Challenge Problem Set 3, Math 291 Fall 2013

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This challenge problem set concerns planetary motion and Newton's derivation of Kepler's Laws from his Universal Theory of Graviation.

As in the text, a star of mass M is a the center of the coordinate system, and $\mathbf{x}(t)$ denote the position of a planet of mass m at time t. We assume m is much less than M, so that the center of mass of the combined system is effectively the center of the star. Then, the equation of motion of the planet is

$$\mathbf{x}''(t) = -\frac{GM}{\|\mathbf{x}(t)\|^3}\mathbf{x}(t) \ , \tag{0.1}$$

where G is the gravitation constant. Because of the universal nature of Newton's theory, this constant is not only relevant in the heavens, but also governs the gravitational attraction between objects on the earth. Therefore, it can be, and has been, measured in laboratories. The value is approximately $G = 6.67 \times 10^{-11}\ Nm^3/kg^2$ in MKS units.

The key to finding the trajectories $\mathbf{x}(t)$ from the differential equation is in the constants of the motion for this equation. Recall that the momentum $\mathbf{p}(t)$, the *angular momentum* $\mathbf{L}(t)$ and the *Runge-Lenz vector* $\mathbf{A}(t)$ of the planet are given by

$$\mathbf{p}(t) = m\mathbf{x}'(t) = m\mathbf{v}(t), \tag{0.2}$$

$$\mathbf{L}(t) = \mathbf{x}(t) \times \mathbf{p}(t) \ . \tag{0.3}$$

and

$$\mathbf{A}(t) = \mathbf{p}(t) \times \mathbf{L}(t) - GMm^2\frac{\mathbf{x}(t)}{\|\mathbf{x}(t)\|} \ . \tag{0.4}$$

Then, if $\mathbf{x}(t)$ solves (0.1),

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{L}(t) = \mathbf{0} \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}(t) = \mathbf{0} \ .$$

There is another constant of the motion, namely the *Energy* E given by

$$E(t) = \frac{\|\mathbf{p}(t)\|^2}{2m} - mMG\frac{1}{\|\mathbf{x}(t)\|} \ .$$

(1) Prove that $\frac{\mathrm{d}}{\mathrm{d}t}E(t) = 0$ for solutions of (0.1).

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(2) Let $\mathbf{x}(t)$ be a negative energy solution of Newton's equation. Let E denote the constant value of the energy. Show that

$$E \geq -nMG\frac{1}{\|\mathbf{x}(t)\|} \ ,$$

and hence that

$$\|\mathbf{x}(t)\| \leq \frac{mMG}{|E|} \ .$$

Thus, negative energy orbits are *bounded.* In fact, as explained in the text, the bounded orbits are ellipses.

In the rest of this problem set, we consider a fixed solution of (0.1) with given values of $\mathbf{L}$, $\mathbf{A}$ and $E < 0$.

(3) Let t be such that $\|\mathbf{x}(t)\|' = 0$. Such times will exists since $\|\mathbf{x}(t)\|$ will take on a minimum and a maximum value on the ellipse. Show that at any such time t,

$$\|\mathbf{L}\| = \|\mathbf{x}(t)\|\|\mathbf{p}(t)\| \ .$$

Then show that at such a time t, $R := \|\mathbf{x}(t)\|$ satisfies the quadratic equation

$$ER^2 + mMGR - \frac{\|\mathbf{L}\|^2}{2m} = 0 \ .$$

(4) Let $R_{\rm min}$ and $R_{\rm max}$ be the minimum and maximum values of $\|\mathbf{x}(t)\|$ on the orbit. Show that

$$\|\mathbf{A}\| = m|E|(R_{\rm max} - R_{\rm min}) \ .$$

Show that the direction of $\mathbf{A}$ is the unit vector pointing from the origin, which is at a focus of the ellipse, to the point of closest approach of the orbit to the center - the *perihelion* in the case of the Earth and the Sun.

Hint: Let t_1 be a time at which $\|\mathbf{x}(t_1)\| = R_{\rm min}$, and let t_2 be such that $\|\mathbf{x}(t_2)\| = R_{\rm max}$. Then $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are at opposite ends of the major axis of the ellipse, and so

$$\frac{\mathbf{x}(t_1)}{\|\mathbf{x}(t_1)\|} = -\frac{\mathbf{x}(t_2)}{\|\mathbf{x}(t_2)\|} \ .$$

Since $\mathbf{A}$ is constant,

$$\mathbf{A} = \frac{1}{2}(\mathbf{A}(t_1) + \mathbf{A}(t_2)) = \frac{1}{2}\left[\mathbf{p}(t_1)\times\mathbf{L} + \mathbf{p}(t_1)\times\mathbf{L}\right] \ .$$

Now show that

$$\|\mathbf{p}(t_1)\times\mathbf{L}\| = R_{\rm min}\|\mathbf{p}(t_1)\|^2 = 2m(mMG + ER_{\rm min}) \ ,$$

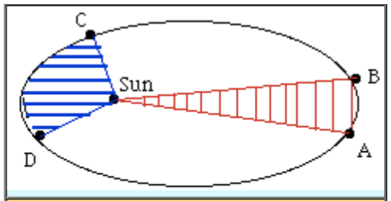
and similarly for $\|\mathbf{p}(t_2)\times\mathbf{L}\|$, and show that $\mathbf{p}(t_1)\times\mathbf{L}$ and $\mathbf{p}(t_2)\times\mathbf{L}$ have the opposite direction.

(5) The *eccentricity* of the elliptic orbit is the quantity e defined by

$$e = \frac{R_{\rm max} - R_{\rm min}}{R_{\rm max} + R_{\rm min}}$$

Express the magnitude of $\mathbf{A}$ in terms of E and e (as well as m, M and G, which are independent of the orbit).

(6) Consider a line drawn from the sun to a planet, As the planet moves around its orbit, this line sweeps out a region. Kepler's Second Law states that the line sweeps out equal areas in equal times. Here is an illustration:



Illustrating Kepler's 2nd law: segments AB and CD take equal times to cover.

Let us prove this. It is essentially equivalent to the fact that the *angular momentum vector is constant.*

By (0.3),

$$\frac{1}{m}\mathbf{L} = \mathbf{x}(t) \times \mathbf{x}'(t) \ .$$

Recall the Sun is at the origin of our coordinate system, For small Δt, the area swept out between times t and $t + \Delta t$ is approximately the area of the triangle with vertices $\mathbf{0}$, $\mathbf{x}(t)$, and $\mathbf{x}(t + \Delta)$.

Show that the area of this triangle is approximately

$$\frac{1}{2}\|\mathbf{x}(t) \times \mathbf{x}(t + \Delta)\| \ .$$

Next, use the tangent line approximation

$$\mathbf{x}(t + \Delta t) \approx \mathbf{x}(t) + \mathbf{x}'(t)\Delta t$$

to show that

$$\mathbf{x}(t) \times \mathbf{x}(t + \Delta) \approx \mathbf{x}(t) \times \mathbf{x}'(t)\Delta t \ .$$

and that therefore, the area swept out in the time interval $(t, \Delta t)$ is approximately

$$\Delta t\frac{1}{2}\|\mathbf{x}(t) \times \mathbf{x}'(t)\| = \Delta t\frac{1}{2m}\|\mathbf{L}\| \ ,$$

and the right hand side is independent of t.

Put this all together to show that if $A(t)$ denotes the cumulative area swept out from time 0 to time t, then

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t) = \frac{1}{2m}\|\mathbf{L}\| \ ,$$

which is constant, and then prove Kepler's Second Law.