Solutions for the even numbered exercises from Chapter 4

Eric A. Carlen[1]
Rutgers University

November 26, 2012

4.2 Let $\mathbf{v}_1 = (1,1)$ and $\mathbf{v}_2 = (3,1)$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable, and suppose that for some $\mathbf{x}_0 \in \mathbb{R}^2$, $\mathbf{v}_1 \cdot \nabla f(\mathbf{x}_0) = -2$ and $\mathbf{v}_2 \cdot \nabla f(\mathbf{x}_0) = 3$.

For $\mathbf{v} = (1,-1)$, compute $\mathbf{v} \cdot \nabla f(\mathbf{x}_0)$.

SOLUTION We will use the information $\mathbf{v}_1 \cdot \nabla f(\mathbf{x}_0) = -2$ and $\mathbf{v}_2 \cdot \nabla f(\mathbf{x}_0) = 3$ to determine $\nabla f(\mathbf{x}_0)$. Let a and b be such that $\nabla f(\mathbf{x}_0) = (a,b)$. Then these equations become

$$\begin{aligned} a + b &= -2 \\ 3a + b &= 3 \ . \end{aligned}$$

Subtracting the first equation from the second, $2a = 5$ so $a = 5/2$ and then $b = -9/2$. Thus,

$$\nabla f(\mathbf{x}_0) = \frac{1}{2}(5,-9) \ .$$

Thus,

$$(1,-1) \cdot \nabla f(\mathbf{x}_0) = \frac{1}{2}(1,-1) \cdot (5,-9) = 7 \ .$$

4.4 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = x^2y + yx - xy^2$. Let $\mathbf{x}(t)$ b given by $\mathbf{x}(t) = (t,t^2)$ Compute $\left.\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x}(t))\right|_{t=1}$.

SOLUTION We us the chain rule:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x}(t))\right|_{t=1} = \nabla f(\mathbf{x}(1)) \cdot \mathbf{x}'(1) \ .$$

We then compute

$$\mathbf{x}(1) = (1,1) \qquad \text{and} \qquad \mathbf{x}'(t) = (1,2) \ ,$$

and then

$$\nabla f(x,y) = (2xy + y - y^2, x^2 + x - 2xy) \ ,$$

so that

$$\nabla f(\mathbf{x}(1)) = (2,0) \ .$$

Therefore,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x}(t))\right|_{t=1} = (2,0) \cdot (1,2) = 2 \ .$$

4.6 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x^2y + yx - xy^2$.

(a) Compute the gradient of f, and find all critical points of f..

(b) Find the equation of the tangent plane to the graph f at the point $(1, 1)$.

SOLUTION We compute, as above, $\nabla f(x, y) = (2xy + y - y^2, x^2 + x - 2xy)$. Thus, is (x, y) a critical point when

$$\begin{aligned} 2xy + y - y^2 &= 0 \\ x^2 + x - 2xy &= 0 \ . \end{aligned}$$

To solve this system, suppose that $x = 0$. Then, from the first equation, $y = y^2$, so either $y = 0$ or $y = 1$. This gives us two solutions so far: $(0, 0)$ and $(0, 1)$.

Now let us look for solutions with $x \neq 0$. Then we may divide the second equation through by x to obtain $x + 1 - 2y = 0$, or

$$x = 2y - 1 \ . \tag{0.1}$$

Using this to eliminate x from the first equation, we obtain

$$3y^2 - 2y + y = 0 \ .$$

This quadratic equation has two roots, namely, $2/3$ and -1. For these values of y, we use (0.1) to determine the corresponding x values. Thus we get the two additional solutions $(1/3, 2/3)$ and $(-3, -1)$. Altogether, we have four critical points which are

$$(0, 0) \ , \quad (0, 1) \ , \quad (1/3, 2/3) \ , \quad (-3, -1) \ .$$

Next, since $f(1, 1) = 1$ and $\nabla f(1, 1) = (2, 0)$, the best affine approximation of f at $(1, 1)$ is given by

$$h(x, y) = f(1, 1) + \nabla f(1, 1) \cdot (x - 1, y - 1) = 1 + (2, 0) \cdot (x - 1, y - 1) = 2x - 1 \ .$$

The tangent plane is the graph of $z = h(x, y)$, so it has the equation $z = 2x - 1$, or $2x - z = 1$.

4.8 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x^3 + y^3 + 3xy$.

(a) Compute the gradient of f, and find all points (x, y) at which the tangent plane to the graph of f is horizontal.

(b) Find the equation of the tangent plane to the graph of f at the point $(1, 2)$.

(c) If you were standing at the point $(1, 1)$ and wanted to climb uphill as directly as possible, in which compass direction would you head? For purposes of this question, take the direction $\mathbf{e}_1$ to be East, and the direction $\mathbf{e}_2$ to be North.

SOLUTION We compute $\nabla f(x, y) = 3(x^2 + y, y^2 + x)$. The tangent plane at (x, y) is horizontal if and only if $\nabla(x, y) = 0$, so we have to find the critical points of f. That is, we have to solve

$$\begin{aligned} x^2 + y &= 0 \\ x + y^2 &= 0 \ . \end{aligned}$$

From the first equation $y = -x^2$. Using this to eliminate y from the second equation, we get $x + x^4 = 0$. Since $x + x^4 = x(1 + x^3)$, the two real solutions are $x = 0$ and $x = -1$. Then from $y = -x^2$, we get the two critical points

$$(0,0) \quad \text{and} \quad (-1,-1)\ .$$

Next, we compute

$$f(1,2) = 15 \qquad \text{and} \qquad \nabla f(1,2) = (9,15)\ .$$

Thus, near $(1,2)$,

$$f(x,y) \approx 15 + (9,15)\cdot(x-1,y-2) = 9x + 15y - 24\ .$$

The equation of the tangent plane then is

$$9x + 15y - z = 24\ .$$

Finally, the steepest downhill direction is the direction of $-\nabla f(1,1) = (-6,-6)$, which points Southwest.

4.10 Let $A := \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$.

(a) Compute $\det(A)$ and the matrix inverse of A.

(b) Find all solutions of the equation $A\mathbf{x} = (1,2)$.

SOLUTION We compute $\det(A) = 5 - 4 = 1$, and

$$A^{-1} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}\ .$$

Since A is invertible, there is exactly one solution of $A\mathbf{x} = (1,2)$, which is

$$A^{-1}(1,2) = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}(1,2) = (-3,8)\ .$$

You can easily check that this is a solution.

4.12 Let $A := \begin{bmatrix} 2 & 0 & 3 \\ -1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$.

(a) Compute $\det(A)$ and the matrix inverse of A.

(b) Find all solutions of the equation $A\mathbf{x} = (1,2,3)$.

SOLUTION We define $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to be the columns of A so that $A = [\mathbf{v}_1, \mathbf{v}_2 \mathbf{v}_3]$. We then compute

$$\mathbf{v}_2 \times \mathbf{v}_3 = (5,3,-6) \qquad \mathbf{v}_3 \times \mathbf{v}_1 = (4,3,-5) \quad \text{and} \quad \mathbf{v}_1 \times \mathbf{v}_2 = (-3,-2,4)\ .$$

It then follows that

$$\det(A) = \mathbf{v}_1 \cdot \mathbf{v}_2 \times \mathbf{v}_3 = 1\ ,$$

and then that

$$A^{-1} = \begin{bmatrix} 5 & 2 & -6 \\ 4 & 3 & -5 \\ -3 & -2 & 4 \end{bmatrix} .$$

Since A is invertible, there is exactly one solution of $A\mathbf{x} = (1, 2, 3brp$, which is

$$A^{-1}(1,2,3) = \begin{bmatrix} 5 & 2 & -6 \\ 4 & 3 & -5 \\ -3 & -2 & 4 \end{bmatrix} (1,2,3) = (-7, -5, 5) .$$

You can easily check that this is a solution.

4.14 Let $f(x, y) = x^2y + (x - 1)(y - 1)^2$.

(a) Find the equations for the tangent planes to the graph of $z = f(x, y)$ at $\mathbf{x}_1 = (2, -1)$ and at $\mathbf{x}_2 = (-1, 1)$.

(b) Parameterize the line that is the intersection of the two planes found in **(a)**.

(c) Let $\mathbf{x}_0 = (-3, 0, 5)$. Compute the distance from $\mathbf{x}_0$ to the line found in **(b)**, and from $\mathbf{x}_0$ to the second tangent plane found in **(a)**. The distance to the plane should be smaller than the distance to the line. Explain why.

SOLUTION For **(a)**, we compute $\nabla f(x, y) = (2xy + (y - 1)^2\ ,\ x^2 + 2(x - 1)(y - 1))$. We the evaluate

$$\nabla f(2, -1) = (0, 0) \qquad \text{and} \qquad \nabla f(-1, 1) = (-2, 1)$$

and

$$f(2, -1) = 0 \qquad \text{and} \qquad f(-1, 1) = 1 .$$

Thus the equation for the tangent plane at $(2, -1)$ is

$$z = 0$$

and the equation for the tangent plane at $(-1, 1)$ is

$$z = 1 + (-2, 1) \cdot (x + 1, y - 1) = -2x + y - 2 .$$

For **(b)**, we substitute $z = 0$ into $z = -2x + y - 2$ to learn $y = 2(x + 1)$. Thus every point on the line has the form $(x, 2(x + 1), 0)$, and the parameterization is

$$\mathbf{x}(t) = (t, 2(t + 1), 0) .$$

For **(c)**, Note the $\mathbf{x}(0) := (0, 2, 0)$ is on the plane, and therefore on both lines. We may take this as the base point for both the plane and the line. Since the normal vector to the plane is $(-2, 1, -1)$, the distance from $(-3, 0, 5)$ to the plane is

$$\frac{|((-3, 0, 5) - (0, 2, 0)) \cdot (-2, 1, -1)|}{\|(-2, 1, -1)\|} = \frac{1}{\sqrt{6}} .$$

Since the direction vector of the line is $(1, 2, 0)$, the distance from $(-3, 0, 5)$ to the line is

$$\frac{|((-3, 0, 5) - (0, 2, 0)) \times (2, 1, 0)|}{\|(2, 1, 0)\|} = \frac{\|(-10, 5, -4)\|}{\|(1, 2, 0)\|} = \frac{\sqrt{141}}{\sqrt{5}} .$$

The distance to the plane is smaller. This is because the distance to plane is them minimum of

$$\|\mathbf{x} - ((-3,0,5)\| = \sqrt{(x+3)^2 + y^2 + (z-5)^2}\ ,$$

where (x,y,z) ranges over the plane, while the distance to the line is the minimum of the *same function* over the line only, which is a subset of the plane. For any non-emety sets $A \subset B$ on which any real valued function f is defined, the greatest lower bound of f on B is certainly a lower bound of f on the smaller set A, and hence is no greater than the gretest lower bound of f on A. If f has minimizers in both A and B, this means that the minimum of f on B is no greater than the minimum of f on A.

4.16 Let $\mathbf{f}(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}))$ where $f(x,y) = x^3 + xy$, and $g(x,y) = 1 - 4y^2 - x^2$. Let $\mathbf{x}_0 = \mathbf{e}_1$.

(a) Compute $[D_{\mathbf{f}}(\mathbf{x})]$ and $[D_{\mathbf{f}}(\mathbf{x}_0)]$.

(b) Use $\mathbf{x}_0$ as a starting point for Newton's method, and compute the next approximate solution $\mathbf{x}_1$.

(c) Evaluate $\mathbf{f}(\mathbf{x}_1)$, and compare this with $\mathbf{f}(\mathbf{x}_0)$.

SOLUTION We compute $\nabla f(x,y) = (3x^2 + y, x)$ and $\nabla g(x,y) = (-2x\ , -8y)$. Therefore

$$[D_{\mathbf{f}}(\mathbf{x})] = \begin{bmatrix} \nabla f(\mathbf{x}) \\ \nabla g(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} 3x^2 + y & x \\ -2x & -8y \end{bmatrix} .$$

Evaluating at $x = 1$, $y = 0$, we find

$$[D_{\mathbf{f}}(\mathbf{x}_0)] = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} .$$

We next compute $\mathbf{f}(\mathbf{x}_0) = (f(1,0), g(1,0)) = (1,0)$, which is not really close to $\mathbf{0}$. To (hopefully) get a better approximation, we solve for

$$\begin{aligned} \mathbf{x}_1 &= (1,0) - \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}^{-1} (1,0) \\ &= (1,0) - \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix} (1,0) \\ &= (1,-1)\ . \end{aligned}$$

We now evaluate $\mathbf{f}(\mathbf{x}_1) = (f(1,-1), g(1,-1)) = (0,-4)$. This is not great. Getting a reasonably good starting point, by graphical methods for example, is important.

4.17 Let $\mathbf{f}(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}))$ where $f(x,y) = \sqrt{x} + \sqrt{y} - 3$, and $g(x,y) = x^2 + y^2 - 18$. Compute $\mathbf{f}(\mathbf{x}_0)$ for $\mathbf{x}_0 = (3,3)$. Does this look like a reasonable starting point? Compute $[D_{\mathbf{f}}(\mathbf{x}_0)]$. What happens if you try to use $\mathbf{x}_0$ as your starting point for Newton's method?

SOLUTION We compute $\mathbf{f}(3,3) = (2\sqrt{3} - 3, 0)$. Since $2\sqrt{3} - 3 \approx 0.54$, this looks reasonable.

Writing $\mathbf{f} = (f,g)$, we compute $\nabla f(x,y) = (x^{-1/2}/2, y^{-1/2}/2)$ and $\nabla g(x,y) = (2x, 2y)$. Thus, $\nabla f(3,3) = (3^{-1/2}/2, 3^{-1/2}/2)$ and $\nabla g(3,3) = (6,6)$, and

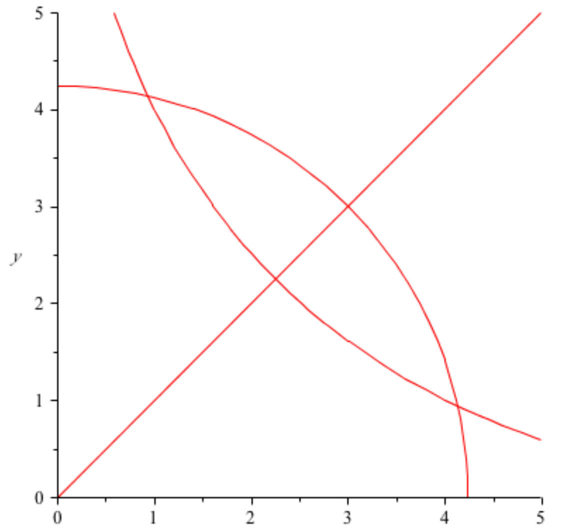
$$[D_{\mathbf{f}}(\mathbf{x}_0)] = \begin{bmatrix} 3^{-1/2}/2 & 3^{-1/2}/2 \\ 6 & 6 \end{bmatrix} .$$

The determinant of this matrix is zero, and even worse,

$$[D_{\mathbf{f}}(\mathbf{x}_0)]\,(\mathbf{x}-\mathbf{x}_0) = -\mathbf{f}(\mathbf{x}_0)$$

has no solution. So for this starting point, Newton's method breaks down at the first step. It was not a good starting point after all.

The reason for this is not hard to see. Both functions f and g are symmetric in x and y: That is, for all $(a,b) \in \mathbb{R}^2$, $f(a,b) = f(b,a)$ and $g(a,b) = g(b,a)$. Therefore, if (a,b) is a solution of the system, so is (b,a), its reflection about the line $y = x$. Indeed, here is a plot of the solutions sets of $f(x,y) = 0$, $g(x,y) = 0$ nad the line $y = x$.



You see there are exactly two solutions, where the two curves intersect. Both of them are equidistant form the starting point $(3,3)$ which is here the line intersects the circle. There is a dilemma: Should then next point move closer to the solution above the line $y = x$, or the solution below this line?

This dilemma is seen to be unresolvable, since again by symmetry it is clear the the gradients of both f and g at any point along the line $y = x$ are multiples of $(1,1)$. Hence the Jacobian of $\mathbf{f}$ cannot be invertible.

4.18: Let $f(x,y)$ and $g(x,y)$ be given by

$$f(x,y) = (x+1)^2 + (y+1)^2 - 4 \qquad \text{and} \qquad g(x,y) = 4(x-1)^2 + (y-1)^2 - 5\ .$$

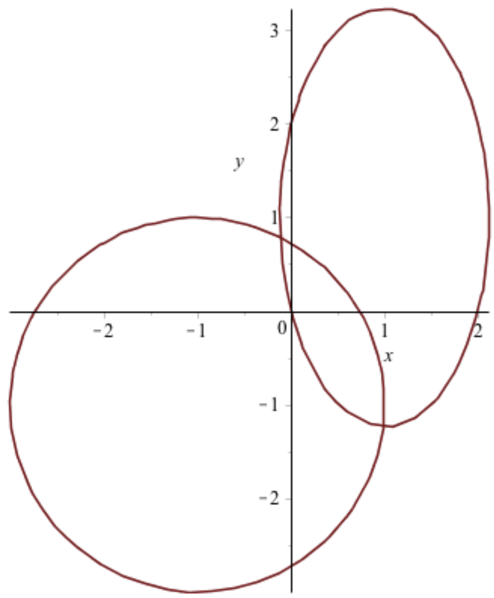
(a) The equation $f(x,y) = 0$ describes a circle. The equation $g(x,y) = 0$ describes an ellipse. Sketch a plot of the circle and the ellipse, and determine how many solutions there are of the system

$$\mathbf{f}(x,y) := (f(x,y), g(x,y)) = \mathbf{0}\ .$$

Note: *It is possible to exactly solve this system by algebraic means, but that is not what is being asked for here.*

(b) Your sketch should show one solution of the system in the lower right hand quadrant not too far from $(1,-1)$. Use $\mathbf{x}_0 = (1,-1)$ as a starting point, and apply one step of Newton's method to find $\mathbf{x}_1$, a better approximate solution, and compute $f(\mathbf{x}_1)$ and $g(\mathbf{x}_1)$.

SOLUTION It is easy to plot ellipses, and even easier when they are circles: Mark the points where the major and minor axes cross the ellipse, and draw an ellipse through these four points. Here is what you get in the case at hand:



Since there are two points of intersection, there are two solutions to the system of equations. This can be seem from even a rather rough sketch.

For **(b)**, indeed we see one point of intersection close to $\mathbf{x}_0 := (1, -1)$. We compute

$$\nabla f(x, y) = 2(x + 1, y + 1) \qquad \text{and} \qquad \nabla g(x, y) = 2(4x - 4, y - 1) \ ,$$

and evaluate

$$\nabla f(1, -1) = (4, 0) \qquad \text{and} \qquad \nabla g(1, -1) = (0, -4) \ .$$

Hence

$$[D_{\mathbf{f}}(1, -1)] = \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix} \qquad \text{and} \qquad [D_{\mathbf{f}}(1, -1)]^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ .$$

We then evaluate

$$\mathbf{f}(\mathbf{x}_0) = (f(1, -1), g(1, -1)) = (0, -1) \ ,$$

and

$$\mathbf{x}_1 = \mathbf{x}_0 - [D_{\mathbf{f}}(1, -1)]^{-1}\mathbf{f}(\mathbf{x}_0) = (1, -5/4) \ .$$

Note from the graph that $\mathbf{x}_1$ is much closer to the solution closest to $\mathbf{x}_0$ than $\mathbf{x}_0$ is. Indeed, $f(\mathbf{x}_1) = g(\mathbf{x}_1) = 1/16$.