Galvin's Failure at $P_\kappa(\lambda)$

Ben-Zion Weltsch

Rutgers University

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Galvin's Theorem

The following, due to Fred Galvin, was published in a paper by Baumgartner, Hajnal, and Maté [1]. Galvin looked at this theorem as generalization of (non)regular ultrafilters.

Galvin's Theorem

Suppose $\kappa^{<\kappa} = \kappa$ and F is a normal filter over κ. Then for whenever $\langle X_\alpha : \alpha < \kappa^+ \rangle \subseteq F$ there is some $Y \in [\kappa^+]^\kappa$ such that $\bigcap_{\alpha \in Y} X_\alpha \in F$.

F is *normal* if every regressive function is constant on a positive set iff F is closed under diagonal intersections of size κ.

We can extract the following combinatorial property from Galvin's theorem:

Definition

Let F be a filter and $\kappa \leq \lambda$. We say $Gal(F, \kappa, \lambda)$ holds iff whenever $\langle X_\alpha : \alpha < \lambda \rangle \subseteq F$ there is a $Y \in [\lambda]^\kappa$ such that $\bigcap_{\alpha \in Y} X_\alpha \in F$.

The Galvin Property

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Examples

1. Galvin's Theorem says that if $\kappa^{<\kappa} = \kappa$ and F is normal then $Gal(F, \kappa, \kappa^+)$.
2. If $Gal(F, \kappa, \lambda)$ holds then we can decrease κ or increase λ.
3. If U is κ-complete on κ then U is Tukey-top iff $\neg Gal(U, \kappa, 2^\kappa)$ [2].

$P_\kappa(\lambda)$ Filters

Definitions

1. $P_\kappa(\lambda) = \{X \subseteq \lambda : |X| < \kappa\}$.
2. A filter F on $P_\kappa(\lambda)$ is **fine** if for all $x \in P_\kappa(\lambda)$, $\hat{x} = \{y \in P_\kappa(\lambda) : x \subseteq y\} \in F$
3. A filter F on $P_\kappa(\lambda)$ is **normal** if whenever $\langle X_\alpha : \alpha < \lambda \rangle \subseteq F$, $\triangle_{\alpha<\lambda} X_\alpha = \{x \in P_\kappa(\lambda) : x \in \bigcap_{\alpha \in x} X_\alpha\} \in U$

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Remarks

1. If U is an ultrafilter over $P_\kappa(\lambda)$ then U is U is normal iff $[id]_U = j_U'' \lambda$.
2. κ is strongly compact iff $P_\kappa(\lambda)$ carries a fine ultrafilter for every $\lambda \geq \kappa$.
3. κ is supercompact iff $P_\kappa(\lambda)$ carries a normal fine ultrafilter for every $\lambda \geq \kappa$.

The Galvin Property on $P_\kappa(\lambda)$ Filters

- Recall Galvin's theorem that whenever $\kappa^{<\kappa} = \kappa$ and F is a normal filter on κ. Then $Gal(F, \kappa, \kappa^+)$.
- Notice the Galvin property makes sense for filters over an arbitrary set, not just filters over ordinals.
- Note that if U is fine then trivially $\neg Gal(U, \kappa, \lambda)$.

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Benhamou and Goldberg asked the following in [3] where they investigate the Galvin property in inner models.

Question

Let U be a fine, normal ultrafilter over $P_\kappa(\kappa^+)$. Must $Gal(U, \kappa, 2^{\kappa^+})$ hold?

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We answer this negatively, in a strong way.

Failure of Galvin Property on $P_\kappa(\lambda)$ Measures

Main Theorem 1

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- Turn this sequence into a counterexample to the Galvin property.

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Even without Dodd-soundness, we can use some tricks to "cover" the sequence in M_U. But we must trade normality of the filter for some cardinal arithmetic, yielding the following theorem.

Main Theorem 2

Let $\kappa < \mathrm{cf}(\lambda)$ and assume $2^{<\lambda} = \lambda$. Let U be a fine, σ-complete $P_\kappa(\lambda)$ ultrafilter. Then $\neg Gal(U, \kappa, 2^\lambda)$.

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- Let M_U denote the ultrapower of V by U. By a theorem of Goldberg, normal ultrafilters are *Dodd-sound*, meaning

$$\mathcal{A} = \{j_U(X) \cap (\sup j_U''\kappa^+) : X \subseteq \kappa^+\} \in M_U$$

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  $j''_U \kappa^+ \in j_U(B_Y)$ by elementarity and the definition of $\mathcal{A}$.
- Suppose towards a contradiction that there is $\{Y_i : i < \kappa\} \subseteq P(\kappa^+)$ such that $\bigcap_{i<\kappa} B_{Y_i} = B \in U$.

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- Hence $\{X \in P_\kappa(\kappa^+) : |\mathcal{A}_X| = 2^{|X \cap \kappa|^+}\} \in U$.

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- But this is a contradiction as $j_U''\kappa^+ \in j_U(B)$ and $|\mathcal{A}| = 2^{\kappa^+}$. $\dashv$

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- Fix $\alpha^* > \sup_{i \neq j < \theta_0^+}(\beta_{i,j})$ below κ^+ such that there is an $X^* \in B$ with $|\mathcal{A}_{X^*}| \leq \theta$ and $\sup(X^*) = \alpha^*$.

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- $i \mapsto Y_i \cap \sup(X^*)$ is a 1-1 map from θ^+ into $\mathcal{A}_{X^*}$, a contradiction. $\square$

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Galvin Property for Posets

Let $\mathbb{P} = (P, \leq)$ be a directed poset. We say $Gal(\mathbb{P}, \kappa, \lambda)$ holds iff whenever $\langle p_\alpha : \alpha < \lambda \rangle \subseteq P$ there is a $Y \in [\lambda]^\kappa$ and a $q \in P$ such that such that $p_\alpha \leq q$ for all $\alpha \in Y$.

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Let $\mathbb{U}^* = (U, \supseteq^*)$ where $\supseteq^*$ is reverse containment modulo the Fine filter and U is a fine $P_\kappa(\kappa^+)$ ultrafilter.

Questions

- Must $Gal(\mathbb{U}^*, \kappa, \kappa^+)$ hold?
- What if U is σ-complete? If U is normal?

Thanks!

Thanks Tom for inviting me to this project, and thanks for your attention!

References

[1] J.E. Baumgartner, András Hajnal, and A. Mate. "Weak saturation properties of ideals". In: *Colloq. Math. Soc. János Bolyai* 10 (Jan. 1973).

[2] TOM BENHAMOU and NATASHA DOBRINEN. "COFINAL TYPES OF ULTRAFILTERS OVER MEASURABLE CARDINALS". In: *The Journal of Symbolic Logic* (Feb. 2024), pp. 1–35. ISSN: 1943-5886. DOI: 10.1017/jsl.2024.12. URL: http://dx.doi.org/10.1017/jsl.2024.12.

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