Supercompactness and Maximal Prikry Trees

Ben-Zion Weltsch

Rutgers University

MAMLS 2025

Prikry Forcing

Classical Prikry Forcing

Let U be a normal, κ-complete ultrafilter on κ. Prikry forcing (with respect to U) consists of pairs (s, A) where s is a finite increasing sequence of ordinals below κ, and $A \in U$. We say $(s, A) \leq (t, B)$ if s extends t, $A \subseteq B$, and $s \setminus t \subseteq B \setminus A$.

Prikry Forcing

Classical Prikry Forcing

Let U be a normal, κ-complete ultrafilter on κ. Prikry forcing (with respect to U) consists of pairs (s, A) where s is a finite increasing sequence of ordinals below κ, and $A \in U$. We say $(s, A) \leq (t, B)$ if s extends t, $A \subseteq B$, and $s \setminus t \subseteq B \setminus A$.

After doing Prikry forcing:

1. If G is the generic, $S = \bigcup\{s : (s, A \in G)\}$ is an ω sequence, a **Prikry sequence**.

Prikry Forcing

Classical Prikry Forcing

Let U be a normal, κ-complete ultrafilter on κ. Prikry forcing (with respect to U) consists of pairs (s, A) where s is a finite increasing sequence of ordinals below κ, and $A \in U$. We say $(s, A) \leq (t, B)$ if s extends t, $A \subseteq B$, and $s \setminus t \subseteq B \setminus A$.

After doing Prikry forcing:

1. If G is the generic, $S = \bigcup\{s : (s, A \in G)\}$ is an ω sequence, a **Prikry sequence**.
2. By density S is cofinal in κ, so $\mathrm{cf}(\kappa) = \omega$

Prikry Forcing

Classical Prikry Forcing

Let U be a normal, κ-complete ultrafilter on κ. Prikry forcing (with respect to U) consists of pairs (s, A) where s is a finite increasing sequence of ordinals below κ, and $A \in U$. We say $(s, A) \leq (t, B)$ if s extends t, $A \subseteq B$, and $s \setminus t \subseteq B \setminus A$.

After doing Prikry forcing:

1. If G is the generic, $S = \bigcup\{s : (s, A \in G)\}$ is an ω sequence, a **Prikry sequence**.
2. By density S is cofinal in κ, so $\mathrm{cf}(\kappa) = \omega$
3. All cardinals are preserved

Tree Prikry Forcing

How to do Prikry forcing with other kinds of ultrafilters:

Tree Prikry Forcing

How to do Prikry forcing with other kinds of ultrafilters:

Tree Prikry Forcing

Let U be a countably complete ultrafilter on a directed set $(D, <)$. The Tree Prikry forcing, denoted $\mathbb{P}_U$, consists of trees $T \subseteq D^{<\omega}$ with a stem s (i.e. for all $t \in T$, either $s \subseteq t$ or $t \subseteq s$) and for all $t \supseteq s$ in T, we have $\{d \in D \colon t^\frown d \in T\} \in U$.

Tree Prikry Forcing

How to do Prikry forcing with other kinds of ultrafilters:

Tree Prikry Forcing

Let U be a countably complete ultrafilter on a directed set $(D, <)$. The Tree Prikry forcing, denoted $\mathbb{P}_U$, consists of trees $T \subseteq D^{<\omega}$ with a stem s (i.e. for all $t \in T$, either $s \subseteq t$ or $t \subseteq s$) and for all $t \supseteq s$ in T, we have $\{d \in D \colon t^\frown d \in T\} \in U$.

A generic G for Tree Prikry forcing also produces a **Prikry sequence** $S = \bigcap\{T : T \in G\}$, and S is cofinal in $(D, <)$ and has order type ω.

Tree Prikry Forcing

How to do Prikry forcing with other kinds of ultrafilters:

Tree Prikry Forcing

Let U be a countably complete ultrafilter on a directed set $(D, <)$. The Tree Prikry forcing, denoted $\mathbb{P}_U$, consists of trees $T \subseteq D^{<\omega}$ with a stem s (i.e. for all $t \in T$, either $s \subseteq t$ or $t \subseteq s$) and for all $t \supseteq s$ in T, we have $\{d \in D\colon t^\frown d \in T\} \in U$.

A generic G for Tree Prikry forcing also produces a **Prikry sequence** $S = \bigcap\{T : T \in G\}$, and S is cofinal in $(D, <)$ and has order type ω. Examples:

1. If $(D, <) = (\kappa, \in)$ and U is a normal ultrafilter on κ

Tree Prikry Forcing

How to do Prikry forcing with other kinds of ultrafilters:

Tree Prikry Forcing

Let U be a countably complete ultrafilter on a directed set $(D, <)$. The Tree Prikry forcing, denoted $\mathbb{P}_U$, consists of trees $T \subseteq D^{<\omega}$ with a stem s (i.e. for all $t \in T$, either $s \subseteq t$ or $t \subseteq s$) and for all $t \supseteq s$ in T, we have $\{d \in D \colon t^\frown d \in T\} \in U$.

A generic G for Tree Prikry forcing also produces a **Prikry sequence** $S = \bigcap\{T : T \in G\}$, and S is cofinal in $(D, <)$ and has order type ω. Examples:

1. If $(D, <) = (\kappa, \in)$ and U is a normal ultrafilter on κ
2. $(\kappa^2, \in \times \in)$ and U^2

Tree Prikry Forcing

How to do Prikry forcing with other kinds of ultrafilters:

Tree Prikry Forcing

Let U be a countably complete ultrafilter on a directed set $(D, <)$. The Tree Prikry forcing, denoted $\mathbb{P}_U$, consists of trees $T \subseteq D^{<\omega}$ with a stem s (i.e. for all $t \in T$, either $s \subseteq t$ or $t \subseteq s$) and for all $t \supseteq s$ in T, we have $\{d \in D\colon t^\frown d \in T\} \in U$.

A generic G for Tree Prikry forcing also produces a **Prikry sequence** $S = \bigcap\{T : T \in G\}$, and S is cofinal in $(D, <)$ and has order type ω. Examples:

1. If $(D, <) = (\kappa, \in)$ and U is a normal ultrafilter on κ
2. $(\kappa^2, \in \times \in)$ and U^2
3. $(P_\kappa(\lambda), \prec)$ and U is a supercompactness measure

Supercompactness

Supercompactness is a very powerful strengthening of measurability.

Supercompactness

Supercompactness is a very powerful strengthening of measurability.

Definitions

1. $P_\kappa(\lambda) = \{x \subseteq \lambda : |x| < \kappa\}$

Supercompactness

Supercompactness is a very powerful strengthening of measurability.

Definitions

1. $P_\kappa(\lambda) = \{x \subseteq \lambda : |x| < \kappa\}$
2. A cardinal κ is *supercompact* if for all $\lambda \geq \kappa$ there is a *normal*, *fine* ultrafilter U on $P_\kappa(\lambda)$. We say U is a *supercompactness* measure.

Supercompactness

Supercompactness is a very powerful strengthening of measurability.

Definitions

1. $P_\kappa(\lambda) = \{x \subseteq \lambda : |x| < \kappa\}$
2. A cardinal κ is *supercompact* if for all $\lambda \geq \kappa$ there is a *normal*, *fine* ultrafilter U on $P_\kappa(\lambda)$. We say U is a *supercompactness* measure.
3. If $x, y \in P_\kappa(\lambda)$ we say $x \prec y$ if $x \subseteq y$ and $|x| < |y \cap \kappa|$.

Supercompactness

Supercompactness is a very powerful strengthening of measurability.

Definitions

1. $P_\kappa(\lambda) = \{x \subseteq \lambda : |x| < \kappa\}$
2. A cardinal κ is *supercompact* if for all $\lambda \geq \kappa$ there is a *normal*, *fine* ultrafilter U on $P_\kappa(\lambda)$. We say U is a *supercompactness* measure.
3. If $x, y \in P_\kappa(\lambda)$ we say $x \prec y$ if $x \subseteq y$ and $|x| < |y \cap \kappa|$.

Supercompactness

Supercompactness is a very powerful strengthening of measurability.

Definitions

1. $P_\kappa(\lambda) = \{x \subseteq \lambda : |x| < \kappa\}$
2. A cardinal κ is *supercompact* if for all $\lambda \geq \kappa$ there is a *normal*, *fine* ultrafilter U on $P_\kappa(\lambda)$. We say U is a *supercompactness* measure.
3. If $x, y \in P_\kappa(\lambda)$ we say $x \prec y$ if $x \subseteq y$ and $|x| < |y \cap \kappa|$.

Supercompact Tree Prikry

We use the term *supercompact Tree Prikry forcing* to denote Tree Prikry forcing with $(U, \prec)$ where U is a supercompactness measure.

Maximality

Maximality Property

Let U be an ultrafilter. $\mathbb{P}_U$ has the **maximality property** if whenever S_1, S_2 are Prikry sequences and $V[S_1] \subseteq V[S_2]$, then S_1 is, modulo an initial segment, a subsequence of S_2.

Maximality

Maximality Property

Let U be an ultrafilter. $\mathbb{P}_U$ has the **maximality property** if whenever S_1, S_2 are Prikry sequences and $V[S_1] \subseteq V[S_2]$, then S_1 is, modulo an initial segment, a subsequence of S_2.

Examples:

1. If U is a normal ultrafilter on κ then $\mathbb{P}_U$ has the maximality property.

Maximality

Maximality Property

Let U be an ultrafilter. $\mathbb{P}_U$ has the **maximality property** if whenever S_1, S_2 are Prikry sequences and $V[S_1] \subseteq V[S_2]$, then S_1 is, modulo an initial segment, a subsequence of S_2.

Examples:

1. If U is a normal ultrafilter on κ then $\mathbb{P}_U$ has the maximality property.
2. If $U = W^2$ for some W then $\mathbb{P}_U$ does not have the maximality property.

Maximality

Maximality Property

Let U be an ultrafilter. $\mathbb{P}_U$ has the **maximality property** if whenever S_1, S_2 are Prikry sequences and $V[S_1] \subseteq V[S_2]$, then S_1 is, modulo an initial segment, a subsequence of S_2.

Examples:

1. If U is a normal ultrafilter on κ then $\mathbb{P}_U$ has the maximality property.
2. If $U = W^2$ for some W then $\mathbb{P}_U$ does not have the maximality property.

Maximality

Maximality Property

Let U be an ultrafilter. $\mathbb{P}_U$ has the **maximality property** if whenever S_1, S_2 are Prikry sequences and $V[S_1] \subseteq V[S_2]$, then S_1 is, modulo an initial segment, a subsequence of S_2.

Examples:

1. If U is a normal ultrafilter on κ then $\mathbb{P}_U$ has the maximality property.
2. If $U = W^2$ for some W then $\mathbb{P}_U$ does not have the maximality property.

Conjecture (Woodin, 90's)

Supercompact Tree Prikry forcing has the maximality property.

What is known?

Conjecture (Woodin, 90's)

Supercompact Tree Prikry forcing has the maximality property.

What is known?

Conjecture (Woodin, 90's)

Supercompact Tree Prikry forcing has the maximality property.

1. If U is a *strongly* normal ultrafilter on $P_\kappa(\lambda)$, then $\mathbb{P}_U$ has the maximality property. (Hamkins, 1997 [1])

What is known?

Conjecture (Woodin, 90's)

Supercompact Tree Prikry forcing has the maximality property.

1. If U is a *strongly* normal ultrafilter on $P_\kappa(\lambda)$, then $\mathbb{P}_U$ has the maximality property. (Hamkins, 1997 [1])
2. Not every supercompactness measure is strongly normal. (Menas, 1985 [2])

What is known?

Conjecture (Woodin, 90's)

Supercompact Tree Prikry forcing has the maximality property.

1. If U is a *strongly* normal ultrafilter on $P_\kappa(\lambda)$, then $\mathbb{P}_U$ has the maximality property. (Hamkins, 1997 [1])
2. Not every supercompactness measure is strongly normal. (Menas, 1985 [2])

What is known?

Conjecture (Woodin, 90's)

Supercompact Tree Prikry forcing has the maximality property.

1. If U is a *strongly* normal ultrafilter on $P_\kappa(\lambda)$, then $\mathbb{P}_U$ has the maximality property. (Hamkins, 1997 [1])
2. Not every supercompactness measure is strongly normal. (Menas, 1985 [2])

Theorem (W.)

Let U be a supercompactness measure on $P_\kappa(\lambda)$ where either λ is regular or $\mathrm{cf}(\lambda) \leq \kappa$. Then Tree Priky forcing with U has the maximality property.

Questions

Questions

Must P_U have the maximality property if:

1. $\kappa < \mathrm{cf}(\lambda) < \lambda$ and U is a supercompactness measure on $P_\kappa(\lambda)$?

Questions

Questions

Must P_U have the maximality property if:

1. $\kappa < \mathrm{cf}(\lambda) < \lambda$ and U is a supercompactness measure on $P_\kappa(\lambda)$?
2. If U is a strongly compact measure?

Questions

Questions

Must P_U have the maximality property if:

1. $\kappa < \mathrm{cf}(\lambda) < \lambda$ and U is a supercompactness measure on $P_\kappa(\lambda)$?
2. If U is a strongly compact measure?
3. If U is Rudin-Keisler minimal among uniform ultrafilters on its underlying set?

Questions

Questions

Must P_U have the maximality property if:

1. $\kappa < \mathrm{cf}(\lambda) < \lambda$ and U is a supercompactness measure on $P_\kappa(\lambda)$?
2. If U is a strongly compact measure?
3. If U is Rudin-Keisler minimal among uniform ultrafilters on its underlying set?

Questions

Questions

Must P_U have the maximality property if:

1. $\kappa < \mathrm{cf}(\lambda) < \lambda$ and U is a supercompactness measure on $P_\kappa(\lambda)$?
2. If U is a strongly compact measure?
3. If U is Rudin-Keisler minimal among uniform ultrafilters on its underlying set?

Conjecture

If U is an ultrafilter on a set of size λ then $\mathbb{P}_U$ does not have the maximality property iff there are uniform ultrafilters W_0, W_1 on λ such that $W_0 \times W_1 \leq_{RK} U$.

Thanks!

Thanks for listening! Thanks to Tom for giving me ideas on this problem, to Dima for her advising, and to Gabe Goldberg for showing me this problem.

[1] Joel David Hamkins. "Canonical Seeds and Prikry Trees". In: *The Journal of Symbolic Logic* 62.2 (1997), pp. 373–396. ISSN: 00224812. URL: http://www.jstor.org/stable/2275538 (visited on 08/08/2025).

[2] Telis K. Menas. "A Combinatorial Property of $P_\kappa(\lambda)$". In: *The Journal of Symbolic Logic* 41.1 (1976), pp. 225–234. ISSN: 00224812. URL: http://www.jstor.org/stable/2272962 (visited on 08/08/2025).