UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. Fiz. № 602 – № 633 (1978), 191 – 193.

628. AN INEQUALITY FOR PERIMETERS IN A SUBDIVIDED TRIANGLE

Richard T. Bumby

Dedicated to Professor D. S. Mitrinović on the ocassion of his seventieth birthday

The following problem attracted some attention in the early 1960's.

Problem. Let triangle ABC be divided into four triangles (CDE, AEF, BFD, DEF) by points D on BC, E on CA, and F on AB; prove that DEF does not have the smalest perimeter.

REMARK 1. If D, E, and F are the midpoints of their respective sides, then all perimeters are equal. We shall show that otherwise at least one of the other triangles has smaller perimeter.

REMARK 2. The solution below was obtained in 1961. but not previously published. Although the problem is mentioned in [**2**] and [**3**], and sclut ons have been published in a variety of places, it was not until recently when I consulted [**1**] (§ 9.2, pp. 181—183) as part of preparing to teach a geometry course that I discovered the full history of the problem. The person presenting the problem to me had thought it to be unsolved at the time; but on the following day he was sure that it was not. He could not recall where he had seen the problem, and others that I asked about it were of no help.

Solution. Suppose the contrary. Then by relabelling, we may assume that

$$p(CDE) \geqq p(AEF) \geqq p(BFD) \geqq p(DEF). \tag{1}$$

Proposition 1. *If* (1) *holds, and not all are equal, then we can find a configuration satisfuing*

$$p(CDE) = p(AEF) > p(BFD) = p(DEF). \tag{2}$$

Proof. We first get $p(CDE)=p(AEF)$. If $p(CED)>p(AEF)$, then we rotate CA about E until we get $p(CDE)>p(C'DE)=p(A'EF)>p(AEF)$ while BFD and DEF are unchanged (see fig. 1).

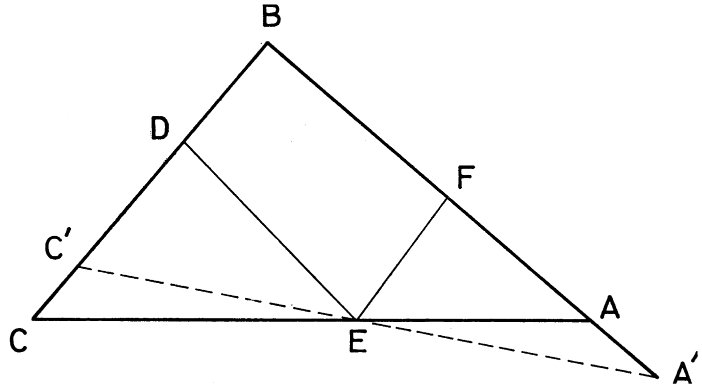

Fig. 1

Now by taking A'' and C'' on $A'C'$ (extended) while keeping DEF fixed we may preserve the relation $p(C''DE)=p(A''EF)$ while decreasing $p(B''FD)$ until it becomes equal to $p(DEF)$ (see fig. 2).

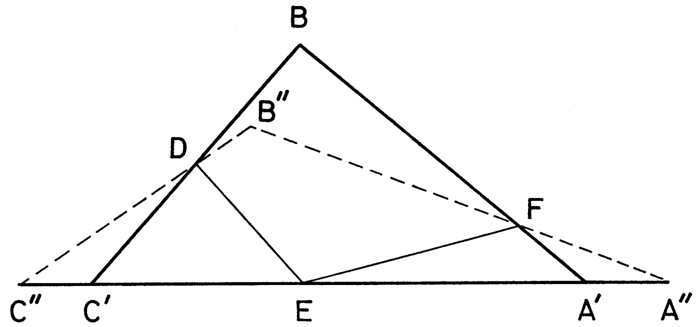

Fig. 2

The „betweenness" properties illustrated in our figures are easily seen to hold.

REMARK 3. Note that DEF has been fixed while ABC has been moved into a standard form. This is a common feature in published proofs. However, our standard form is different from those exploited elsewhere.

Since

$$p(CDE)+p(AEF)+p(BFD)=p(ABC)+p(DEF),$$

(2) requires $p(CDE)=p(AEF)=(1/2)p(ABC)>p(BFD)$. Hence a point G on BC with $p(CDG)=(1/2)p(ABC)$ would be between D and C. If all perimeters are equal, then $G=D$.

Proposition 2. *Given ABC: if D on BC and E on CA are related by $p(CED)=(1/2)p(ABC)$, then there is a projective transformation of BC to CA taking D to E.*

Proof. Let $2a=|BC|$, $2b=|CA|$, $2c=|AB|$ and establish coordinates on BC and CA. The special role of midpoints suggests that the midpoint of each segment have coordinate 0 and the endpoints have coordinates ± 1. With x being the coordinate on BC and y the coordinate on CA we have $|DC|=a(1-x)$, $|CE|=b(1+y)$ and hence $|DE|=a+b+c-a(1-x)-b(1+y)=ax-by+c$. The law of cosines gives

$$c^2=a^2+b^2-2ab\cos\gamma,$$

$$(ax-by+c)^2=a^2(1-x)^2+b^2(1+y)^2-2ab(1-x)(1+y)\cos\gamma.$$

Thus

$$(ax-by+c)^2=a^2(1-x)^2+b^2(1+y)^2+(c^2-a^2-b^2)(1-x)(1+y).$$

The coefficients of x^2 and y^2 are the same on both sides so this gives y as a fractional linear transformation of x, proving the proposition.

The formula for the transformation of proposition 2 is

(3) $$y=\frac{(c+a-b)\,x}{(a+b-c)\,x+(b+c-a)}.$$

REMARK 4. There is nothing special about the factor 1/2 in Proposition 2. Any condition of the form $p\,(CDE)=\lambda\,p\,(ABC)$ leads to the same conclusion. The proof is just as easy.

REMARK 5. The formula does not tell the whole story as there are values of x for which the construction can not be performed. Clearly, if this construction can be performed, then $xy\geqq 0$.

Repeating (3) we get that F has coordinate z and G has coordinate x' where

$$z=\frac{(a+b-c)\,y}{(b+c-a)\,y+(c+a-b)},$$

$$x'=\frac{(b+c-a)\,z}{(c+a-b)\,z+(a+b-c)}.$$

The condition on G translates into $x'\geqq x$, but

$$x'=\frac{(b+c-a)\,x}{(a+b+c)\,x+(b+c-a)},$$

or

(4) $$(b+c-a)\,(x'-x)=-(a+b+c)\,xx'.$$

Now by Remark 5 the right side of (4) is non-positive while we desire the left side to be non-negative. Thus $x'=x=0$ is the only possibility and D, E, and F are midpoints of their respective sides.

REMARK 6. In Proposition 2 a Euclidean construction is seen to yield a projective transformation by examining the effect on coordinates. It would be more natural to exhibet a geometric reasoo for the transformation to be projective. Thus one should note that the lines DE are the tangents to a circle which is tangent to both BC and CA at points whose distance from C is $(1/4)\,p\,(ABC)$. However, it still appears that the use of coordinates facilitates the analysis of the mapping $D\rightarrow G$.

REFERENCES

1. O. BOTTEMA, R. Ž. ĐORĐEVIĆ, R. R. JANIĆ, D. S. MITRINOVIĆ, P. M. VASIĆ: *Geometric Inequalities.* Groningen 1969.
2. H. DEBRUNNER: *Problem* 260. Elem. Math. **11** (1956), 20.
3. E. TROST and A. BAGER: *Problem* 4964. Amer. Math. Monthly **68** (1961), 384.