Challenge Problem Set 1, Math 291 Fall 2013

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This challenge problem set concerns finding the distance between lines and planes in four dimensional space. Along the way, it will introduce some methodology for producing orthonormal sets of vectors in higher dimensional spaces.

0.1 Finding a unit vector that is orthogonal to two given vectors in $\mathbb{R}^3$

Let $\mathbf{v}$ and $\mathbf{w}$ be two vectors in $\mathbb{R}^3$, neither one a multiple of the other. We know one way to find a unit vector $\mathbf{u}$ that is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$: Use the cross product. By properties of the cross product

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\times\mathbf{w}\|}\mathbf{v}\times\mathbf{w}$$

works, as does its negative, which is what you would get using the vectors in the other order.

For examples, if $\mathbf{v} = (2,1,2)$ and $\mathbf{w} = (1,1,4)$, we compute

$$\mathbf{v}\times\mathbf{w} = (2,-6,1) \ ,$$

and so

$$\mathbf{u} = \frac{1}{\sqrt{41}}(2,-6,1) \ ,$$

or its negative is the desired unit vector.

In this challenge problem set, we will need to do something similar in four dimensional space. But the cross product is specific to three dimensions. So we need another way. Let us work with dot products instead, since they can be used to test for orthogonality in any number of dimensions.

A vector $\mathbf{x} = (x,y,z)$ is orthogonal to $\mathbf{v} = (2,1,2)$ if and only if $\mathbf{v}\cdot\mathbf{x} = 0$, which works out to

$$2x + y + 2z = 0 \ .$$

Likewise, we find that $\mathbf{x}$ is orthogonal to $\mathbf{w} = (1,1,4)$ if and only if

$$x + y + 4z = 0 \ .$$

Therefore, $\mathbf{x}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$ if and only x, y and z solve the system of equations

$$\begin{aligned} 2x + y + 2z &= 0 \\ x + y + 4z &= 0 \end{aligned}$$

We can find all solutions of this system of equations, which will be a line in $\mathbb{R}^3$, by elimination of variables. If we subtract the top equation from the bottom one, we get the equivalent system

$$\begin{aligned} 2x + y + 2z &= 0 \\ -x + 2z &= 0 \end{aligned}$$

It is equivalent because we can recover the original system bay adding the second equation back onto the first. But now we have eliminated the variable y from the second equation. It now tells us that

$$x = 2z.$$

Substituting this into the third equation, we find

$$y + 6z = 0 \ .$$

We have learned that for any solution (x, y, z), $x = 2z$ and $y = -6z$ so that

$$(x, y, z) = (2, -6, 1)z \ ,$$

where z is any real number, is the general form of the solution. We have, in fact, parameterized the solution set, which is a line: Changing the name of z to t, just so it looks more like a parameter, we have found that the points $\mathbf{x} = (x, y, z)$ that solve the system $\mathbf{x} \cdot \mathbf{v} = 0$ and $\mathbf{x} \cdot \mathbf{w} = 0$ are exactly the points on the line through the origin parameterized by

$$\mathbf{x}(t) = t(2, -6, 1) \ .$$

There are two unit vectors on this line, namely

$$\pm \frac{1}{\sqrt{41}}(2, -6, 1) \ ,$$

which is what we found before.

Let us compare approaches. The cross product is more complicated than the dot product, but using it we can find our orthogonal unit vector by *direct computation*, we do not need to solve any equations. Using the dot product, we have to solve equations, but they are simple systems of linear equations, and so this is easily accomplished. Moreover, this approach works in any dimension.

0.2 Finding a unit vector that is orthogonal to three given vectors in $\mathbb{R}^4$

Consider the three vectors in $\mathbb{R}^4$

$$\mathbf{v} = (1, 2, 3, 1) \ , \qquad \mathbf{w} = (1, 3, 1, 0) \ , \qquad \mathbf{z} = (2, 2, 1, 4) \ .$$

If $\mathbf{x} = (x, y, z, w)$ is any vector in $\mathbb{R}^4$ that is orthogonal to each of these three vectors, then:

$$\begin{aligned} \mathbf{v} \cdot \mathbf{x} &= 0 \\ \mathbf{w} \cdot \mathbf{x} &= 0 \\ \mathbf{z} \cdot \mathbf{x} &= 0 \end{aligned}$$

Writing this system out explicitly in terms of x, y, z and w, we find

$$\begin{aligned} x + 2y + 3z + w &= 0 \\ -x + 3y + z &= 0 \\ 2x + 2y + z + 4w &= 0 \end{aligned}$$

By successively subtracting multiples of one equation from another, we find (check this) that this system of equations is equivalent to

$$\begin{aligned} x + 2y + 3z + w &= 0 \\ y - 2z - w &= 0 \\ -9z &= 0 \end{aligned}$$

The last equation tells us $z = 0$, and then, using this, the next to the last tells us $y = w$, and then the first equation then tells us that $x = -3w$. Thus

$$\mathbf{x} = (x, y, z, w) = w(-3, 1, 0, 1) \ ,$$

and the solution set is the line through the origin and $(-3, 1, 0, 1)$. The two unit vectors in the solution sets are

$$\mathbf{u} = \pm\frac{1}{\sqrt{11}}(-3, 1, 0, 1) \ .$$

You can (and should) easily check that these are unit vectors orthogonal to the three given vectors.

Exercise 1: Let

$$\mathbf{v} = (1, 1, -1, 1) \ , \qquad \mathbf{w} = (1, 3 - 1, 3) \ , \qquad \mathbf{z} = (1, 0, 1, 2) \ .$$

Find both unit vectors in $\mathbb{R}^4$ that are orthogonal to each of $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{z}$.

0.3 Lines and planes in $\mathbb{R}^4$

In $\mathbb{R}^3$ a line is either parallel to a plane, or else it intersects it at some point, in which case we would say that the distance between the line and the plane is zero. If the line is parallel to the plane, then all points on the line are the same distance form the plane, and we can compute the distance between the line and the plane by computing the distance between an arbitrary point on the line and the plane, and we know how to do this.

In $\mathbb{R}^4$ a plane is a set of points parameterized by

$$\mathbf{x}(s, t) = \mathbf{x}_0 + s\mathbf{v} + t\mathbf{w} \tag{0.1}$$

where $\mathbf{v}$ and $\mathbf{w}$ are two vectors in $\mathbb{R}^4$ such that neither is a multiple of the other. As the parameters s and t vary, $\mathbf{x}(s, t)$ traces out a plane in $\mathbb{R}^4$.

Likewise, in $\mathbb{R}^4$ a line is a set of points parameterized by

$$\mathbf{y}(u) = \mathbf{y}_0 + u\mathbf{z} \tag{0.2}$$

where $\mathbf{z}$ is some non-zero vector in $\mathbb{R}^4$. As the parameter u varies $\mathbf{y}(u)$ traces out a line in $\mathbb{R}^4$.

Now, generically, in $\mathbb{R}^4$ a line and plane will not intersect. Given a line and a plane in $\mathbb{R}^4$, let $\mathbf{y}(u)$ and $\mathbf{x}(s,t)$ be parameterizations of them. We then ask how small can can the distance between $\mathbf{y}(u)$ and $\mathbf{x}(s,t)$ become as the parameters vary? That is, we want to compute

$$\min_{u,s,t}\{\|\mathbf{y}(u)-\mathbf{x}(s,t)\|\ \}\ .$$

Exercise 2. Let the line be parameterized by (0.2) and the plane by (0.1). Let $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4\}$ be an orthonormal basis of $\mathbb{R}^4$ in which $\mathbf{u}_1$ is orthogonal to $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{z}$. Show that in this case

$$\|\mathbf{y}(u)-\mathbf{x}(s,t)\|^2 \geq ((\mathbf{y}_0-\mathbf{x}_0)\cdot\mathbf{u}_1)^2$$

no matter what s, t, and u are. The key is to recall that for any orthonormal basis $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4\}$, and any vector $\mathbf{a}$ in $\mathbb{R}^4$,

$$\|\mathbf{a}\|^2 = \sum_{j=1}^{4}(\mathbf{a}\cdot\mathbf{u}_j)^2\ .$$

What we have done so far shows that the distance between the line and the plane cannot be any smaller than

$$|(\mathbf{y}_0-\mathbf{x}_0)\cdot\mathbf{u}_1|\ ,$$

but it does not yet show that there are values of s, t and u for which

$$\|\mathbf{y}(u)-\mathbf{x}(s,t)\| = |(\mathbf{y}_0-\mathbf{x}_0)\cdot\mathbf{u}_1|\ .$$

To see if that is the case, it helps to make a careful choice of $\mathbf{u}_2$ and $\mathbf{u}_3$: So far we have only made a choice for $\mathbf{u}_1$. Let us choose $\mathbf{u}_2$ to be orthogonal to $\mathbf{u}_1$, $\mathbf{v}$ and $\mathbf{w}$. That is, we choose $\mathbf{u}_2$ so that it is orthogonal to $\mathbf{u}_1$, which is necessary for $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4\}$ to be orthonormal, and also to the direction vectors of the plane.

Next, choose $\mathbf{u}_3$ to be orthogonal to $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{v}$.

Exercise 3. Show that with the choice of the orthonormal basis made above,

$$\begin{aligned}\|\mathbf{y}(u)-\mathbf{x}(s,t)\|^2 &= ((\mathbf{y}_0-\mathbf{x}_0)\cdot\mathbf{u}_1)^2\\ &+ ((\mathbf{y}_0-\mathbf{x}_0)\cdot\mathbf{u}_2+u\mathbf{z}\cdot\mathbf{u}_2)^2\\ &+ ((\mathbf{y}_0-\mathbf{x}_0)\cdot\mathbf{u}_3+u\mathbf{z}\cdot\mathbf{u}_3-t\mathbf{w}\cdot\mathbf{u}_3)^2\\ &+ ((\mathbf{y}_0-\mathbf{x}_0)\cdot\mathbf{u}_4+u\mathbf{z}\cdot\mathbf{u}_4-t\mathbf{w}\cdot\mathbf{u}_4-s\mathbf{v}\cdot\mathbf{u}_4)^2\ .\end{aligned}$$

Then show that provided $\mathbf{z}\cdot\mathbf{u}_2\neq 0$, $\mathbf{w}\cdot\mathbf{u}_3\neq 0$ and $\mathbf{v}\cdot\mathbf{u}_4\neq 0$, there are uniquely determined values of u_0, t_0 and s_0 such that

$$\|\mathbf{y}(u_0)-\mathbf{x}(s_0,t_0)\|^2 = ((\mathbf{y}_0-\mathbf{x}_0)\cdot\mathbf{u}_1)^2\ .$$

Thus, the distance between the line and the plane is $|(\mathbf{y}_0-\mathbf{x}_0)\cdot\mathbf{u}_1|$, and the two closest points are $\mathbf{y}(u_0)$ and $\mathbf{x}(s_0,t_0)$.

Exercise 4. Let

$$\mathbf{v} = (1,1,-1,1)\ , \qquad \mathbf{w} = (1,3-1,3)\ , \qquad \mathbf{z} = (1,0,1,2)\ .$$

Let

$$\mathbf{x}_0 = (1,1,1,1)\ , \qquad \mathbf{y}_0 = (3,1,2,3)\ .$$

Compute the distance between the line parameterized by $\mathbf{y}(u) = \mathbf{y}_0 + u\mathbf{z}$ and the plane parameterized by $\mathbf{x}(s,t) = \mathbf{x}_0 + s\mathbf{v} + t\mathbf{w}$. Also, compute the point $\mathbf{y}(u_0)$ on the line that comes closest to the plane, and the point $\mathbf{x}(s_0,t_0)$ on the plane that comes closest to the line.

Exercise 5. (Extra Credit) In working out Exercise 4, you will have seen that the conditions

$$\mathbf{z}\cdot\mathbf{u}_2 \neq 0\ , \qquad \mathbf{w}\cdot\mathbf{u}_3 \neq 0 \quad \text{and} \quad \mathbf{v}\cdot\mathbf{u}_4 \neq 0 \tag{0.3}$$

were all satisfied.

Show that the conditions (0.3) are always satisfied provided $\mathbf{z}$ is not a linear combination of $\mathbf{v}$ and $\mathbf{w}$, so that the line is not parallel to the plane, which means that the line and plane are not parallel. (We continue to assume also that $\mathbf{z}$ is non-zero and that neither of $\mathbf{v}$ or $\mathbf{w}$ is a multiple of the other.)

To get started, recall that by definition, $\mathbf{u}_2$ is orthogonal to $\mathbf{u}_1$, $\mathbf{v}$ and $\mathbf{w}$. Show that $\mathbf{u}_2\cdot\mathbf{z} = 0$ if and only if $\mathbf{z}$ is a linear combination of $\mathbf{u}_1$, $\mathbf{v}$ and $\mathbf{w}$. One way to do this is to use the fact that

$$\left\{\mathbf{u}_1\ ,\ \mathbf{u}_2\ ,\ \frac{1}{\|\mathbf{v}\|}\mathbf{v}\ ,\ \frac{1}{\|\mathbf{w}_\perp\|}\mathbf{w}_\perp\right\}$$

is an orthonormal basis, where we take the perpendicular component of $\mathbf{w}$ with respect to $\mathbf{v}$, . Since any vector can be written as a linear combination of the vectors in *any* orthonormal basis, we have

$$\mathbf{z} = a\mathbf{u}_1 \ + b\mathbf{u}_2 + c\frac{1}{\|\mathbf{v}\|}\mathbf{v} + d\frac{1}{\|\mathbf{w}_\perp\|}\mathbf{w}_\perp$$

for some numbers a, b, c and d – that you know how to compute. Using this knowledge, show that $a = b = 0$. From this you can conclude that $\mathbf{z}$ would be a linear combination of $\mathbf{v}$ and $\mathbf{w}_\perp$. From here you should be able to show that $\mathbf{z}$ would be a linear combination of $\mathbf{v}$ and $\mathbf{w}$. Thus, whenever that is not the case, $\mathbf{z}\cdot\mathbf{u}_2 \neq 0$.

Proceeding this way, establish the validity of the other two conditions. This proves that the method we have developed always works whenever the line a plane are not parallel. To conclude, find a formula for this case too.