Solutions for the Even Exercises from Chapter 5

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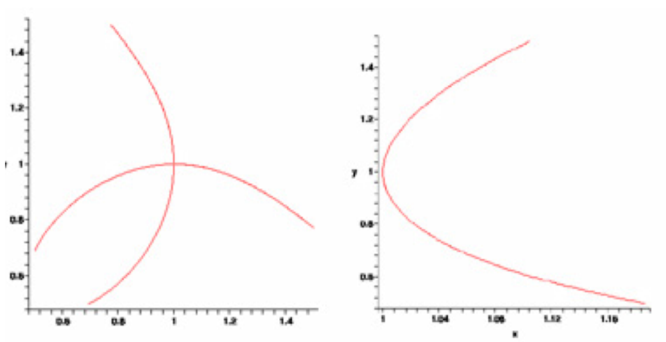
5.2: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = x^2y + yx - xy^2 \ .$$

(a) Compute the gradient of f, and find all critical points of f.

(b) Find a parameterization of the tangent line to the level curve of f through the point $(1, 1)$.

(c) Could either of the following be a contour plot of f? If so, which could, and which could not? Explain your answer.



SOLUTION: For (**a**), we first compute

$$\nabla f(x, y) = ((2x + 1 - y)y \ , \ (x + 1 - 2y)x) \ .$$

The set of critical points is the solution set of the system of equations

$$\begin{aligned} (2x + 1 - y)y &= 0 \\ (x + 1 - 2y)x &= 0 \ . \end{aligned}$$

From the first equation, either $y = 0$ or $2x + 1 - y = 0$ (or both). If $y = 0$, the second equation reduces to $(x + 1)x = 0$ and so

$$(-1, 0) \qquad \text{and} \qquad (0, 0)$$

are critical points with $y = 0$.

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Likewise, from the second equation, either $x = 0$ or $x + 1 - 2y = 0$ (or both). If $x = 0$, the first equation reduces to $(1 - y)y = 0$ and so

$$(0, 1)$$

is another critical point with $x = 0$, besides $(0, 0)$ which we already have. Finally if $x \neq 0$ and $y \neq 0$, the system reduces to

$$\begin{aligned} 2x + 1 - y &= 0 \\ x + 1 - 2y &= 0 \ . \end{aligned}$$

Subtracting twice the second equation from the first, we get $3y = 1$, or $y = 1/3$, and then from either equation we see $x = -1/3$. Thus the only other critical point is $(-1/3, 1/3)$.

For (**b**), we evaluate

$$\nabla f(1, 1) = (2, 0) \ .$$

Hence the tangent line is parameterized by

$$\mathbf{x}(t) = (1, 1) + t(2, 0)^{\perp} = (1, 1 + 2t) \ .$$

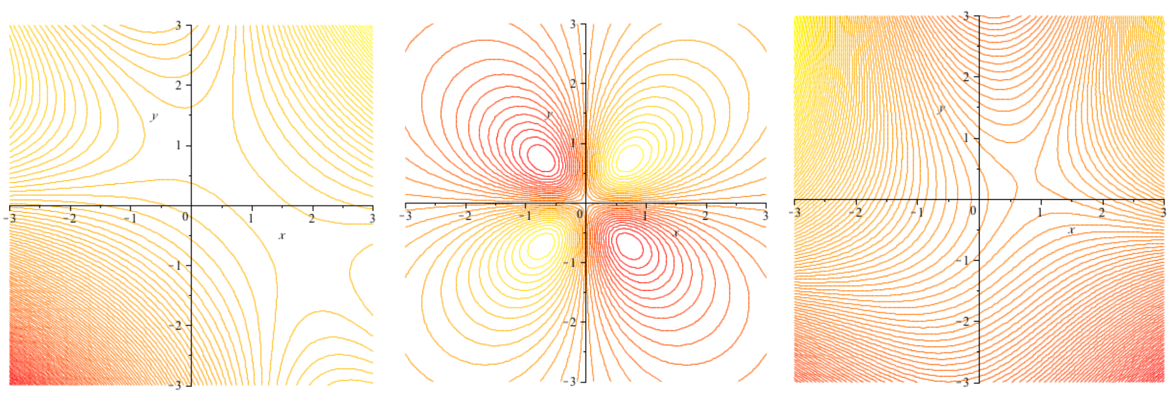
For (**c**), the first graph shows contour lines crossing at $(1, 1)$, which would mean that $(1, 1)$ is a critical point. But we see from part (**a**) (or (**b**)) that $(1, 1)$ is not a critical point of f. Hence the first graph cannot be a contour plot of f.

The second plot shows a curve passing through $(1, 1)$ with a vertical tangent line there. Since we found in part (**b**) that the level curve of f passing through $(1, 1)$ has a vertical tangent line, the second graph could (and does) show a contour curve of f.

5.4: Let $f(x, y) = \dfrac{xy}{(1 + x^2 + y^2)^2}$.

(a) Find all of the critical points of f, and find the value of f at each of the critical points.

(b) One of the following is a contour plot for f. Which one is it? Explain your answer to receive credit.



SOLUTION: For (**a**), we comute

$$\nabla f(x, y) = \frac{1}{(1 + x^2 + y^2)^3} \left(y(3x^2 - y^2 - 1) \ \ x(1 + x^2 - 3y^2) \right) \ .$$

The system of equations for the critical points is

$$\begin{aligned} y(3x^2 - y^2 - 1) &= 0 \\ x(1 + x^2 - 3y^2) &= 0 \ . \end{aligned}$$

If $y = 0$, the second equation reduces to $x(1 + x^2) = 0$, so then $x = 0$. If $x = 0$, the first equation reduces to $y(1 + y^2) = 0$, so then $y = 0$. Hence the only critical point with *either* $x = 0$ or $y = 0$ is $(0, 0)$.

To find the other critical points, we need only solve the reduced system

$$\begin{aligned} 3x^2 - y^2 - 1 &= 0 \\ 1 + x^2 - 3y^2 &= 0 \ . \end{aligned}$$

Adding these equations, we get $4x^2 - 4y^2 = 0$, and so $x^2 = y^2$ at any critical point. Using this to eliminate y^2 from the first equation, we have $2x^2 = 1$ or $x = \pm 2^{-1/2}$. We then have $y = \pm 2^{-1/2}$ without any correlation of the signs. Hence we get four more critical points:

$$(\pm 2^{-1/2}, \pm 2^{-1/2}) \ .$$

We compute $f(0, 0) = 0$, and

$$f(\pm 2^{-1/2}, \pm 2^{-1/2}) = \pm \frac{1}{8}$$

where the sign on the right is positive if both signs on the left are the same, and is negative otherwise.

For (**b**), only the middle plot shows 5 critical points, nad they are all in the right places.

5.6 Let $f(x, y) = (x + y)^4 + (x - y)^2$. Find the minimum and maximum values of f in the unit circle $x^2 + y^2 = 1$, and all of the points in the unit circle at which f takes on these values.

SOLUTION: We define $g(x, y) = x^2 + y^2 - 1$ so that the constraint curve is the solution set of $g(x, y) = 0$. We then compute

$$\nabla f(x, y) = 2\big(2(x + y)^3 + (x - y) \ , \ 2(x + y)^3 - (x - y)\big) \ ,$$

and

$$\nabla g(x, y) = 2(x, y) \ .$$

Hence, after some algebra,

$$\det\left(\begin{bmatrix} \nabla f(x, y) \\ \nabla g(x, y) \end{bmatrix}\right) = 4(x^2 - y^2)(2(x + y)^2 + 1) \ .$$

Since $4(2(x + y)^2 + 1)$ is never zero, the system of equations solved by the points whihc might possibly be optimizers is

$$\begin{aligned} x^2 - y^2 &= 0 \\ x^2 + y^2 &= 1 \end{aligned}$$

Adding the equations, we see $2x^2 = 1$ and so $x = \pm 2^{-1/2}$, and then from the first equation, also $y = \pm 2^{-1/2}$, and there is no correlation of the signs. Thus we have four candidates:

$$(\pm 2^{-1/2}, \pm 2^{-1/2}) \ .$$

We compute $f(\pm 2^{-1/2}, \pm 2^{-1/2}) = 4$ if the signs on the left are the same, and $f(\pm 2^{-1/2}, \pm 2^{-1/2}) = 2$ if the signs on the left are different. Hence $(2^{-1/2}, 2^{-1/2})$ and $(-2^{-1/2}, -2^{-1/2})$ are the maximizers, while $(-2^{-1/2}, 2^{-1/2})$ and $(2^{-1/2}, -2^{-1/2})$ are the minimizers.

5.8: Let $f(x,y) = \dfrac{xy}{(1+x^2+y^2)^2}$. Find the minimum and maximum values of f in the set where

$$|x| + |y| \leq 1 \ .$$

Also, find all of the minimizers and maximizers.

SOLUTION: We have already computed the gradient of f and all critical points of f in **5.4**, finding:

$$\nabla f(x,y) = \frac{1}{(1+x^2+y^2)^3}(y(1+y^2-3x^2) \ , \ x(1+x^2-3y^2))$$

and finding $(0,0)$ and $(\pm 2^{-1/2}, \pm 2^{-1/2})$ as the critical points.

However, if $(x,y) = (\pm 2^{-1/2}, \pm 2^{-1/2})$, then $|x| + |y| = 2^{1/2} > 1$, so none of the critical points $(\pm 2^{-1/2}, \pm 2^{-1/2})$ lie in the region D. The only critical point in D is $(0,0)$.

Next, we look for solutions of Lagrange's equations on the boundary of the region D. The equation for the boundary is $g(x,y) = 0$ where

$$g(x,y) = |x| + |y| - 1 \ .$$

Let $\mathrm{sgn}(t)$ be given by

$$\mathrm{sgn}(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \\ 0 & t = 0 \end{cases} \ .$$

Then for $x \neq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}x}|x| = \mathrm{sgn}(x)$$

while $|x|$ is not differentiable at $x = 0$. Since the same is true for y in place of x,

$$\nabla g(x,y) = (\mathrm{sgn}(x) \ , \ \mathrm{sgn}(y))$$

provided neither x nor y is zero, and otherwise $\nabla g(x,y)$ is undefned.

Lagrange's system of equations is

$$\begin{aligned} \det\left(\begin{bmatrix} \nabla f(x,y) \\ \nabla g(x,y) \end{bmatrix}\right) &= 0 \\ g(x,y) &= 0 \ , \end{aligned} \tag{0.1}$$

The first equation in the system reduces to

$$\mathrm{sgn}(x)y(1+x^2-3y^2) = \mathrm{sgn}(y)x(1+y^2-3x^2) \ . \tag{0.2}$$

If x and y have the same sign,

$$\operatorname{sgn}(x)y = |y| \qquad \text{and} \qquad \operatorname{sgn}(y)x = |x| \ .$$

Otherwise,

$$\operatorname{sgn}(x)y = -|y| \qquad \text{and} \qquad \operatorname{sgn}(y)x = -|x| \ .$$

Either way,

$$|y|(1 + x^2 - 3y^2) = |x|(1 + y^2 - 3x^2) \ . \tag{0.3}$$

This means that $x^2 - 3y^2 = y^2 - 3x^2$, and hence $x^2 = y^2$. Combining this with $|x| + |y| = 1$, we deduce $(x, y) = \pm(1/2, 1/2)$. On the other hand, if x and y have the opposite sign, (0.2) becomes

$$1 + x^2 - 3y^2 = 3x^2 - y^2 - 1 \ .$$

which is

$$x^2 + y^2 = 1 \ .$$

5.10 Let $\mathcal{S}$ be closed upper hemisphere of the unit sphere in $\mathbb{R}^3$. Let $f(x, y, z) = xyz$. Find the minimum and maximum values of f on $\mathcal{S}$, and all of the points at which f takes on these values. Explain how you are taking into account both of the constraints $x^2 + y^2 + z^2 = 1$ and $z \geq 0$.

SOLUTION: First, we compute

$$\nabla f(x, y, z) = (yz, xz, xy) \ .$$

We can write the constraint that (x, y, z) lies on the unit sphere as $g(x, y, z) = 0$ where

$$g(x, y, z) = x^2 + y^2 + z^2 = 1 \ .$$

Hence

$$\nabla g(x, y, z) = 2(x, y, z) \ .$$

We then compute

$$\nabla f \times \nabla g(x, y, z) = 2\big(x(z^2 - y^2), y(x^2 - z^2), z(y^2 - x^2)\big) \ .$$

Hence the candidates for maximizers or minimizers of f on the whole sphere are the solutions of

$$\begin{aligned} x(z^2 - y^2) &= 0 \\ y(x^2 - z^2) &= 0 \\ z(y^2 - x^2) &= 0 \\ x^2 + y^2 + z^2 &= 1 \ . \end{aligned}$$

If $x = 0$, then from the second (or third) equation, either $y = 0$ or $z = 0$, in any solution where one coordinate is zero, there must be at least two zero coordinates. The final equation precludes having three zero coordinates, so the solutions where some coordinate is zero are

$$(\pm 1, 0, 0) \ , \quad (0, \pm 1, 0) \quad \text{and} \quad (0, 0, \pm 1) \ .$$

When no coordinate is zero, the system reduces to

$$\begin{aligned} z^2 - y^2 &= 0 \\ x^2 - z^2 &= 0 \\ y^2 - x^2 &= 0 \\ x^2 + y^2 + z^2 &= 1 \ . \end{aligned}$$

and hence

$$x^2 = y^2 = z^2 = \frac{1}{3} \ .$$

Thus, these solutions are

$$(\pm 3^{-1/2}, \pm 3^{-1/2}, \pm 3^{-1/2}) \ .$$

We can now find the minimizers and maximizers of f on the *whole sphere* by evaluating f at each of the candidate points. We find

$$f(\pm 1, 0, 0) = f(0, \pm 1, 0) = f(0, 0, \pm 1) = 0 \ ,$$

and

$$f(\pm 3^{-1/2}, \pm 3^{-1/2}, \pm 3^{-1/2}) = \pm 3^{-3/2}$$

where the sign on the right is positive if the are an even number of negative signs on the left, and is negative otherwise.

Since $(3^{-1/2}, 3^{-1/2}, 3^{-1/2}$ and $(-3^{-1/2}, -3^{-1/2}, 3^{-1/2}$ are the only maximizers of f on the whole sphere that lie in the upper hemisphere, there are the only maximizers of f in $\mathcal{S}$. Likewise, since $(-3^{-1/2}, 3^{-1/2}, 3^{-1/2}$ and $(3^{-1/2}, -3^{-1/2}, 3^{-1/2}$ are the only minimizers of f on the whole sphere that lie in the upper hemisphere, there are the only maximizers of f in $\mathcal{S}$. The maximum value is $3^{-3/2}$ and the minimum value is $-3^{-3/2}$.