1}{6} \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{27}} (\sqrt{36 - r^2} - 3) r\, dr\, d\theta = \frac{2\pi}{6} \left(-\frac{1}{3}(36 - r^2)^{3/2} - \frac{3r^2}{2} \right) \Bigg|_0^{\sqrt{27}} = \frac{\pi}{3} \left(63 - \frac{81}{2} \right) = \frac{15\pi}{2}.$$

Putting it all together, $\displaystyle\int_{\mathcal{S}} \mathbf{F} \bullet \mathbf{N}\, dS = -3A - 2V = -\pi((27/2) + 15) = -57\pi/2.$

9.14. $\operatorname{div} \mathbf{F} = 2$ and $\operatorname{curl} \mathbf{F} = (0, 0, 1) = \mathbf{e}_3$ so the divergence theorem and Stokes' theorem may be useful.

(a) By the divergence theorem, $\displaystyle\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{N}\, dS = \int_{\mathcal{V}} 2\, dV.$

The change of coordinates $u = 2x$, $v = 3y$, $w = z$ transforms the ellipsoid into the sphere $u^2 + v^2 + w^2 = 36$ in u, v, w-space, and transforms $\mathcal{V}$ into the part $\mathcal{V}'$ of that sphere above the plane $w = 3$. Also $[D_{\mathbf{x}}(\mathbf{u})] = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ so $\det[D_{\mathbf{x}}(\mathbf{u})] = 1/6$. The corresponding u, v, w-integral can be evaluated by cylindrical or spherical coordinates, we use cylindrical, i.e., $u = r\cos\theta$ and $v = r\sin\theta$, $w = w$:

$$\int_{\mathcal{V}} 2\, dV = \int_{\mathcal{V}'} 2 \cdot (1/6)\, du\, dv\, dw = \int_{\theta=0}^{2\pi} \int_{r=0}^{3\sqrt{2}} \int_{w=3}^{\sqrt{36-r^2}} (1/3) r\, dw\, dr\, d\theta = 2\pi \int_{r=0}^{3\sqrt{2}} (1/3)(\sqrt{36 - r^2} - 3) r\, dr$$

Substituting $t = 36 - r^2$ gives

$$\frac{2\pi}{3}\frac{(-1)}{2}\int_{36}^{18}(\sqrt{t}-3)\,dt = \frac{\pi}{3}\left(\frac{2}{3}t^{3/2}-3t\right)\Bigg|_{18}^{36} = \frac{\pi}{3}(90-36\sqrt{2})$$

(b) We use Stokes' Theorem; the oriented surface in Stokes' Theorem, $\mathcal{S}_1$, whose boundary is C, can be taken to be the ellipse $\mathcal{S}_1$ defined by $4x^2+9y^2 \leq 27$ in the plane $z=3$, with $\mathbf{N}$ pointing up, so that $\mathbf{N} = \mathbf{e}_3$. Since also $\operatorname{curl}\mathbf{F} = \mathbf{e}_3$, $\operatorname{curl}\mathbf{F}\cdot\mathbf{N}\,dS = \mathbf{e}_3\cdot\mathbf{e}_3\,dx\,dy = dx\,dy$ and so

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_{\mathcal{S}} dx\,dy = \text{ area of } \mathcal{S} = \pi(\sqrt{27}/2)(\sqrt{27}/3) = 9\pi/2.$$

9.16. (a) $\operatorname{div}\mathbf{F} = 2(x+y)$, $\operatorname{curl}\mathbf{F} = \mathbf{0}$, $\operatorname{div}\mathbf{G} = 0$, $\operatorname{curl}\mathbf{G} = (2-2z, 2z-2, 0)$.

(b) Let $\mathcal{V}$ be the solid unit sphere. Then by the Divergence Theorem,

$$\int_{\mathcal{S}} \mathbf{G}\cdot\mathbf{N}\,dS = \int_{\mathcal{V}} \operatorname{div}\mathbf{G}\,dV = 0.$$

(c) $\mathbf{F}$ is a gradient since $\operatorname{curl}\mathbf{F} = \mathbf{0}$ in all of $\mathbf{R}^3$. By the method of Example 150, or by inspired guessing $\mathbf{F} = \nabla\varphi$ for $\varphi(x,y,z) = xy + xz^2 + yz^2$.

(d) Since $\mathbf{F}$ is a gradient, its circulation around any closed path is 0.

9.18. See 9.24.

9.20. We assume $\mathcal{S}$ is the part of the cone lying *inside* the cylinder. Notice that $\operatorname{div}\mathbf{G} = 0$ (!). So we replace $\mathcal{S}$ by an easier surface $\mathcal{S}_1$ with the same boundary curve, namely $\mathcal{S}_1$ is the circular disk $x^2+y^2 \leq 4$ in the plane $8 - z = 2$, i.e. the plane $z = 6$. Together $\mathcal{S}$ and $\mathcal{S}_1$ form the boundary of a solid cone $\mathcal{V}$ resting on the base $\mathcal{S}_1$. By the divergence theorem

$$\int_{\mathcal{S}} \mathbf{G}\bullet\mathbf{N}\,dS + \int_{\mathcal{S}_1} \mathbf{G}\bullet\mathbf{N}\,dS = \int_{\mathcal{V}} \operatorname{div}\mathbf{G}\,dV = 0,$$

where $\mathbf{N}$ is the outward normal; thus in the second surface integral, $\mathbf{N}$ points down.

Using the parametrization $x = x$, $y = y$, $z = 6$ for $\mathcal{S}_1$ gives $\mathbf{N} = \pm\mathbf{e}_3$, so $\mathbf{N} = -\mathbf{e}_3$. Let D be the disk $x^2+y^2 \leq 4$ in the x,y-(parameter) plane. Then putting it all together,

$$\int_{\mathcal{S}} \mathbf{G}\bullet\mathbf{N}\,dS = -\int_{\mathcal{S}_1} \mathbf{G}\bullet(-\mathbf{e}_3)\,dS = \int_D x^2y\,dx\,dy = 0,$$

the last equality because D is symmetric with respect to the x-axis but the integrand is odd (changes sign) with respect to y.

9.22. (a) $\operatorname{curl} \mathbf{F} = (-8y, 0, 4)$ and $\operatorname{curl} \mathbf{G} = \mathbf{0}$.

(b) $\mathbf{G}$ is a gradient because its curl is $\mathbf{0}$ on all of $\mathbf{R}^3$ (and $\mathbf{R}^3$ is simply connected). To find φ such that $\mathbf{G} = \nabla\varphi$, use the method of 9.8 above or a good guess to get $\varphi(x, y, z) = 2xy + xz - 2y^2 z$.

(c) Let D be the unit disk in the x, y-plane. Given the usual counterclockwise orientation of the boundary C of D, it must be that $\mathbf{N}$, the preferred normal for Stokes' Theorem, points in the positive z-direction, i.e., $\mathbf{N} = (0, 0, 1)$. Then $\mathbf{N}\, dS = (0, 0, 1)\, dx\, dy$. By Stokes' Theorem the requested circulation is

$$\int_C \mathbf{F} \bullet \mathbf{T}\, ds = \int_D \operatorname{curl} \mathbf{F} \bullet \mathbf{N}\, dS = \int_D (-8y, 0, 4) \bullet (0, 0, 1)\, dx\, dy = \int_D 4\, dx\, dy = 4\pi.$$

Of course the line integral could be evaluated directly, but it would be a longer calculation.

9.24. $\operatorname{curl} \mathbf{G} = (2 - 2z, 2z - 2, 0)$. Use Stokes' Theorem, with the surface $\mathcal{S}$ being the triangle with the three given vertices, part of the plane $x + y + z = 1$. With the given direction of circulation the normal $\mathbf{N}$ should point in the direction of increasing x, y, and z, so $\mathbf{N} = (1/\sqrt{3})(1, 1, 1)$. But then $\operatorname{curl} \mathbf{G} \bullet \mathbf{N} = 0$. By Stokes' Theorem,

$$\int_C \mathbf{G} \bullet \mathbf{T}\, ds = \int_{\mathcal{S}} \operatorname{curl} \mathbf{G} \bullet \mathbf{N}\, dS = 0.$$

9.26. On the intersection $x^2 = 4 - y^2$, so we can use the polar coordinate θ to parametrize x and y, and $z = x^2$: $x = 2\cos\theta$, $y = 2\sin\theta$, $z = 4\cos^2\theta$, $0 \le \theta \le 2\pi$. With this $\mathbf{x} = \mathbf{x}(\theta)$,

$$\int_C \mathbf{F} \bullet \mathbf{T}\, ds = \int_0^{2\pi} \mathbf{F}(\mathbf{x}(\theta)) \bullet \mathbf{x}'(\theta)\, d\theta = \int_0^{2\pi} (2\cos\theta, 2\cos\theta, 4\cos^2\theta) \bullet (-2\sin\theta, 2\cos\theta, -8\cos\theta\sin\theta)\, d\theta$$

$$= 4\int_0^{2\pi} -\cos\theta\sin\theta + \cos^2\theta - 8\cos^3\theta\sin\theta\, d\theta = 0 + 4\pi + 0 = 4\pi.$$

9.28. $\operatorname{curl} \mathbf{F} = (0, 0, 1)$. We'll use Stokes' Theorem.

Let $\mathcal{S}$ be the triangle surrounded by C. The plane in which $\mathcal{S}$ lies contains $(0, 0, 1) - (0, 0, 0) = (0, 0, 1)$. Therefore $\operatorname{curl} \mathbf{F}$ is parallel to $\mathcal{S}$. So $\operatorname{curl} \mathbf{F} \bullet \mathbf{N} = 0$ for any normal vector $\mathbf{N}$ to $\mathcal{S}$. With Stokes' Theorem,

$$\int_C \mathbf{F} \bullet \mathbf{T}\, ds = \int_{\mathcal{S}} \operatorname{curl} \mathbf{F} \bullet \mathbf{N}\, dS = 0.$$