# Challenge Problem Set 2, Math 291 Fall 2013

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This challenge problem set concerns the Frenet-Serret formulae, the Gram-Schimdt orthonormalization process, and the generalization of the Frenet-Serret formulae to higher dimension.

### 0.1 The Gram-Schimdt orthonormalization process in $\mathbb{R}^3$

Suppose we are given 3 non-zero vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ in $\mathbb{R}^3$, and we want to construct an orthonormal basis of $\mathbb{R}^3$ out of these vectors. There is a very useful procedure for doing this that works whenever all three vectors do not lie in the same plane through the origin. We shall soon apply this procedure to $\mathbf{x}'(t_0)$, $\mathbf{x}''(t_0)$ and $\mathbf{x}'''(t_0)$ for a thrice continuously differentiable curve $\mathbf{x}(t)$, and shall see how this process yields the Frenet-Serret basis.

Here is how the procedure works: Let us assume that $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ do not all lie in any plane through the origin.

*(1)* Since $\mathbf{v}_1 \neq \mathbf{0}$, we define the unit vector.

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 \ ,$$

*(2)* Define

$$\mathbf{w}_2 = \mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{u}_1)\mathbf{u}_1 \ ,$$

i.e., take $\mathbf{w}_2$ to be the component of $\mathbf{v}_2$ that is orthogonal to $\mathbf{v}_1$, so that $\{\mathbf{u}_1, \mathbf{w}_2\}$ is orthogonal, but not necessarily orthonormal. Provided $\|\mathbf{w}_2\| \neq 0$, we can normalize $\mathbf{w}_2$ by dividing by its length.

Were it the case that $\mathbf{w}_2 = \mathbf{0}$, this would mean that $\mathbf{v}_2$ would be multiple of $\mathbf{v}_1$ but then all three vectors would lie in the plane though $\mathbf{v}_1$, $\mathbf{v}_3$ and $\mathbf{0}$, contrary to assumption. Hence $\|\mathbf{w}_2\| \neq 0$, and we define

$$\mathbf{u}_2 := \frac{1}{\|\mathbf{w}_2\|}\mathbf{w}_2 \ .$$

By Construction, $\{\mathbf{u}_1, \mathbf{u}_2\}$ is orthonormal. Note that since $\mathbf{u}_1$ is a multiple of $\mathbf{v}_1$, $\mathbf{w}_2$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Since $\mathbf{u}_2$ is a multiple of $\mathbf{w}_2$, it too is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

*(3)* Define

$$\mathbf{w}_3 := \mathbf{v}_3 - (\mathbf{v}_3 \cdot \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{v}_3 \cdot \mathbf{u}_2)\mathbf{u}_2 \ .$$

We readily compute

$$\mathbf{w}_3 \cdot \mathbf{u}_j = \mathbf{v}_3 \cdot \mathbf{u}_j - [(\mathbf{v}_3 \cdot \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{v}_3 \cdot \mathbf{u}_2)\mathbf{u}_2] \cdot \mathbf{u}_j = \mathbf{v}_3 \cdot \mathbf{u}_j - \mathbf{v}_3 \cdot \mathbf{u}_j = 0 \ ,$$

for $j = 1, 2$, and so $\mathbf{w}_3$ is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$.

Next, note that if $\mathbf{w}_3 = \mathbf{0}$, then $\mathbf{v}_3 = (\mathbf{v}_3 \cdot \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{v}_3 \cdot \mathbf{u}_2)\mathbf{u}_2$, so $\mathbf{v}_3$ would be a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$. But $\mathbf{u}_1$ is a multiple of $\mathbf{v}_1$, and $\mathbf{u}_2$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, as we have observed in the second step, and so this would mean that $\mathbf{v}_3$ would be a linear combination of $\mathbf{v}_!$ and $\mathbf{v}_2$, and hence would lie in the plane through $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{0}$, contrary to our assumption. Thus, $\|\mathbf{w}_3\| \neq 0$.

We then define

$$\mathbf{u}_3 := \frac{1}{\|\mathbf{w}_3\|}\mathbf{w}_3 \ .$$

Then by construction, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis of $\mathbb{R}^3$.

As we have observed along the way, $\mathbf{u}_1$ is a multiple of $\mathbf{v}_1$, and $\mathbf{u}_2$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. We have proved:

**0.1 THEOREM.** *Given $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ in $\mathbb{R}^3$ such that all three vectors do not lie in any plane through the origin, the Gram-Schmitdt orthonormailization procedure always yields an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ of $\mathbb{R}^3$ such that $\mathbf{u}_1$ is a multiple of $\mathbf{v}_1$ and $\mathbf{u}_2$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.*

There is one final point to make: We do not need to check the condition that all three vectors do not lie in any plane through the origin before beginning to apply the algorithm. If the condition is not satisfied, than along the way we shall find either $\mathbf{w}_2 = \mathbf{0}$ or $\mathbf{w}_3 = \mathbf{0}$. As we have seen, this happens only when the condition is not satisfied, and then of course the procedure breaks down. If it does not break down, it yields the desired orthonormal basis.

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**Exercise 1:** Let

$$\mathbf{v}_1 = (1, 2, -2) \ , \qquad \mathbf{v}_2 = (-1, 0, 1) \ , \qquad \mathbf{v}_3 = (1, 1, 1) \ .$$

Apply the Gram-Schmidt orthonormalization procedure to construct an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ such that $\mathbf{u}_1$ is a multiple of $\mathbf{v}_1$ and $\mathbf{u}_2$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

**Exercise 2:** Show that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is already orthonormal, applying the Gram-Schmidt orthonormalization procedure to it leaves it unchanged.

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## 0.2 Application to the Frenet-Serret basis

Let $\mathbf{x}(t)$ be a thrice continuously differentiable curve defined on $(a, b)$. Fix some $t_0 \in (a, b)$, and let us apply the Gram-Schmidt procedure to $\mathbf{v}_1 := \mathbf{x}'(t_0)$, $\mathbf{v}_2 := \mathbf{x}''(t_0)$ and $\mathbf{v}_3 := \mathbf{x}'''(t_0)$.

We find

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{x}'(t_0)\|}\mathbf{x}'(t_0) = \mathbf{T}(t_0) \ .$$

Next

$$\mathbf{w}_2 = \mathbf{x}''(t_0) - (\mathbf{x}''(t_0) \cdot \mathbf{T}(t_0))\mathbf{T}(t_0) = \mathbf{a}_\perp(t_0) \ ,$$

and hence

$$\mathbf{u}_2 = \frac{1}{\|\mathbf{a}_\perp(t_0)}\mathbf{a}_\perp(t_0) = \mathbf{N}(t_0) \ .$$

Now as long as $\mathbf{x}'''(t_0)$ is not a linear combination of $\mathbf{T}(t_0)$ and $\mathbf{N}(t_0)$, we will have

$$\mathbf{w}_3 = \mathbf{x}'''(t_0) - (\mathbf{x}'''(t_0)\cdot\mathbf{T}(t_0))\mathbf{T}(t_0) - -(\mathbf{x}'''(t_0)\cdot\mathbf{N}(t_0))\mathbf{N}(t_0) \neq \mathbf{0} ,$$

and thus we can define

$$\mathbf{u}_3 = \frac{1}{\|\mathbf{w}_3\|}\mathbf{w}_3 .$$

The orthonormal basis may be either left handed or right handed, and so we may have either $\mathbf{u}_3 = \mathbf{B}(t_0)$, which will be the case if $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is right handed, or else $\mathbf{u}_3 = -\mathbf{B}(t_0)$, which will be the case if $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is left handed.

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**Exercise 3:** Consider the helix

$$\mathbf{x}(t) := (r\cos t, r\sin t, bt)$$

where $r > 0$ but where $b$ may be either positive or negative. Show that the Gram-Schmidt orthonormalization procedure applied to $\mathbf{x}'(t)$, $\mathbf{x}''(t)$ and $\mathbf{x}'''(t)$ yields $\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\}$ if $b > 0$, and yields $\{\mathbf{T}(t), \mathbf{N}(t), -\mathbf{B}(t)\}$ if $b < 0$.

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## 0.3 Differentiating time dependent orthonormal bases

Let $\mathbf{u}_j(t)$ be continuously differentiable in $\mathbb{R}^3$ on $(a, b)$ for $j = 1, 2, 3$. Suppose that $\{\mathbf{u}_1(t), \mathbf{u}_2(t), \mathbf{u}_3(t)\}$ is orthonormal for each $t$.

Since $\{\mathbf{u}_1(t), \mathbf{u}_2(t), \mathbf{u}_3(t)\}$ is an orthonormal basis, we may expand $\mathbf{u}_i'(t)$ as a linear combination of these basis vectors for each $i = 1, 2, 3$:

$$\mathbf{u}_i'(t) = \sum_{j=1}^{3} [\mathbf{u}_i'(t)\cdot\mathbf{u}_j(t)]\mathbf{u}_j(t) .$$

There is a convenient way to express this conclusion: First, define

$$A_{i,j}(t) := \mathbf{u}_i'(t)\cdot\mathbf{u}_j(t) \qquad \text{for } 1 \leq i, j \leq 3 .$$

We can then arrange the 9 functions $A_{i,j}$, $1 \leq i, j \leq 3$ in a *matrix*; i.e., a rectangular array in which we place $A_{i,j}$ in the $i$th row and the $j$th column. Call this matrix $A(t)$. By definition we have

$$A(t) := \begin{bmatrix} A_{1,1}(t) & A_{1,2}(t) & A_{1,3}(t) \\ A_{2,1}(t) & A_{2,2}(t) & A_{2,3}(t) \\ A_{3,1}(t) & A_{3,2}(t) & A_{3,3}(t) \end{bmatrix} .$$

Then we can write

$$\mathbf{u}_i'(t) = \sum_{j=1}^{3} A_{i,j}(t)\mathbf{u}_j(t) \qquad j = 1, 2, 3 .$$

It turns out that the matrix $A(t)$ has a relatively simple form:

$$A(t) = \begin{bmatrix} 0 & A_{1,2}(t) & A_{1,3}(t) \\ -A_{1,2}(t) & 0 & A_{2,3}(t) \\ -A_{1,3}(t) & -A_{2,3}(t) & 0 \end{bmatrix} . \tag{0.1}$$

That is, all of the diagonal elements are zero, and the matrix is *antisymmetric*, meaning that

$$A_{i,j}(t) = -A_{j,i}(t) \qquad \text{for all } 1 \le i, j \le 3 \ . \tag{0.2}$$

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**Exercise 4:** Since $\{\mathbf{u}_1(t), \mathbf{u}_2(t), \mathbf{u}_3(t)\}$ is orthonromal for all $t \in (a, b)$,

$$\mathbf{u}_i(t) \cdot \mathbf{u}_j(t)$$

is constant: The value is 1 if $i = j$ and 0 if $i \neq j$.

Therefore

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{u}_i(t) \cdot \mathbf{u}_j(t)] \ .$$

Use this to prove (0.2) and then (0.1) for all $t \in (a, b)$.

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So far, the time dependent orthonormal basis $\{\mathbf{u}_1(t), \mathbf{u}_2(t), \mathbf{u}_3(t)\}$ was arbitrary, apart from the requirement that each $\mathbf{u}_i(t)$ be differentiable.

Now let us suppose we are given a thrice continuously differentiable curve $\mathbf{x}(t)$ deined on $(a, b)$, and let us suppose that that $\{\mathbf{u}_1(t), \mathbf{u}_2(t), \mathbf{u}_3(t)\}$ is obtained by applying the Gram-Schmidt orthonormalization procedure to $\{\mathbf{x}'(t), \mathbf{x}''(t), \mathbf{x}'''(t)\}$ for all $t \in (a, b)$, In partiicular, we are assuming that for no $t \in (a, b)$ do all three vectors in $\{\mathbf{x}'(t), \mathbf{x}''(t), \mathbf{x}'''(t)\}$ lie in a common plan through the origin. In this case, there is further simplification.

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**Exercise 5:** Recalling that if $\{\mathbf{u}_1(t), \mathbf{u}_2(t), \mathbf{u}_3(t)\}$ is obtained by applying the Gram-Schmidt orthonormalization procedure to $\{\mathbf{x}'(t), \mathbf{x}''(t), \mathbf{x}'''(t)\}$, then $\mathbf{u}_1(t)$ is a mutiple (time dependent) of $\mathbf{x}'(t)$ so that $\mathbf{u}_1'(t)$ is a linear combination of $\mathbf{x}'(t)$ and $\mathbf{x}''(t)$. Using this, show that

$$\mathbf{u}_1'(t) \cdot \mathbf{u}_3(t) = 0$$

for all $t \in (a, b)$.

In particular, when $\{\mathbf{u}_1(t), \mathbf{u}_2(t), \mathbf{u}_3(t)\}$ is obtained in this way,

$$A(t) = \begin{bmatrix} 0 & A_{1,2}(t) & 0 \\ -A_{1,2}(t) & 0 & A_{2,3}(t) \\ 0 & -A_{2,3}(t) & 0 \end{bmatrix} \ . \tag{0.3}$$

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Now if we define

$$\kappa(t) := A_{1,2}(t) \qquad \text{and} \qquad \tau(t) := A_{2,3}(t) \ ,$$

we have

$$A(t) = \begin{bmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{bmatrix} \ ,$$

and therefore that

$$\begin{aligned} \mathbf{u}_1'(t) &= \kappa(t)\mathbf{u}_2(t) \\ \mathbf{u}_2'(t) &= -\kappa(t)\mathbf{u}_1(t) + \tau(t)\mathbf{u}_3(t) \\ \mathbf{u}_3'(t) &= -\tau(t)\mathbf{u}_2(t) \ . \end{aligned}$$

As we have seens above, if $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is right handed, then

$$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\} ,$$

and (0.4) reduces to the usual Frenet-Serret equations.

However, if $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is left handed, then

$$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \{\mathbf{T}(t), \mathbf{N}(t), -\mathbf{B}(t)\} ,$$

and in this case if we change $\tau(t)$ to $-\tau(t)$, (0.4) reduces to the usual Frenet-Serret equations.

Thus, if we define

$$\tau(t) = \mathbf{u}_2'(t) \cdot \mathbf{u}_3(t) ,$$

$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ satisfies the Frenet-Serret equations *whethr or not* the basis produced by the Gram-Schmidt orthonmormalization prcedure is right handed. However, if the basis is left handed, $\tau$ will have the opposite sign from the usual definition in which we give preference to right handed bases.

## 0.4 Higher dimesions

We have not used any cross products at any point in our analysis so far. It has all been based on the dot product, and thus it extends to arbitrary dimensions.

The first step is to extend the Gram-Schmidt orthonormalization procedure to higher dimension. Given a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of non-zero vectors in $\mathbb{R}^n$, define

$$\mathbf{u}_1 := \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 .$$

We now proceed recursively. Suppose $\mathbf{u}_j$, $j = 1, \ldots, m$ are already defined. We then define

$$\mathbf{w}_{m+1} = \mathbf{v}_{m+1} - \sum_{j=1}^{m} [\mathbf{v}_{m+1} \cdot \mathbf{u}_j]\mathbf{u}_j .$$

and then provided that

$$\mathbf{w}_{m+1} \neq \mathbf{0} ,$$

we define

$$\mathbf{u}_{m+1} = \frac{1}{\|\mathbf{w}_{m+1}\|}\mathbf{w}_{m+1} .$$

As long as $\|\mathbf{w}_j\| \neq 0$ for any $j = 2, \ldots, n$, so that we are not required to devide by zero, the procedure produces an orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ of $\mathbb{R}^n$. If for some $j = 2, \ldots, n$, $\mathbf{w}_j = \mathbf{0}$, then we say the procedure *terminates early.*

Since the construction of $\mathbf{u}_j$ involves only $\{\mathbf{v}_1, \ldots \mathbf{v}_j\}$ one might expect that $\mathbf{u}_j$ is a linear combination of only the vectors in $\{\mathbf{v}_1, \ldots \mathbf{v}_j\}$. In fact:

**0.2 THEOREM.** *Let $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ be a set of $n$ vectors in $\mathbb{R}^n$, and suppose that the Gram-Schmidt orthonormalization procedure does not terminate early, but instead produces an orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$. Then, for each $j = 1, \ldots, n$, $\mathbf{u}_j$ is a linear combintation of $\{\mathbf{v}_1, \ldots, \mathbf{v}_j\}$. and conversely that for each $j = 1, \ldots, n$, $\mathbf{v}_j$ is a linear combintation of $\{\mathbf{u}_1, \ldots, \mathbf{u}_j\}$.*

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**Exercise 6:** Let

$$\mathbf{v}_1 = (1,1,1,1) \ , \quad \mathbf{v}_2 = (0,0,2,2) \ , \quad \mathbf{v}_3 = (2,0,2,0) \ , \quad \mathbf{v}_4 = (2,0,0,2) \ . \tag{0.4}$$

Compute the orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ that the Gram-Schmidt orthonormalization procedure yields starting from the four vectors given in (0.4).

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**Exercise 7:** Prove Theorem 0.2.

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Now suppose that

$$\{\mathbf{v}_1(t), \ldots, \mathbf{v}_n(t)\} = \{\mathbf{x}'(t), \ldots . \mathbf{x}^{(n)}(t)\}$$

where $\mathbf{x}^{(j)}(t)$ denotes the $j$th derivative of $\mathbf{x}(t)$ with respect to $t$.

By the theorem proved in the previous exercise, the Gram-Schmidt orthonormalization procedure produces (provided it does not teminate early) an othonormal basis $\{\mathbf{u}_1(t), \ldots, \mathbf{u}_n(t)\}$ in which for each $j = 1 \ldots, n$, $\mathbf{u}_j(t)$ is a linear combination of the vectors in $\{\mathbf{x}'(t), \ldots . \mathbf{x}^{(j)}(t)\}$.

Then for each $i = 1, \ldots, n-1$, $\mathbf{u}_i'(t)$ is a linear combination of $\{\mathbf{x}'(t), \ldots, \mathbf{x}^{(i+1)}(t)\}$, and hence of $\{\mathbf{u}_1(t), \ldots, \mathbf{u}_{i+1}(t)\}$. In particular,

$$\mathbf{u}_i'(t) \cdot \mathbf{u}_j(t) = 0 \qquad \text{for all } j > i+1 \ .$$

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**Exercise 8: (Extra Credit)** Let $\mathbf{x}(t)$ be an $n$ times continuously differentiable curve in $\mathbb{R}^n$ defined on $(a,b)$. Suppose that on this interval, the Gram-Schmidt orthonormalization procedure produces an othonormal basis $\{\mathbf{u}_1(t), \ldots, \mathbf{u}_n(t)\}$ out of $\{\mathbf{x}'(t), \ldots, \mathbf{x}^{(n)}(t)\}$. Define

$$A_{i,j} = \mathbf{u}_i'(t) \cdot \mathbf{u}_j(t) \qquad \text{for } 1 \le i,j \le n \ .$$

Show that

$$A_{i,j}(t) = -A_{j,i}(t) \qquad \text{for all} 1 \le 1, j \le n \ ,$$

and that

$$A_{i,j}(t) = 0 \qquad \text{for all } j > i+1 \ .$$

Then show that if $A(t)$ is the $n \times n$ matrix whose entry in the $i$th row and $j$th colum is $A_{i,j}(t)$,

$$A(t) = \begin{bmatrix} 0 & A_{1,2}(t) & 0 & 0 & 0 & \ldots \\ -A_{1,2}(t) & 0 & A_{2,3} & 0 & 0 & \ldots \\ 0 & -A_{2,3}(t) & 0 & A_{3,4}(t) & 0 & \ldots \\ 0 & 0 & -A_{3,4}(t) & 0 & A_{4,5}(t) & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} .$$

Define the $n-1$ *genralized curvatures* $\kappa_j$, $j = 1, \ldots, n-1$ by

$$\kappa_j(t) = \frac{1}{v(t)} \mathbf{u}_j'(t) \cdot \mathbf{u}_{j+1}(t) \ ,$$

where $v(t) = \|\mathbf{x}'(t)\|$. Define

$$\kappa_0(t) = \kappa_{n+1}(t) = 0 \ .$$

Likewise define $\mathbf{u}_0(t) = \mathbf{u}_{n+1}(t) = \mathbf{0}$.

Show that for $i = 1, \dots n$,

$$\mathbf{u}_i'(t) = v(t)[\kappa_{i+1}(t)\mathbf{u}_{i+1}(t) - \kappa_{i-1}(t)\mathbf{u}_{i-1}(t)] \ . \tag{0.5}$$

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The equations (0.5) are the *generalized Frenet-Serret formulae* for dimension $n \geq 3$. In particular, when $n = 4$, we have

$$A(t) = v(t)\begin{bmatrix} 0 & \kappa_1(t) & 0 & 0 \\ -\kappa_1(t) & 0 & \kappa_2(t) & 0 \\ 0 & -\kappa_2(t) & 0 & \kappa_3(t) \\ 0 & 0 & -\kappa_3(t) & 0 \end{bmatrix}$$

In this case $\kappa_1(t) = \kappa(t)$, the curvature, as we have defined it for all $n$. The "generalized curvatures" $\kappa_2(t)$ and $\kappa_3(t)$ are analogs of the torsion, describing the motion of the osculating plane in $\mathbb{R}^4$.