Solutions for the even numbered exercises from Chapter 3

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October 11, 2012

3.2 A function $\mathbf{f}$ defined domain $U \subset \mathbb{R}^n$ with values in $\mathbb{R}$ is called a *Lipschitz continuous* function in case there is some number M so that

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| \le M\|\mathbf{x} - \mathbf{y}\| \tag{0.1}$$

for all $\mathbf{x}$ and $\mathbf{y}$ in U.

(a) Show that a Lipschitz continuous function is continuous by finding a valid margin of error on the input; i.e., a valid $\delta(\epsilon)$.

(b) For $R > 0$, let U denote the ball of radius R about the origin; i.e., $U = B_R(\mathbf{0})$. Let $f(\mathbf{x})$ be defined on U by $f(\mathbf{x}) = \|\mathbf{x}\|^2$. Using the identity

$$\|\mathbf{x}\|^2 - \|\mathbf{y}\|^2 = (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y})$$

and the Cauchy-Schwarz inequality, show that f is Lipschitz on U with Lipschitz constant $2R$.

(c) Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ have the form $\mathbf{f}(\mathbf{x}) = (\mathbf{a}_1 \cdot \mathbf{x}, \dots, \mathbf{a}_m \cdot \mathbf{x})$ for some set of vectors $\{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ in $\mathbb{R}^n$. Show that is Lipschitz continuous on $\mathbb{R}^n$.

SOLUTION If $\|\mathbf{x} - \mathbf{x}_0\| \le \epsilon/M$, and $\mathbf{x}, \mathbf{x}_0 \in U$, then $\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0\| \le M(\epsilon/M) = \epsilon$, so

$$\delta(\epsilon) = \frac{\epsilon}{M}$$

works.

Next, for $f(\mathbf{x}) = \|\mathbf{x}\|^2$, by the Cauchy-Scwarz inrqality,

$$|f(\mathbf{x}) - f(\mathbf{x}_0)| = |(\mathbf{x} - \mathbf{x}_0) \cdot (\mathbf{x} + \mathbf{x}_0)| \le \|\mathbf{x} + \mathbf{x}_0\|\|\mathbf{x} - \mathbf{x}_0\|.$$

Next, note that, by the triangle inequality,

$$\|\mathbf{x} + \mathbf{x}_0\| \le \|\mathbf{x}\| + \|\mathbf{x}_0\|$$

so that if $\mathbf{x}, \mathbf{x}_0 \in U$, $\|\|\|\mathbf{x} + \mathbf{x}_0\| \le 2R$, and then putting things together, we have

$$|f(\mathbf{x}) - f(\mathbf{x}_0)| \le 2R\|\mathbf{x} - \mathbf{x}_0\|,$$

which means that f is Lipschitz on U with Lipschitz constant $2R$.

Finally, we compute

$$\begin{aligned}
\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)\| &= \left\| \sum_{j=1}^{m} \mathbf{a}_j \cdot (\mathbf{x} - \mathbf{x}_0) \right\| \\
&\le \left(\sum_{j=1}^{m} (\mathbf{a}_j \cdot (\mathbf{x} - \mathbf{x}_0))^2 \right)^{1/2} \\
&\le \left(\sum_{j=1}^{m} \|\mathbf{a}_j\|^2 \|\mathbf{x} - \mathbf{x}_0\|^2 \right)^{1/2} \\
&= \left(\sum_{j=1}^{m} \|\mathbf{a}_j\|^2 \right)^{1/2} \|\mathbf{x} - \mathbf{x}_0\| \ .
\end{aligned}$$

Thus $\mathbf{f}$ is Lipschitz with Lipschitz constant $\left(\sum_{j=1}^{m} \|\mathbf{a}_j\|^2 \right)^{1/2}$.

3.4 Let $f(x, y)$ be given by

$$f(x,y) = \begin{cases} \dfrac{x^2 \sin(xy)}{x^6 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \ . \end{cases}$$

(a) For any $a, b \in \mathbb{R}$, define the sequence $\{\mathbf{x}_n\}$ by $\mathbf{x}_n = (a/n, b/n)$. Compute $\lim_{n \to \infty} f(\mathbf{x}_n)$.

(b) For any $a, b \in \mathbb{R}$, define the sequence $\{\mathbf{x}_n\}$ by $\mathbf{x}_n = (a/n, b/n^3)$. Compute $\lim_{n \to \infty} f(\mathbf{x}_n)$.

(c) Is the function f continuous? justify your answer.

SOLUTION For **(a)** when $(a, b) \neq (0, 0)$,

$$f(\mathbf{x}_n) = \frac{a^2 \sin(ab/n^2)}{a^6/n^4 + b^2} \ .$$

Since $0 \le |\sin(ab/n^2)| \le ab/n^2$, by the squeeze principle,

$$\lim_{n \to \infty} \sin(ab/n^2) = 0 \ ,$$

and clearly $\lim_{n \to \infty} a^6/n^4 = 0$. Thus, when $(a, b) \neq (0, 0)$,

$$\lim_{n \to \infty} f(\mathbf{x}_n) = 0 \ .$$

This is also true when $(a, b) = (0, 0)$, but then because each term in the sequence is zero.

For **(b)** when $(a, b) \neq (0, 0)$,

$$f(\mathbf{x}_n) = \frac{a^2 n^4 \sin(ab/n^4)}{a^6/ + b^2} \ .$$

$$\lim_{t \to 0} \frac{\sin(t)}{t} = 1$$

Defining $t = ab/n^t$,

$$n^4 \sin(ab/n^4) = \frac{1}{ab}\frac{\sin(t)}{t} \ .$$

Therefore,

$$\lim_{n\to\infty} n^4 \sin(ab/n^4) = \frac{1}{ab} \ .$$

Finally then

$$\lim_{n\to\infty} f(\mathbf{x}_n) = \frac{a}{b}\frac{a^6+b^2}{}.$$

The limit is zero when $(a,b) = (0,0)$, but then because each term in the sequence is zero.

For **(c)**, no, it is not continuous, since the different sequences in part **(b)** give different limits – although, as we saw in part **(a)**, along all sequences that approach the origin through a *fixed direction*, the function does go to zero.

3.5 Consider the function defined by

$$f(x,y) = \begin{cases} (x+y)\ln(x^2+y^2) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Is this function is continuous? Justify your answer.

SOLUTION Since the logarithm function is continuous on $(0,\infty)$, and since products of continuous functions are continuous, and polynomials are continuous, f is clearly continuous at every $\mathbf{x}_0$ for $\mathbf{x}_0 \neq \mathbf{0}$.

After trying a few ways of approaching $\mathbf{0}$, this one looks continuous at $\mathbf{0}$ too. To prove that it is, we use a squeeze principle argument. Note that $|x+y| \leq \sqrt{2}(x^2+y^2)^{1/2}$, so

$$|f(x,y)| \leq \sqrt{2}(x^2+y^2)^{1/2}\ln(x^2+y^2) = 2^{3/2} g(\|\mathbf{x}\|)$$

where $g(t) = 2^{3/2} t\ln(t)$. Since $\lim_{t\to 0} g(t) = 0$, $\lim_{x\to 0} g(\|\mathbf{x}\|) = 0$, and then since

$$0 \leq f(\mathbf{x}) \leq g(\|\mathbf{x}\|) \ ,$$

we have that

$$\lim_{\mathbf{x}\to 0} f(\mathbf{x}) = 0 \ ,$$

and therefore, f is continuous everywhere.

3.6 Consider the function defined by

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^4} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Is this function is continuous? Justify your answer.

SOLUTION Since rational functions are continuous at all points where the denominator is not zero, f is continuous at every $\mathbf{x}_0$ for $\mathbf{x}_0 \neq \mathbf{0}$.

After trying a few ways of approaching $\mathbf{0}$, this one looks continuous at $\mathbf{0}$ too. To prove that it is, we use a squeeze principle argument. Note that for $(x, y) \neq (0, 0)$,

$$|f(x,y)| = \frac{x^2|y|}{x^2+y^4} \le \frac{x^2|y|}{x^2} = |y| \le \sqrt{x^2+y^2} = g(\|\mathbf{x}\|)$$

where $g(t) = t$. Since $\lim_{t\to 0} g(t) = 0$, $\lim_{x\to 0} g(\|\mathbf{x}\|) = 0$, and then since

$$0 \le f(\mathbf{x}) \le g(\|\mathbf{x}\|) \ ,$$

we have that

$$\lim_{\mathbf{x}\to 0} f(\mathbf{x}) = 0 \ ,$$

and therefore, f is continuous everywhere.

3.8 Let $\mathbf{a}$ and $\mathbf{b}$ be given vectors in $\mathbb{R}^3$ such that neither is a multiple of the other. Define a function $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(\mathbf{x}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{x}) \ .$$

Define

$$\mathbf{x}_0 = \frac{1}{\|\mathbf{a}\times\mathbf{b}\|}\mathbf{a}\times\mathbf{b} \ .$$

Show that

$$f(\mathbf{x}) \le f(\mathbf{x}_0)$$

for all unit vectors $\mathbf{x} \in \mathbb{R}^3$. In other words, show that $\mathbf{x}_0$ is the maximizer of f on the unit sphere in $\mathbb{R}^3$.

SOLUTION By the triple product identity, and then the Cauchy-Scwarz inequality,

$$|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{x}| = |(\mathbf{a}\times\mathbf{b})\cdot\mathbf{x}| \le \|\mathbf{a}\times\mathbf{b}\|\|\mathbf{x}\| = \|\mathbf{a}\times\mathbf{b}\|$$

for any unit vector $\mathbf{x}$. On the other hand, for $\mathbf{x}_0 = \dfrac{1}{\|\mathbf{a}\times\mathbf{b}\|}\mathbf{a}\times\mathbf{b}$,

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{x}_0) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{x}_0 = \frac{(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b})}{\|\mathbf{a}\times\mathbf{b}\|} = \|\mathbf{a}\times\mathbf{b}\|.$$

Thus, for all unit vectors $\mathbf{x}$,

$$f(\mathbf{x}_0) = \|\mathbf{a}\times\mathbf{b}\| \ge f(\mathbf{x}) \ ,$$

and so $\mathbf{x}_0$ maximizes f on the unit sphere in $\mathbb{R}^3$.

3.10 Let $\mathbf{f}$ be any function from $\mathbb{R}^n$ to $\mathbb{R}^m$. For any set $A \subset \mathbb{R}^m$, define $f^{-1}(A)$ to be the set of all points $\mathbf{x}$, if any, in $\mathbb{R}^n$ such that $\mathbf{f}(\mathbf{x}) \in A$. The set $f^{-1}(A)$, which may be the empty set, is called the *preimage of A under* $\mathbf{f}$. Do not be misled by the notation: $f^{-1}(A)$ is defined whether or not the function $\mathbf{f}$ itself is invertible.

(a) Prove that $\mathbf{f}$ is continuous if and only if whenever A is an open set in $\mathbb{R}^m$, then $f^{-1}(A)$ is an open set in $\mathbb{R}^n$. This result provides a way to talk about continuity without explicitly bringing ϵ and δ into the discussion. It also has other uses:

(b) Use the result of part **(a)** to give a short proof that whenever $\mathbf{f}$ is a continuous function from $\mathbb{R}^n$ to $\mathbb{R}^m$, and $\mathbf{g}$ is a continuous function from $\mathbb{R}^m$ to $\mathbb{R}^\ell$, then $\mathbf{g} \circ \mathbf{f}$ is a continuous function from $\mathbb{R}^n$ to $\mathbb{R}^\ell$.

SOLUTION For **(a)**, suppose first that $\mathbf{f}$ is continuous. Let A be any open set in $\mathbb{R}^m$. We must show that $\mathbf{f}^{-1}(A)$ is open, which means that for each $\mathbf{x} \in \mathbf{f}^{-1}(A)$, there is an $r > 0$ so $B_r(\mathbf{x}) \subset \mathbf{f}^{-1}(A)$.

Therefore, consider any $\mathbf{x}_0 \in \mathbf{f}^{-1}(A)$. Then $\mathbf{f}(\mathbf{x}_0) \in A$. Since A is open, there is some $\epsilon > 0$ so that $B_\epsilon(\mathbf{f}(\mathbf{x}_0)) \subset A$. Then since $\mathbf{f}$ is continuous, there is a $\delta_\epsilon > 0$ such that

$$\|\mathbf{x} - \mathbf{x}_0\| < \delta_\epsilon \quad \Rightarrow \quad \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)\| < \epsilon \ .$$

Therefore

$$\|\mathbf{x} - \mathbf{x}_0\| < \delta_\epsilon \quad \Rightarrow \quad \mathbf{f}(\mathbf{x}) \in A \ , \quad \text{and hence} \quad \mathbf{x} \in \mathbf{f}^{-1}(A) \ .$$

Therefore, with $r := \delta_\epsilon$, $B_r(\mathbf{x}) \subset \mathbf{f}^{-1}(A)$.

Conversely, suppose that whenever $A \subset of \mathbb{R}^m$ is open, so is $\mathbf{f}^{-1}(A)$. We must show that for all $\mathbf{x}_0$, and all $\epsilon > 0$, there exists a $\delta_\epsilon > 0$ such that

$$\|\mathbf{x} - \mathbf{x}_0\| < \delta_\epsilon \quad \Rightarrow \quad \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)\| < \epsilon \ .$$

Fix any $\mathbf{x}_0$ and any $\epsilon > 0$. The set $B_\epsilon(\mathbf{f}(\mathbf{x}_0)$, the open ball of radius ϵ about $\mathbf{f}(\mathbf{x}_0)$, is open. By hypothesis, $\mathbf{f}^{-1}(B_\epsilon(\mathbf{f}(\mathbf{x}_0))$ is open, and it contains $\mathbf{x}_0$. Hence, by the definition of "open", there is an $r > 0$ so that

$$B_r(\mathbf{x}_0) \subset \mathbf{f}^{-1}(B_\epsilon(\mathbf{f}(\mathbf{x}_0)) \ .$$

But this mean that whenever $\|\mathbf{x} - \mathbf{x}_0\| < r$, then $\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)\| < \epsilon$, and so with $\delta_\epsilon := r$, we have what we require, and $\mathbf{f}$ is continuous at $\mathbf{x}_0$. Since $\mathbf{x}_0$ was an arbitrary point, $\mathbf{f}$ is continuous.

For **(b)**, if $\mathbf{f}$ and $\mathbf{g}$ are both continuous, and A is any open set in $\mathbb{R}^\ell$, then

$$(\mathbf{g} \circ \mathbf{f})^{-1}(A) = \mathbf{f}^{-1}(\mathbf{g}^{-1}(A)) \ .$$

This is open since $\mathbf{g}^{-1}(A)$ is open by the continuity of $\mathbf{g}$, and then $\mathbf{f}^{-1}(\mathbf{g}^{-1}(A))$ is open by the continuity of $\mathbf{f}$. But then since $(\mathbf{g} \circ \mathbf{f})^{-1}(A)$ is open whenever A is open, $\mathbf{g} \circ \mathbf{f}$ is continuous.

3.12 Let $K \subset \mathbb{R}^n$ be compact, and let $\mathbf{f}$ be a continuous function from $\mathbb{R}^n$ to $\mathbb{R}^m$. Define $L \subset \mathbb{R}^m$ by

$$L := \{\mathbf{y} \in \mathbb{R}^m \ : \ \mathbf{y} = \mathbf{f}(\mathbf{x}) \text{ for some } \mathbf{x} \in K \ \} \ .$$

Is L necessarily compact? Justify your answer.

SOLUTION By the definition of "compact" we must show that L is both bounded and closed.

To show that L is bounded, consider the function

$$g(\mathbf{x}) := \|\mathbf{f}(\mathbf{x})\| \ .$$

Since $\mathbf{f}$ is continuous, and since the length function $\mathbf{y} \rightarrow \|\mathbf{y}\|$ is continuous, and since the composition of continuous functions is continuous, g is continuous. Since K is compact, by the theorem on existence of maximizers, there is an $\mathbf{x}_0$ in K so that $g(\mathbf{x}) \leq g(\mathbf{x}_0)$ for all $\mathbf{x} \in K$. That is

$$\|\mathbf{f}(\mathbf{x})\| \leq \|\mathbf{f}(\mathbf{x}_0)\|$$

for all $\mathbf{x} \in K$. But $\mathbf{y}_0 := \mathbf{f}(\mathbf{x}) \in L$ and by the definition of L, every $\mathbf{y} \in L$ has the form $\mathbf{y} = \mathbf{f}(\mathbf{x})$ for some $\mathbf{x} in K$. Hence

$$\|\mathbf{y}\| \leq \|\mathbf{y}_0\| := R$$

for all $\mathbf{y} \in L$. This shows that L is bounded: It is contained in $B_R(\mathbf{0})$.

To show that L is closed, let $\{\mathbf{y}_n\}$ be any convergent sequence in L such that

$$\mathbf{z} = \lim_{n\to\infty} \mathbf{y}_n$$

We must show that $\mathbf{z} \in L$. By the definition of L, every $\mathbf{y} \in L$ has the form $\mathbf{y} = \mathbf{f}(\mathbf{x})$, and hence there is a sequence $\{\mathbf{x}_n\}$ in K such that $\mathbf{y}_n = \mathbf{f}(\mathbf{x}_n)$ for all n.

Since every sequence in a compact set has a subsequence converging to an element of that compact set, there is a subsequence $\{\mathbf{x}_{n_k}\}$ of $\{\mathbf{x}_n\}$ that converges to some $\mathbf{w}$ in K. That is

$$\lim_{k\to\infty} \mathbf{x}_{n_k} = \mathbf{w} \in K \ .$$

But then since $\mathbf{f}$ is continuous,

$$\lim_{k\to\infty} \mathbf{y}_{n_k} = \lim_{k\to\infty} \mathbf{f}(\mathbf{x}_{n_k})\mathbf{f}\left(\lim_{k\to\infty} \mathbf{x}_{n_k}\right) = \mathbf{f}(\mathbf{w}) \in L \ .$$

But since $\lim_{n\to\infty} \mathbf{y}_n$, $\lim_{k\to\infty} \mathbf{y}_{n_k} = \mathbf{z}$ also, and so $\mathbf{z} = \mathbf{f}(\mathbf{w}) \in L$. This proves L is closed, and completes the proof that L is compact.