ROBIN’S THEOREM, PRIMES, AND A NEW ELEMENTARY
REFORMULATION OF THE RIEMANN HYPOTHESIS

JONATHAN SONDOW

Let $\sigma(n)$ denote the sum of the divisors of $n$, and for $n > 1$ set

$$G(n) := \frac{\sigma(n)}{n \log \log n}.$$ 

In 1913 Gronwall [3], [4, Theorem 323] found that the maximal order of $G$ is

$$\limsup_{n \to \infty} G(n) = e^\gamma,$$

where $\gamma$ is Euler’s constant. In 1915 Ramanujan [5, 6] proved that if the Riemann Hypothesis (RH) is true, then $G(n) < e^\gamma$ for all large $n$. In 1984 Robin [7] sharpened this by showing that

$$\text{RH} \iff G(n) < e^\gamma \quad (n > 5040).$$

Recently Geoffrey Caveney, Jean-Louis Nicolas and I [2] used Robin’s theorem to prove that the RH holds if and only if $4$ is the only composite number $N$ satisfying

$$G(N) \geq \max \{G(N/p), G(aN)\}$$

for all prime factors $p$ of $N$ and all multiples $aN$ of $N$. An alternate proof of one step depends on two properties of superabundant numbers derived from those of Alaoglu and Erdős [1] in 1944.

REFERENCES