1 Solutions for the even exercises from Chapter Two

2.2 Let $\mathbf{x}(t) = \left(t^{-2}\ ,\ 4/\sqrt{t}\ ,\ t\right)$ for $t > 0$.

(a) Compute $\mathbf{v}(t) = \mathbf{x}'(t)$ and $\mathbf{a}(t) = \mathbf{x}''(t)$.

(b) Compute $v(t)$ and $\mathbf{T}(t)$.

(c) Find the tangent line to this curve at $t = 1$.

SOLUTION: Differentiating, we find

$$\mathbf{v}(t) = \left(-2t^{-3}, -2t^{-3/2}, 1\right) \qquad \text{and} \qquad \mathbf{a}(t) = \left(6t^4, 3t^{-5/2}, 0\right) .$$

Then

$$v(t) = \|\mathbf{v}(t)\| = \sqrt{1 + 4t^{-3} + 4t^{-6}} = 1 + 2t^{-3} ,$$

and so

$$\mathbf{T}(t) = \frac{1}{v(t)}\mathbf{v}(t) = \frac{1}{1 + 2t^{-3}}\left(-2t^{-3}, -2t^{-3/2}, 1\right) .$$

The tangent line is given by

$$\mathbf{x}(1) + t\mathbf{v}(1) = \left(1 - 2t, 4 - 2t, 1 + t\right) .$$

2.4 Let $\mathbf{x}(t) = \left(\cos(t)\ , \sin(t)\ , t/r\right)$ where $r > 0$. The curve $\mathbf{x}(t)$ is a helix in $\mathbb{R}^3$.

(a) Compute $\mathbf{v}(t)$ and $\mathbf{a}(t)$.

(b) Compute $v(t)$ and $\mathbf{T}(t)$.

(c) Compute the curvature $\kappa(t)$ and the torsion $\tau(t)$, as well as $\mathbf{N}(t)$ and $\mathbf{B}(t)$.

(d) Compute the Darboux vector $\boldsymbol{\omega}(t)$.

(e) Find the tangent line to this curve at $t = \pi/4$, and the equation of the osculating plane to the curve at $t = \pi/2$. Find the intersection of this line and plane.

SOLUTION: (We give here the solution of a slightly more general problem in which the third component of the curve it t/r instead of t/π. Simply replace r by π below to get the solution for the special case asked for in the text.) Differentiating, we find

$$\mathbf{v}(t) = \left(-\sin(t), \cos(t), 1/r\right) \qquad \text{and} \qquad \mathbf{a}(t) = \left(-\cos(t), -\sin(t), 0\right) .$$

Then

$$v(t) = \|\mathbf{v}(t)\| = \frac{\sqrt{r^2+1}}{r},$$

and so

$$\mathbf{T}(t) = \frac{1}{v(t)}\mathbf{v}(t) = \frac{r}{\sqrt{r^2+1}}\left(-\sin(t), \cos(t), 1/r\right) .$$

To find the curvature and $\mathbf{N}$, we first compute

$$\mathbf{a}_\perp := \mathbf{a} - (\mathbf{a}\cdot\mathbf{T})\mathbf{T} = \frac{\mathbf{a}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} .$$

By the computations made above, $\mathbf{a}\cdot\mathbf{v} = 0$, so all of the acceleration is orthogonal, and so *in this case*

$$\kappa(t) = \frac{\|\mathbf{a}(t)\|}{v(t)^2} = \frac{r^2}{r^2+1} \ ,$$

and

$$\mathbf{N}(t) = \frac{1}{\|\mathbf{a}_\perp(t)\|}\mathbf{a}_\perp(t) = \frac{1}{\|\mathbf{a}(t)\|}\mathbf{a}(t) = \mathbf{a}(t) = \big(-\cos(t), -\sin(t), 0\big) \ .$$

We next compute $\mathbf{B} = \mathbf{T}\times\mathbf{N}$, and find

$$\mathbf{B}(t) = \frac{1}{\sqrt{r^2+1}}\big(\sin(t), -\cos(t), r\big) \ .$$

Since $\mathbf{B}$ is so simple, the easiest way to compute the torsion is to compute $\mathbf{B}'(t)$, and to use the definition $\mathbf{B}'(t) = -v(t)\tau(t)\mathbf{N}(t)$. We readily find

$$\tau(t) = \frac{r}{r^2+1} \ .$$

We are now ready to compute the Darboux vector, $\boldsymbol{\omega} = \tau\mathbf{T} + \kappa\mathbf{B}$. We find

$$\begin{aligned}
\boldsymbol{\omega} &= \frac{r}{r^2+1}\frac{r}{\sqrt{r^2+1}}\big(-\sin(t), \cos(t), 1/r\big) + \frac{r^2}{r^2+1}\frac{1}{\sqrt{r^2+1}}\big(\sin(t), -\cos(t), r\big)\\
&= \frac{r}{\sqrt{r^2+1}}(0,0,1)
\end{aligned}$$

Finally, we compute that tangent line to $\mathbf{x}(t)$ at $t = \pi/4$, finding it is parameterized by

$$\mathbf{x}(\pi/4) + t\mathbf{v}(\pi/4) = \big((1-t)/\sqrt{2}, (1+t)/\sqrt{2}, (\pi+4t)/4r\big) \ . \tag{1.1}$$

The equation of the osculating plane at $t = \pi/2$ is

$$\mathbf{B}(\pi/2)\cdot(\mathbf{x} - \mathbf{x}(\pi/2)) = 0 \ .$$

This simplifies to $(1, 0, r)\cdot(x, y-1, z-\pi/(2r)) = 0$ which then further simplifies to

$$x + rz = \pi/2 \ .$$

Plugging the parameterized line (1.1) into the equation of the plane and solving for t, we find that the intersection occurs at

$$t_0 = -\frac{2+\pi}{8-2\sqrt{2}} \ .$$

Plugging this value of t into (1.1) gives the intersection.

2.6: Let $\mathbf{x}(t)$ be the curve given by

$$\mathbf{x}(t) = (t^{3/2}, 3t, 6t^{1/2})$$

for $t > 0$.

(a) what is the arc length along the curve between $\mathbf{x}(1)$ and $\mathbf{x}(4)$?

(b) Compute curvature $\kappa(t)$ and torsion $\tau(t)$ as a function of t.

(c) Find an equation for the osculating plane at $t = 1$, and find a parameterization of the tangent line to the curve at $t = 1$.

SOLUTION: Differentiating, we find

$$\mathbf{v}(t) = 3(t^{1/2}/2, 1, t^{-1/2}) \qquad \text{and} \qquad \mathbf{a}(t) = \frac{3}{4}(t^{-1/2}, 0, 2t^{-3/2}) \ .$$

Then

$$v(t) = \|\mathbf{v}(t)\| = \frac{3}{2}\sqrt{4 + t + t/4} = \frac{3}{2}(t^{1/2} + 2t^{-1/2}) \ .$$

The arc length along the curve between $\mathbf{x}(1)$ and $\mathbf{x}(4)$ then is

$$\int_1^4 v(t)\mathrm{d}t = \frac{3}{2}\frac{26}{3} = 13 \ .$$

Computing the curvature and torsion, we find

$$\kappa(t) = \frac{2}{3}(t+2)^{-2} \qquad \text{and} \qquad \tau = -\frac{2}{3}(t+2)^{-2} \ .$$

Since $\mathbf{a} \times \mathbf{v}$ is a multiple of $\mathbf{B}$, the equation of the osculating plane at $t = 1$ is

$$(\mathbf{a}(1) \times \mathbf{v}(1) \cdot (\mathbf{x} - \mathbf{x}(1) = 0 \ .$$

Computing, this becomes

$$\frac{9}{4}(2, -2, 1) \cdot (x - 1, y - 3, z - 6) = 0 \ ,$$

and this further simplifies to

$$2x - 2y + z = 2 \ .$$

The tangent line is parameterized by

$$\mathbf{x}(1) + t\mathbf{v}(1) = (1 + 3t/2, 3 + 3t, 6 + 3t) \ .$$

2.8: Let $\mathbf{x}(t)$ be the curve given by

$$\mathbf{x}(t) = (2t, t^2, t^3/3) \ .$$

(a) Compute the arc length $s(t)$ as a function of t, measured from the starting point $\mathbf{x}(0)$.

(b) Compute curvature $\kappa(t)$ and torsion $\tau(t)$ as a function of t.

(c) Find equations for the osculating planes at time $t = 0$ and $t = 1$, and find a parameterization of the line formed by the intersection of these planes.

SOLUTION: Differentiating, we find

$$\mathbf{v}(t) = (2, 2t, t^2) \qquad \text{and} \qquad \mathbf{a}(t) = (0, 2, 2t) \ .$$

Then

$$v(t) = \|\mathbf{v}(t)\| = 2 + t^2 \ .$$

The arc length along the curve between $\mathbf{x}(0)$ and $\mathbf{x}(t)$ then is

$$\int_0^t v(r)\mathrm{d}r = \int_0^t (2+r^2)\mathrm{d}r = 2t + 26^3/3 \ .$$

Computing the curvature and torsion, we find

$$\kappa(t) = \frac{\sqrt{(2+t^2)^2+4}}{(2+t^2)^2}(t+2)^{-2} \qquad \text{and} \qquad \tau = -\frac{4}{(2+t^2)^2}\frac{1}{\sqrt{(2+t^2)^2+4}} \ .$$

Since $\mathbf{a} \times \mathbf{v}$ is a multiple of $\mathbf{B}$, the equation of the osculating plane at $t = 1$ is

$$\mathbf{a}(1) \times \mathbf{v}(1) \cdot (\mathbf{x} - \mathbf{x}(1) = 0 \ .$$

Computing, this becomes

$$\frac{9}{4}(2,-2,1) \cdot (x-1, y-3, z-6) = 0 \ ,$$

and this further simplifies to

$$2x - 2y + z = 2 \ .$$

The tangent line is parameterized by

$$\mathbf{x}(1) + t\mathbf{v}(1) = (1 + 3t/2, 3 + 3t, 6 + 3t) \ .$$

Finally since $\mathbf{a}(t) \times \mathbf{v}(t)$ is a multiple of $\mathbf{B}(t)$ (as is $\mathbf{a}(t) \times \mathbf{v}(t)$), we can write an equation for the normal plane at time t as

$$[\mathbf{a}(t) \times \mathbf{v}(t)] \cdot [\mathbf{x} - \mathbf{x}(t)] = 0 \ .$$

Doing this for $t = 0$, we find $(0,0,-3) \cdot (x,y,z) = 0$, which reduces to

$$z = 0 \ .$$

Doing this for $t = 0$, we find $(\,-2, 4, -4) \cdot (x-2, y-1, z-1/3) = 0$, which reduces to

$$2x - 4y + 4z = \frac{4}{3} \ .$$

Plugging $z = 0$ (from the first equation) into the second equation, we find that the line is the line lies in the x, y plane, where is is given by

$$y = \frac{x}{2} - \frac{1}{3}$$

This line can also be prameterized as

$$\mathbf{x}(t) = (t, t/2 - 1/3, 0) \ .$$

2.10 Let $\mathbf{x}(t)$ be the curve given by $\mathbf{x}(t) = (t \ , \ \sqrt{2}\ln(t) \ , \ 1/t)$ for $t > 0$.

(a) Find the arc length along the curve from $\mathbf{x}(1)$ to $\mathbf{x}(3)$.

(b) Find the arc length along the curve from $\mathbf{x}(1)$ to $\mathbf{x}(t)$ as a function of t.

(c) Find the arc length parameterization $\mathbf{x}(s)$ of this curve.

SOLUTION: We compute

$$\mathbf{v}(t) = \left(1, \sqrt{2}/t, -1/t^2\right) \ ,$$

and hence

$$v(t) = \|\mathbf{v}(t)\| = \sqrt{1 + \frac{2}{t^2} + \frac{1}{t^4}} = \sqrt{(1+t^{-2})^2} = 1 + t^{-2} \ .$$

Then the arc length along the curve from $\mathbf{x}(1)$ to $\mathbf{x}(3)$ is

$$\int_1^3 v(t)\mathrm{d}t = \int_1^3 (1+t^{-2})\mathrm{d}t = \frac{8}{3} \ .$$

Likewise, the arc length along the curve from $\mathbf{x}(1)$ to $\mathbf{x}(t)$ is, for $t \geq 1$,

$$\int_1^t v(r)\mathrm{d}r = \int_1^t (1+r^{-2})\mathrm{d}r = t - 1/t \ .$$

For $0 < t < 1$, it is

$$\int_t^1 v(r)\mathrm{d}r = \int_t^1 (1+r^{-2})\mathrm{d}r = -(t - 1/t) \ .$$

Combining the formulas, the along the curve from $\mathbf{x}(1)$ to $\mathbf{x}(t)$ is $|t - 1/t|$.

To find the arc length parameterization, we compute

$$s(t) = \int_1^t (1+r^{-2})\mathrm{d}r = t - 1/t \ ,$$

for all $t > 0$, and then solve

$$s = t - 1/t$$

to determine $t(s)$. We find $ts = t^2 - 1$, or $t^2 - st = 1$ or $(t - s/2)^2 = 1 + s^2/4$, so

$$t(s) = \sqrt{1 + s^2/4} - s/2 \ .$$

Then

$$\mathbf{x}(s) = \mathbf{x}(t(s)) = \left(t\sqrt{1+s^2/4} - s/2 \ , \ \sqrt{2}\ln\left(\sqrt{1+s^2/4} - s/2\right) \ , \ 1/\left(\sqrt{1+s^2/4} - s/2\right)\right) \ .$$

2.12 Find the arc length parameterization of the curve given by $\mathbf{x}(t) = \left(t^{-2} \ , \ 4/\sqrt{t} \ , \ t\right)$ for $t > 0$. What is the arc length along the segment of the curve joining $\mathbf{x}(1)$ and $\mathbf{x}(4)$?

SOLUTION: We compute

$$\mathbf{v}(t) = \left(-2/t^3, -2/t^{3/2}, 1\right) \ ,$$

and hence

$$v(t) = \|\mathbf{v}(t)\| = \sqrt{1 + \frac{4}{t^3} + \frac{4}{t^6}} = \sqrt{(1+2t^{-3})^2} = 1 + 2t^{-3} \ .$$

We then pick some time, say $t = 1$ as a refrence starting time, and define

$$s(t) = \int_1^t v(r)\mathrm{d}r = \int_1^t (1 + 2r^{-3}\mathrm{d}r = t - t^{-2} \ .$$

We must then solve

$$t - t^{-2} = s \tag{1.2}$$

to find $t(s)$, We then plug $t(s)$ in to $\mathbf{x}(t)$ to find $\mathbf{x}(s)$. Solving (1.2), we find

$$t(s) = \frac{1}{6}\left(108 + 8s^2 + 12\sqrt{81 + 12s^3}\right)^{1/3} + \frac{2}{3}\frac{s^2}{\left(108 + 8s^2 + 12\sqrt{81 + 12s^3}\right)^{1/3}} + \frac{1}{3}s \ .$$

Plugging this in, one has the (rather messy) answer.

On the other hand, the arc length along the curve from $\mathbf{x}(1)$ to $\mathbf{x}(4)$ is

$$\int_1^4 v(t)\mathrm{d}t = \int_1^3 (1 + 2t^{-3})\mathrm{d}t = \frac{63}{16} \ .$$

Moral of the story: *Arc length parameterization, while useful conceptually, is very messy in practice. Computing arc length between two points may be very much simpler.*

2.14 Let $\mathbf{b} = (4, 7, 4)$. Let $\mathbf{x}(t)$ be the curve given satisfying the initial value problem

$$\mathbf{x}'(t) = \mathbf{b} \times \mathbf{x}(t) \qquad \text{and} \qquad \mathbf{x}(0) = (2, 2, 1) \ .$$

(a) Compute $\mathbf{x}(\pi)$ and find the arc length along the curve from $\mathbf{x}(0)$ to $\mathbf{x}(\pi)$.

(b) Compute the curvature and torsion for this curve as a function of t.

SOLUTION: This is the rotation equation. Since $\|\mathbf{b}\| = 9$, $\mathbf{x}(\pi)$ is what you get rotating $\mathbf{x}(0)$ through and angle 9π about the axis along $\mathbf{b} = (4, 7, 4)$, But since rotation by 2π is the identity, $\mathbf{x}(\pi)$ is the same as the rotation of $\mathbf{x}(0)$ through the angle π. Let us decompose

$$\mathbf{x}(0) = (\mathbf{x}(0))_{\|} + (\mathbf{x}(0))_{\perp}$$

into its components parallel and perpendicular to $\mathbf{b}$. Since $(\mathbf{x}(0))_{\|}$ lies along the axis of rotation, the rotation leave it unchanged. Since $(\mathbf{x}(0))_{\perp}$ lies in the plane of rotation – the plane orthogonal to the axis of rotation, and since rotation any vector in this plane through the angle π takes it to minus itself, we have

$$\mathbf{x}(\pi) = (\mathbf{x}(0))_{\|} - (\mathbf{x}(0))_{\perp} = \frac{2}{\|\mathbf{b}\|^2}(\mathbf{x}(0)\cdot\mathbf{b})\mathbf{b} - \mathbf{x}(0) \ .$$

We now compute $\mathbf{x}(0)\cdot\mathbf{b} = 26$ and so

$$\mathbf{x}(\pi) = \frac{52}{81}(4, 7, 4) - (2, 2, 1) = \frac{1}{81}(46, 202, 127) \ .$$

As for the second part, the curve is a circle of radius

$$\|(\mathbf{x}(0))_{\perp}\| = \sqrt{9 = \|(\mathbf{x}(0))_{\|}\|^2} = \sqrt{9 - 26/9} = \frac{\sqrt{55}}{9} \ .$$

Hence the curvature has the constant value $\kappa = 9/.\sqrt{55}$ and since the curve is planar, the torsion has the constant value $\tau = 0$. (Note that this curve is a "degenerate helix": the speed of motion along the axis of rotation is zero, so it is simply a circle.)

2.16 Let $\mathbf{x}(t)$ be the curve given by $\mathbf{x}(t) = (\cos t + 1, \cos t + \sin t, \sin t + 1)$.

(a) Compute curvature $\kappa(t)$ and torsion $\tau(t)$ as a function of t.

(b) Find an equation for the osculating plane at time $t = 0$

(c) Find the distance between the plane given by $x - y + z = 0$ to $\mathbf{x}(t)$ as a function of t.

SOLUTION: We compute

$$\mathbf{x}'(t) = (-\sin t, \cos t - \sin t, \cos t) \qquad \text{and} \qquad \mathbf{x}''(t) = (-\cos t, -\cos t - \sin t, -\sin t)\ .$$

Thus

$$v^2(t) = 2 - 2\sin t \cos t = 2 - \sin(2t)\ .$$

Next,

$$\mathbf{x}'(t) \times \mathbf{x}''(t) = (1, -1, 1)\ .$$

It follows that $\mathbf{B}$ is the constant vector

$$\mathbf{B} = \frac{1}{\sqrt{3}}(1, -1, 1)\ ,$$

and thus that the torsion τ is zero – the curve is planar. Since $\|\mathbf{x}'(t) \times \mathbf{x}''(t)\| = \sqrt{3}$, we find

$$\kappa(t) = \frac{\sqrt{3}}{(2 - \sin(2t))^{3/2}}\ .$$

For part **(b)**, the vector $(1, -1, 1)$ is normal to the osculating plane at *all* times t. Since $\mathbf{x}(0) = (2, 1, 1)$, the equation of the osculating plane at *all* t is

$$(\mathbf{x} - (2, 1, 1)) \cdot (1, -1, 1) = 0$$

which is

$$x - y + z = 2\ .$$

Finally, for part **(c)**, the plane $x - y + z = 0$ is evidently parallel to the plane containing the planar curve $\mathbf{x}(t)$ at all t. Hence the distance in question is simply the distance between these two planes. This is turn, again since the planes are parallel, is the distance form any point in one of the planes to the other plane. Since $(0, 0, 0)$ lies in the plane $x - y + z = 0$, the distance is $2/\sqrt{3}$.