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The Continued Fraction Algorithm Approached Through Quadratic Forms

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1. INTRODUCTION : At first glance the arithmetic continued fraction appears to be a very well understood animal. The familiar Euclidean algorithm is a close relative, and the continued fraction itself has been domesticated since the time of Euler (1737). Almost every textbook in Number Theory has a section dealing with continued fractions and one can even find books which are the equivalents of 'Continued Fractions as Pets' (e.g. Olds [13]) or 'The Care and Feeding of Continued Fractions' (e.g. Perron [14]).

This common domestic contined fraction is completely at home wherever there is any interest in rational approximations to real numbers. When fed a real number, the continued fraction algorithm rewards its master with an unlimited number of rational numbers which are known as 'best approximations' to the given number.

For many years, mathematicians have been staking out territories less confining than the real line. As more of us dwell in those lands, the questions arise: 'Can I take my continued fraction with me?' ; or 'Do continued fractions live there already?'. To a certain extent, these questions lead back to the more fundamental question : "What is a continued fraction?" Traditionally, continued fractions are described very arithmetically. one starts with the Euclidean algorithm, which in an ALGOL-type language would consist of the following steps.

(Initialization) integers p, q and a list A are required; p and q are arguments of the procedure ; by changing signs of and interchanging p, q (as required), the condition $p \geq q \geq 0$ is established and A is set to an initial value which indicates which transformations are performed in establishing this condition.

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(Loop) while $q > 0$ the following steps are repeated :

(Division) integers a , r are obtained which are characterized by $p = aq+r$ and $0 \leq r \leq p$;

(Substitution) replace p by q and q by r ;

(Concatenation) append a to the list A.

Since each value of the fraction p/q is equal to a plus the reciprocal of the next value of p/q , the term 'continued fraction' is suggested. When the algorithm terminates : $q = 0$; p is the greatest common divisor of the original integers p, q ; and A is a list of integers (preceded by some special character) from which the original value of p/q may be recovered.

It is not necessary for p and q to be integers for the division step to be performed. If p and q are allowed to be real numbers , there is stilll a unique integer a and real number r satisfying the requirements of the division step. The algorithm is still well-defined, but it need not terminate. In Theorem 3, it will be shown that the infinite list A determines the original value of p/q. This allows the notation p/q = [A].

Remark on notation : Various conventions will be used in writing the sequence A. One that deserves special mention because it differs from customary usage is that periodicity will be indicated by enclosing the period in parentheses.

The substitution step may now be written

$$\begin{bmatrix} p \\ q \end{bmatrix}_{old} = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}_{new} \quad \textbf{(1)}$$

The list $A = < a_0 , ... , a_n >$ at any stage of the algorithm leads to a matrix

$$\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \cdots\cdots \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix}$$

which transforms the current value of the column $(p\ q)'$ to its initial value. The elements a_i of the list A are called 'partial quotients'. The fact that this matrix has determinant $(-1)^{n+1}$ plays a major role in the classical theory.

If $p \geq q \geq 0$, the initial value of A will be taken to be null. Some examples are: $8/3 = [2,1,2]$.

For $p/q = (\sqrt{5} + 1)/2$, division gives $a = 1$ and $q/r = (\sqrt{5} + 1) / 2$. Thus $(\sqrt{5} + 1)/2 = [(1)]$ (see ' Remark on Notation' above).
From the analytic theory of continued fractions (which shall not otherwise be mentioned here), it follows that $(e + 1)/(e - 1) = [2, 6, 10, 14, ...]$.

If the continued fraction does not terminate, then the initial segments of the infinite list A are the continued fractions of rational numbers which are called the convergents of the initial value of p/q. One view of continued fractions (see Richards [15]) is that the continued fraction is this sequence of convergents.

The convergents $p_n/q_n = [a_0 , ..., a_n]$ for $n \geq 1$ are also called best approximants to $\alpha = [A]$. The term is justified by the following heuristics.

The effort required to specify a rational number may be assumed to be a monotonic function of the denominator. Indeed, a "complexity measure" giving the total number of symbols used in writing p/q is of the order of $\log p + \log q$. When p/q is close to α, p is essentially just $q\alpha$, so that our measure can be nicely estimated by a function of q alone. A best approximant must then minimize some measure of approximation for all rational numbers whose denominators are subject to some fixed bound. If q is fixed, it is always possible to choose p so that $|q\alpha - p| \leq 1/2$. Usually this determines p uniquely - the only exception occurs if $2q\alpha$ is an odd integer. This allows $\varepsilon(q) = \min \{|q\alpha - p| : p \in \mathbb{Z}\}$ to be determined. (An actual attained minimum, not just a greatest lower bound.)

The interesting values of q are those for which $\varepsilon(q)$ is smaller than all previous values. A proof of the equivalence of the ideas of "continued fraction" and "best approximant" must include a proof that $\varepsilon(q)$ is not bounded below. This is accomplished by Theorem 1. Indeed that result shows that $q.\varepsilon(q)$ is infinitely often smaller than some fixed bound. The elegant simplicity of the proof has changed little since Dirichlet [7]. The availability of this method makes $\varepsilon(q)$ a more useful measure of approximation than the naive measure $|\alpha - 1/q|$.

The relation between the two measures has been explored in depth (see Perron [14, chap. II]).

It should be pointed out that the method of Theorem 1 can show with equal ease that for all real numbers α, β, there are integers p, q, r such max (|p|, |q|, |r|) $\leq$ N and max ($|q\alpha - p|$, $|q\beta - r|$) $\leq C(\alpha, \beta) N^{-1/2}$. However, many "multidimensional continued fractions" failed to guarantee that they can produce approximations p/q, r/q to (α, β) for which $|q\alpha - p|$ and $|q\beta - r|$ are less than a fixed bound. (see Brentjes [1].)

The expression $q.\varepsilon(q)$ which will be shown to be bounded is unnecessarily special. Indeed, the only reason for taking the first factor to be exactly q was that all measures of complexity are roughly the same when $\varepsilon(q)$ is small. The natural object of study seems to be an indefinite real binary quadratic form. Such an object may be defined word-by-word :

form	= homogeneous polynomial
quadratic	= of second degree
binary	= in two variables
real	= with real coefficient
indefinite	= taking both positive and negative values.

This level of generality seems well suited to studies of 'best approximations' and 'continued fractions'.

In plain symbols, a binary quadratic form may be represented as a combination of x^2, xy, and y^2. If indefinite, the form may also be considered as the product of two linear factors. In this article, the letters occurring in the expression

$$f(x, y) = Az^2 + Bxy + Cy^2 = (\alpha x + \beta y)(\gamma x + \delta y) \qquad \textbf{(2)}$$

will be reserved for describing a quadratic form. When several forms occur, subscripts will be used. The arithmetic theory of such forms deals with $\{f(x, y): x \in \mathbb{Z}, y \in \mathbb{Z}, (x, y) \neq (0, 0)\}$. In particular, one has

Theorm 1 : There are infinitely many pairs of integers (x , y) such that $|f(x, y)| \leq (|\alpha| + |\beta|)(|\gamma| + |\delta|)$.

Proof : For any integer $N \geq 0$, the (N + 1) points with $0 \leq x \leq N$, $0 \leq y \leq N$ satisfy $LN \leq (\alpha x + \beta y) \leq UN$, where L is the sum of those of $\{\alpha, \beta\}$ which are negative and U is the sum of those which are positive. Notice that $U-L = |\alpha| + |\beta|$ Divide this interval into $N^2 + 2N$ pieces of equal length. One of these intervals must contain two of the values of $(\alpha x + \beta y)$, say $(\alpha x_0 + \beta y_0)$ and $(\alpha x_1 + \beta y_1)$. Now, $|x_0 - x_1| \leq N$ and $|y_0 - y_1| \leq N$. From this it follows that

$$|\gamma (x_0 - x_1) + \delta(y_0 - y_1)| \leq (|\gamma| + |\delta|) N.$$

The construction has given

$$|\alpha(x_0 - x_1) + \beta(y_0 - y_1)| \leq (|\alpha| + |\beta|) /(N + 2).$$

Thus $(x , y) = ((x_0 - x_1), (y_0 - y_1))$ satisfies the requirements of the theorem.

If β/α is irrational, each point can arise in this way for only finitely many values on N, so this construction leads to infinitely many points, as required. If, however, β/α is rational, then one has a non-zero point (x , y) with $\alpha x + \beta y = 0$. In this case the non-zero multiples of (x , y) satisfy the theorem.

Remark : Much more precise results of this nature will appear in this article. However, the method of proof of Theorem 1 has proved generally valuable in this type of Number Theory. In many applications, the fairly crude estimates of this method differ from the best possible bounds by a bounded factor. In this case, the continued fraction method leads to uncountably many forms, easily described in terms of continued fractions, for which $|f(x , y)| \geq | \alpha\delta - \beta\gamma| / \sqrt{12}$ for all integers x , y not both zero. The difference between the upper bound provided by Theorem 1 and this lower bound is not particularly large.

Clearly, multiplying every coefficient of f by some constant, also multiplies the value of $f(x , y)$ by the same constant. This effect can be suppressed by considering the ratio of $f(x , y)$ to some fixed expression , such as the $(|\alpha| + |\beta|)(|\gamma| + |\delta|)$ used in Theorem 1, which has the same property. The quantity $\alpha\delta - \beta\gamma$ is a better choice (it is the discriminant of the form).

Definition 1 : (The Markoff value of a form)

$$M(f) = |\alpha\delta - \beta\gamma| / \inf \{ |f (x , y)| : x , y \in \mathbb{Z} ; f(x , y) \neq 0\}.$$

Remark : Since values of $f(x , y)$ occur in the denominator, M(f) is large (possibly infinite) if f takes small values.

The geometry of quadratic forms enriches the language available to describe the properties of 'best approximations'. From the geometric viewpoint, x and y are coordinates in the plane. The factorization $(\alpha x + \beta y)(\gamma x + \delta y)$ represents $|f(x, y)|$ as a constant multiple of the product of the distances from (x, y) to two lines through the origin. The points (x, y) with x and y integers form something called a lattice in the plane. The origin also belongs to this lattice, hence it is a distinguished point of this geometry. The plane should then be thought of as the vector space R^2 with the origin distinguished as the identity of the additive group of this vector space. The distinguished lines are fixed one-dimensional subspaces. The lattice is a discrete subgroup with two generators. These generators are then also a basis for the vector space.

For each point (u, v) of the plane, define its parallelogram

$$p(u, v) = \{ (x, y) : |\alpha x + \beta y| \le |\alpha u + \beta v| ; |\delta x + \delta y| \le |\gamma u + \delta v| \}$$

(as f will be fixed in this discussion, the dependence on f will not be indicated explicityly). The set P(u, v) is always compact, so there are only finitely many lattice points in P(u, v).

Definition 2 : If $(x, y) \neq (0, 0)$ is a lattice point such that the (relative) interior of P(x, y) contains no lattice point other than (0, 0), then (x, y) is said to be a minimal point of the form f.

Remark : If β/α is rational, so that there are infinitely many lattice points (x, y) with $\alpha x + \beta y = 0$, then I want those points on this line distinct from the origin and closest to the origin to be called minimal points. In effect, this will allow me to ignore such forms (called zero-forms because of their non-trivial representation of zero) in the rest of the article. The theorems will be true, as stated, but the non-trivial reperesentation of zero will exclude these forms as soon as I limit attention to forms which are bounded away from zero on the minimal points.

Lemma 1 : If $(x, y) \neq (0, 0)$ is a lattice point, then P(x, y) contains a minimal point.

Proof : Either (x, y) is itself a minimal point or there is a lattice point (x', y')

$\neq (0, 0)$ in the interior of $P(x, y)$. As there can be only finitely many points in each parallelogram and $(x, y) \in P(x', y')$, the lemma follows by induction.

Corollary : $M(f) = |\alpha\delta - \beta\gamma| / \text{Inf}\{f(x, y) : (x, y) \text{ minimal}\}$.

Remark : Remember this! It will be needed in Theorem 5.

The minimal points of f may be ordered by the value of $|\alpha x + \beta y|$. In particular, if $|\alpha x + \beta y| < |\alpha x' + \beta y'|$, then (x, y) is said to precede (x', y') and (x', y') is said to follow (x, y).

Given two points $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$ where P_0 precedes P_1, the parallelogram

$$B = \{ (x, y) : |\alpha x + \beta y| \leq |\alpha x_1 + \beta y_1|, |\gamma x + \delta y| \leq |\gamma x_0 + \delta y_0| \}$$

contains both P_0 and P_1 on its boundary. If P_1 is any minimal point not on the line $\alpha x + \beta y = 0$, then one can find a point P_0 preceding P_1, using the method of Lemma 1, so that B will contain no lattice point in its interior other than (0, 0). In this case, P_0 and P_1 are said to be consecutive. If α/β and γ/δ are both irrational, this leads to a chain of minimal points $P_n = (x_n, y_n)$, $n \in \mathbb{Z}$, with each pair P_n, P_{n+1} consecutive. If one or both of the ratios are rational, the modifications are straightforward - an exercise for the reader.

The minimal points P_i and P_{i+1} will be seen (Theorem 2) to always form a basis for the lattice. A suitable choice of one of $\pm P_i$ for each i will then give a chain of bases. Adjacent bases will be related by matrices

$$\begin{bmatrix} a_i & 1 \\ 1 & 0 \end{bmatrix}$$

for some integers $a_i > 0$. The chain of integers $\langle a_i \rangle$ contains the information about small values of $| f |$ on lattice points.

Remark on notation : Angular brackets will be used to delimit a chain (i.e. sequence indexed by positive and negative integers).

Definition 3 : Given a chain $A = \langle a_i \rangle$, denote

$$[a_n, a_{n+1}, a_{n+1}] + [a_n, a_{n-1}, a_{n-2}, ...] - a_n$$

by $M_n(A)$ and denote $\sup_n M_n(A)$ by $M(A)$

Section 3 is devoted to the study of the values taken on by M(A) as A runs through all chains of positive integers. Notice that M(A) is finite if and only if the a_n are bounded. It is not difficult to prove that the real numbers whose continued fractions have bounded partial quotients form a set of measure zero (see Koksma [12, IV. 5]). However, emphasizing the sequence (resp. chain) of partial quotients rather than the number (resp. form) represented shows that there are already uncountably many examples with each $a_i \in \{1, 2\}$. This ability to synthesize examples by exhibiting the chain of partial quotients is a powerful tool.

The functions M(f) of Definition 1 and M(A) of Definition 3 will be related in section 2. As you might expect, if A is the chain determined by the form f, then M(A) = M(f) (see Theorem 5). In particular, showing $M(A) \leq \sqrt{12}$ if all $a_i \in \{1, 2\}$ will lead to family of forms mentioned in the Remark that followed Theorem 1.

In order to recover information about rational approximations to real numbers, some modifications are necessary. The traditional questions translate into the behavior of $|y|\,|\alpha y - x|$ <u>for large values of</u> y. This requires substituting "lim inf" for the "inf" in Definition 1. See Cusick [5] for more information on this topic.

2. Reduction Theory : Much of modern mathematics strives to describe geometric objects in a coordinate-free manner. Unfortunately, a quadratic form requires coordinates in order to be written as a polynomial. However, the concept of a minimal point, and even the value of the form at that minimal point can be defined by a picture and so is, in some sense, independent of coordinates. <u>The equivalence of forms</u> is introduced to allow certain <u>admissible changes of coordinates</u>. <u>The reduction theory</u> seeks to define a special coordinate system determined by the geometry of the picture in reference. Each minimal point will then be associated with a reduced form. In general, a form has infinitely many minimal points, so that there will be infinitely many reduced forms associated with (and equivalent to) a given form. These forms are related in a way which gives rise to the continued fraction.

The continued fraction associated to a form f gives rise to matrices which represent changes of coordinates in the plane. These changes of coordinates all preserve the origin, which is the point of intersection of the two lines on which f = 0, as well as being a point of the lattice which carries the arithmetic aspect of the study.

Such changes of coordinates may be given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m & n \\ r & s \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \tag{3}$$

In particular, $(\alpha x + \beta y)$ becomes

$$\alpha(mx' + ny') + \beta(rx' + sy') = (\alpha m + \beta r)x' + (\alpha n + \beta s)y'$$

and similarly for $(\gamma x + \delta y)$. Thus $f(x, y) = f'(x', y')$ where f' is determined by α', β', γ', and δ' satisfying

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} m & n \\ r & s \end{bmatrix} = \begin{bmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{bmatrix} \tag{4}$$

If m, n, r, s are integers, then the lattice of points with (x', y') coordinates in $\mathbb{Z}$ will have (x, y) coordinates in $\mathbb{Z}$. If, in addition, $ms - nr = \pm 1$ then the lattices with respect to the two coordinate systems will be the same. From this it follows that the sets of values of f and f' on the integer lattice are the same. If f and f' are related by (4), and (x, y) and (x', y') are points related by (3), then one might say that f is equivalent to f' with (x, y) corresponding to (x', y'). (The construction really gives an equivalence relation on factorizations, so one must be careful in speaking of equivalence of forms.)

A reduction theory of quadratic forms is a procedure for producing all forms equivalent to a form f which have some specific nice property. These are the reduced forms. The theory will also produce all interrelations between the reduced forms. Before defining a reduced form, I will take you half-way with the following definition (which is not standard).

Definition 4 : A form for which (0, 1) is a minimal point is called semi-reduced.

Proposition : For each form f and each minimal point (x_0, y_0) of f, there is a semi-reduced form f' such that f' is equivalent to f with (0, 1) corresponding to (x_0, y_0).

Proof : Since (x_0, y_0) is a minimal point, it is impossible for x_0 and y_0 to have a common factor greater than 1. The Euclidean algorithm then yields integers x_1 and y_1 such that $x_0y_1 - y_0x_1 = 1$. Put

$$\begin{bmatrix} m & n \\ r & s \end{bmatrix} = \begin{bmatrix} x_1 & x_0 \\ y_1 & y_0 \end{bmatrix}$$

in (3) and (4), to obtain the required equivalence.

Proposition : The form $f(x, y) = (\alpha x + \beta y)(\gamma x + \delta y)$ is semi-reduced if and only if there are at least two integers in the closed interval with endpoints $-\alpha/\beta$ and $-\gamma/\delta$.

Proof : It is immediate from the definitions that f(x, y) is semi-reduced if and only if there is no simultaneous solution to

$$|\alpha x + \beta y| < |\beta| \text{ and } |\gamma x + \delta y| < |\delta|$$

except (0, 0). For x = 0, only y = 0 satisfies either inequality. Given $x \neq 0$, each of these inequalities demands that y should be within 1 of some real number. The number solutions of simultaneous diophantine inequalities of this type depends only on the number of integers between the numbers.

If no integers between the numbers, then two solutions.

If one integers between the numbers, then one solutions.

If more than one integer between the numbers, then no solutions.

The numbers in question are $-\alpha x/\beta$ amd $-\gamma x/\delta$. The "only if" part of the proposition follows from taking x = 1. To prove the converse, suppose there are at least two integers between x = 1. To prove the converse, suppose there are at least two integers between $-\alpha/\beta$ and $-\gamma/\delta$.

This has the direct consequence that there are no solutions with $x = \pm 1$, and the additional result that $|(-\alpha/\beta) - (-\gamma/\delta)| \geq 1$. Thus, if $|x| \geq 2$, $|(-\alpha x/\beta) - (-\gamma x/\delta)| \geq 2$. This guarantees that the numbers have more than one integer between them, and hence that the inequalities have no solution with this value of x.

Theorem 2 : If $f(x, y)$ is semi-reduced, and if $(0, 1)$, (x_1, y_1) are consecutive minimal points, then $x_1 = \pm 1$.

Proof : The above proposition shows that there is more than one integer between $-\alpha/\beta$ and $-\gamma/\delta$. Let y_* be the integer in this interval closest to $-\gamma/\delta$. In particular, $|y_* + \gamma/\delta| < 1$ and, hence, $|\gamma + \delta y_*| < \delta$. Furthermore, for any x, the integer xy_* lies between $-\alpha/\beta$ and $-\gamma/\delta$.

Now let $(0, 1)$ and (x_1, y_1) be consecutive minimal points. Recall that this means that there are no non-zero solutions of

$$|\alpha x + \beta y| < |\alpha x_1 + \beta y_1| \text{ and } |\gamma x + \delta y| < \delta.$$

In other words, (x_1, y_1) minimizes $|\alpha x + \beta y|$ among all (x, y) with $|y + \gamma x/\delta| < 1$. If x_1 were known, y would be one of the integers adjacent to $-\gamma x_1/\delta$. Indeed, since $|\alpha x + \beta y|$ is to be minimized, y must be the integer between $-\alpha x_1/\beta$ and $-\gamma x_1/\delta$ closest to $-\gamma x_1/\delta$. Now $x_1 y_*$ is also an integer between $-\alpha x_1/\beta$ and $-\gamma x_1/\delta$. Since y_1 is closest to one end, $x_1 y_*$ is at least as close to the other end, i.e. $|\alpha x_1/\beta + x_1 y_*| \leq |\alpha x_1/\beta + y_1|$.
Thus,

$$|\alpha x_1 + \beta y_1| \geq |\alpha x_1 + \beta y_1 y_*| \geq |x_1| \, |\alpha . \beta y_*|$$

If $|x_1| > 1$, this would contradict the defining property of (x_1, y_1).

Corollary : If (x_0, y_0) and (x_1, y_1) are consecutive minimal points of f, then there is a form f^* equivalent to f with $(0, 1)$ and $(1, 0)$ corresponding to (x_0, y_0) and (x_1, y_1).

Proof : There is no loss of generality in assuming that f is already semireduced and that $(x_0, y_0) = (0, 1)$. Theorem 2 then says that x_1 may be taken to be 1. The required equivalence is then given by the matrix

$$\begin{bmatrix} 1 & 0 \\ y_1 & 1 \end{bmatrix}$$

in equations (3) and (4).

If (0, 1) and (1, 0) are already consecutive minimal points, then the construction used in Theorem 2 must lead to $y = 0$. That is, 0 must be the integer between $(-\alpha/\beta)$ and $(-\gamma/\delta)$ which is closest to $(-\gamma/\delta)$. An application of the transformation $x' = x$, $y' = -y$ to f will change the signs of the ratios (α/β) and (γ/δ) while keeping (0, 1) and (1, 0) as consecutive minimal points. Performing this change if necessary allows one to enforce the normalization

$$1 > -\gamma/\delta \geq 0 - 1 \geq -\alpha/\beta \qquad \textbf{(5)}$$

Definition 5 : A form satisfying (5) is said to be <u>reduced</u>.

The effect of all this is to give, for each form f and minimal point P_n of f, a reduced form f_n equivalent to f with (0, 1) corresponding to P_n. The chain of minimal points of f induces a chain of reduced forms $\langle f_n \rangle$ equivalent to f. In order to determine the relation between consecutive elements in such a chain, it suffices to take a reduced form f and determine the reduced form which corresponds to the minimal point (1, 0). Using the method of Theorem 2, let a be adjacent to $-\delta/\gamma$ on the interval $(-\beta/\alpha, -\delta/\gamma)$. Then (1, 0) and (a, 1) are consecutive minimal points. From (5), one has $a = [-\delta/\gamma]$. Thus the transformation from one reduced form to the next in a chain is given by a matrix of the form

$$\begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}$$

With $a_n = [-\delta_n/\gamma_n]$, this gives

$$\alpha_{n+1} = a_n\alpha_n + \beta_n \qquad \gamma_{n+1} = a_n\gamma_n + \delta_n$$

$$\beta_{n+1} = \alpha_n \qquad \delta_{n+1} = \gamma_n \qquad \textbf{(6)}$$

$$\alpha_{n+1}/\beta_{n+1} = a_n + 1/(\alpha_n/\beta_n) \qquad -\delta_{n+1}/\gamma_{n+1} = 1/((-\delta_n/\gamma_n) - a_n)$$

In particular, $a_n = [\alpha_{n+1}/\beta_{n+1}]$. The step from f_{n+1} to f_n can thus be recovered from the factored form of f_{n+1}. In this way, a given form f_0 leads to the entire chain $<a_k>$. It also follows from (6) that

$$(-\delta_0/\gamma_0) = [a_0, a_1, \ldots\ldots\ldots, a_n, (-\delta_{n+1}/\gamma_{n+1})]$$

$$(\alpha_0/\beta_0) = [a_{-1}, a_{-2}, \ldots\ldots\ldots, a_{-n}, (\alpha_{-n}/\beta_{-n})] \tag{7}$$

(These ratios were asserted to be greater than 1 in (5).)

Theorem 3 : If $\xi = [a_0, \ldots\ldots\ldots\ldots a_n, \xi_{n+1}]$ with each $a_j \geq 1$ and $1 \leq \xi_{n+1}, \leq \infty$, then ξ is confined to a closed interval. The intervals obtained in this manner from an infinite sequence of positive integers $<a_j>$ are nested and their intersection is a single point.

Proof : Using (1) and the fact that

$$\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \cdots\cdots \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} \text{ has the form } \begin{bmatrix} P_{n+1} & P_n \\ q_{n+1} & q_n \end{bmatrix}$$

(where all P_k and q_k are non-negative. and all but q_0 are actually positive) gives $\xi = P_{n+1}\,\xi_{n+1} + q_n / q_{n+1}\xi_{n+1} + q_n$. The denominator of this expression is always positive. Furthermore, ξ is the function of, $\xi_{n+1,}$ whose derivative is always of the same sign as the constant $P_{n+1}\, q_n - P_n q_{n+1}$ (which just happens to be $(-1)^{n+1}$) so it is monotonic. Specifying the value of a_{n+1} has the same effect as confining ξ_{n+1} to the interval $a_{n+1} \leq \xi_{n+1} \leq 1 + a_{n+1.}$ This proves all but the last assertion of the theorem. To show that the intersection of the intervals is a single point, it suffices to show that the length of the n-th interval determined by $<a_k>$ approaches zero as n goes to infinity. This length is easily seen to be $1/q_{n+1}\,(q_{n+1} + q_n)$. Since $q_0 = 0$, $q_1 = 1$ and $q_{k+1} = a_k q_k + q_{k-1} \geq q_k + q_{k+1,}$ it follows easily that $q_n \geq ((1+\sqrt{5})/2)^{n-2}$ for $n \geq 1$. This completes the proof.

Corollary : The chain $<a_n>$ determines the form f_0 up to a constant factor.

Proof : The a_n with $n \geq 0$ determine the ratio γ/δ, and those with $n \leq 0$ determine α/β.

Consider the special case in which $<a_n>$ is periodic, and suppose the period is a_1, ..., a_n. Let

$$M = \begin{bmatrix} a_1 & 0 \\ 1 & 0 \end{bmatrix} \cdots\cdots \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix}$$

Consider the form f_1 in the chain of reduced forms determined by $<a_n>$. It follows from Theorem 3 that M transforms f_1 into a multiple of itself. Thus the form $f_1(x,y) = (\alpha x + \beta y)(\gamma x + \delta y)$ where (α,β) and (γ,δ) are row eigenvectors of M. Since the form is reduced, α/β is positive and γ/δ is negative. It is thus immediate that the two eigenvectors are independent. The corresponding eigenvalues are then real numbers whose product is det $(M) = (-1)^n$. Since all $a_i \geq 1$, it follows that $tr(M) \geq 1$ for all M and, moreover, if $\det(M) = +1$, then $tr(M) \geq 3$. Thus the eigenivalues are necessarily real quadratic irrationals. (Exercise: fill in the details here.) Since the eigenvalues are conjugate over the rational, the same must be true of their corresponding (row) eigenvectors; but these are known to be (α,β) and (γ,δ). In others words, f is a multiple of a form with integer coefficients.

Conversely, suppose that a given form $f(x,y) = Ax^2 + Bxy + Cy^2$ has integer coefficients. All equivalent forms must also have integer coefficients. In addition, equivalence preserves the discriminant $B^2 - 4AC$. Periodicity follows from the following result of Lagrange (1770)

Proposition : There are only finitely many reduced forms with integer coefficients and fixed discriminant.

Proof : Suppose that $C > 0$. Then $F(z) = Cz^2 + Bz + A$ has $(-\alpha/\beta)$ and $(-\gamma/\delta)$ as roots, so that (5) yields: $F(1) > 0$; $F(0) \leq 0$; $F(-1) \leq 0$. Evaluation of F at 1, 0, and -1 then gives : $A + B + C > 0$; $A \leq 0$; $A - B + C \leq 0$. Thus $B \geq |A+C| \geq 0$ While A and C have opposite signs.

For reduced forms, it follows that B^2 is no larger than the discriminant. Hence there are only a finite number of values of B if the discriminant is fixed. For each value of B, one gets a fixed value of $-4AC$. If this is not zero, then there are only finitely many possible integer values of A and C. In the unlikely case that the discriminant is a perfect square, so that this quantity could be zero, the condition $|A+C| \leq B$ gives only finitely many values of A and C with $A \leq 0 < C$.

Corollary : The chain associated with a form with integer coefficients is purely periodic.

Proof : For each reduced form with the given discriminant, the construction of its immediate successor or predecessor in any chain depends only on the form itself. Thus this operation induces a permutation of the reduced forms.

Theorem 4 : A purely periodic continued fraction $[(a_0,..., a_n)]$ represents a quadratic irrational ξ, whose conjugate (in $Q(\xi)$) η satisfies $-1/\eta = [(a_n, ..., a_0)]$.

Proof : Consider the form represented by the chain $<(a_0,....a_n)>$. It has already been noted that this is a multiple of a form with integer coefficients, and that it does not factor over the rationals. Formula (7) gives the conclusion with $\xi = (-\delta_0/\gamma_0)$ and $\eta = (-\beta_0/\alpha_0)$, its conjugate.

Corollary : If A is purely periodic, then $M(A)^2$ is rational.

Proof : $M(A) = M_k(A)$ for some k since $M_n(A)$ assumes only finitely many values as n varies. There is no loss of generality in assuming $M(A) = M_0(A)$. Substituting the result of Theorem 4 into Definition 3 gives $M_0(A) = \xi - \eta$ where ξ and η are two roots of a quadratic equation with rational coefficients. The result follows.

Remarks : Theorem 4 is due to Galois (1829). See Koksma [12, § 23] for more details. Equation (5) expresses the fact that the form represented by A is reduced as a condition on the conjugates ξ, η. It is now easy to use Theorem 3 to show that the continued fraction of any quadratic irrational is eventually periodic. Theorem 4 will then imply that the conjugates have periods which are the reverses of each other. Herzog [11] has given a complete analysis of the 28 possible behaviours of the preperiods of continued fractions of conjugate numbers.

Theorem 5 : If the form f corresponds to the chain A, then $M(f) = M(A)$.

Proof : Applying (7) gives $M_n(<a_n>) = \dfrac{\beta_n\gamma_n - \alpha_n\delta_n}{\alpha_n\gamma_n}$ (necessarily positive from definition 3). But $|\beta_n\gamma_n - \alpha_n\delta_n| = |(\alpha_0\delta_0 - \beta_0\gamma_0)|$ and $\alpha_n\gamma_n = f_n(1,0) = f_0(p_n)$. The corollary of lemma 1 gives the desired result.

3. The Markoff Spectrum : The set of values of M(f) (from def. 1), or equivalently by Theorem 5, $M(<a_n>)$ (from def. 3) is called the Markoff Spectrum. My own work in this area has emphasized the approach through the chain $<a_n>$.

It is useful in constructing examples to know that every value in the Markoff spectrum can be realized by a chain A for which all $M_n(A) \leq M_0(A)$, so that $M(A) = M_0(A)$. This is a version, within the continued fraction approach, of a compactness theorem in the geometry of the numbers due to Mahler (see[4, chapter 5]) .Such a 'compactness theorem' has figured prominantly in the writing on the Markoff spectrum since 1970, but did not seem to be noticed earlier. Various writers seem to have discovered it independently. I learned it from Hall [10].

Lemma 2 : For each chain $A = <a_n>$ with all $a_n \leq M$, there is a chain $A^* = <a_n^*>$ such that :

i) every finite string in A^* occurs somewhere in A;

ii) $M(A^*) = M_0(A^*) = M(A)$.

Proof : Construct the elements of A^* in the ('spiral') order $a_0^*, a_1^*, a_{-1}^*, a_2^*, a_{-2}^*$, ... using an inductive process. At each stage, one will have an initial segment of this sequence (the null segment at stage 0) for which it is true that

$$M(A) = \sup\{M_n(A) : a_{n+i} = a_i^* \text{ for all } i \text{ with } a_i^* \text{ defined}\} \qquad (8)$$

Note that at stage 0, before any of the a^* have been defined, this is exactly the definition of M(A). The inductive step requires that a value be assigned to the first free a^* in the spiral ordering. Condition (i) restricts attention to a set of possible values for a_i^* which is <u>finite</u> since $a_i^* \leq M$. To show this set is <u>non-empty</u>, find any n such that $a_{n+j} = a_j^*$ for all previously defined a_j^*. Then

a_{n+i} is a possible value of a_i^*. For each possible value, form the supremum (8) (the supremum over the null set may be taken to be zero). Now M(A) is the maximum of the finitely many values obtained in this way. Thus, at least one of these values is equal to M(A). This allows one to choose a_i^* so that (8) is satisfied and to continue the induction.

It is immediate that $M(A^*) \geq M_0(A^*) \geq M(A)$ for the sequence constructed in this way. As $M(A^*) \leq M(A)$ follows from (i), the lemma is proved.

As an application of this, consider those chains A for which $M(A) = M_0(A)$ and $a_0 = N$. It is classical, and will be proved below, that this entails $\sqrt{N^2+4} \leq M(A) \leq \sqrt{N^2+4N}$. When N = 1, this allows only $M(A) = \sqrt{5}$; When $N = 2$, $\sqrt{8} \leq M(A) \leq \sqrt{12}$; and when $N \geq 3$, $\sqrt{13} \leq M(A)$. Lemma 2 guarantees that every value in the Markoff spectrum lies in one of these intervals. Hence:

i) the smallest value in the Markoff spectrum is $\sqrt{5}$;
ii) there are no values in the spectrum between $\sqrt{5}$ and $\sqrt{8}$;
iii) there are no values in the spectrum between $\sqrt{12}$ and $\sqrt{13}$.

Items (i) and (ii) were known to Markov and are part of his result that the portion of the spectrum below 3 is discrete (see[3]). Item (iii), is due to perron (see Koksma [12, III.2]). Let's begin with the special cases needed for this result.

Proposition : If $a_0 = N$ and $M(A) = M_0(A)$, then $M(A) \leq \sqrt{N^2+4N}$. The upper bound is attained for $A = \langle(N,1)\rangle$.

Proof : $M_0(A)$ is an increasing function of each a_{2i} and a decreasing function of each a_{2i+1}. If it were known that all $a_k \leq N$, the given chain would certainly give the largest value of M(A). To prove the result as stated, let $x = [a_0, a_1, ...]$ and $y = [a_0, a_{-1}, a_{-2} ...]$ so that $M_0(A) = x + y - N$. Without loss of generality, it may be assumed that $x \geq y$. Now: how large can x be? If $a_1 > 1$, then $y \leq x < [N,2]$ which requires that $M_0(A) < N + 1$, which is much smaller than $M(\langle(N,1)\rangle)$. One may then assume $a_1 = 1$. Introduce $u = [a_2, a_3, ...]$ and observe that

$$M_0(A) = y + \frac{u}{u+1} \qquad M_2(A) = u + \frac{y}{y+1}.$$

The assumption that $M_0(A) \geq M_2(A)$ is equivalent to $y \geq u$. Hence, $u \leq y \leq x = [N, 1, u]$ which forces $u \leq [(N, 1)]$.

This shows the use of the compactness theorem in analyzing the Markoff Spectrum. For lower bounds, the problem is different: it is desirable to show that the presence of certain strings in A force M(A) to be large . The method of the above proposition does not do this since it assumes that $M(A) = M_0(A)$. For example, it would be useful to know that the presence of any 4 in A force $M(A) > \sqrt{20}$. An approach which relied on the compactness theorem could obtain such a result only after a long analysis of chains A with $a_0 = 3$.

Proposition : If A is any chain, write $\overset{+}{M}_n(A) = \max\ (M_{n-1}(A), M_n(A), M_{n+1}(A))$. If $a_n = N$, then $\overset{+}{M}_n \geq \sqrt{N^2+4}$.

Proof : Write $u = [a_{n-1}, a_{n-2}, ...]$ and $y = [a_{n+1}, a_{n+2}, ...]$. Then $N.M_{n-1}(A) = (Nu + 1) - 1/(Ny + 1)$ and $N.M_{n+1}(A) = (Ny + 1) - 1/(Nu + 1)$. This shows the equivalence of $M_{n-1}(A) = M_{n+1}(A)$ with the simpler condition $u = y$ (since $z + 1/z$ is increasing function for $z > 1$). As in the previous proposition $M_n(A) \geq M_{n+1}(A)$ is equivalent to $N + 1/u \geq y$ and $M_n(A) \geq M_{n-1}(A)$ is equivalent to $N + 1/y \geq u$. A figure can be drawn to illustrates the manner in which the quadrant where $u > 1$ and $y > 1$ is dissected into regions where the three numbers $M_{n-1}(A)$, $M_n(A)$, $M_{n+1}(A)$ occur in a particular order of size . The three curves meet at the point where $u = y = [(N)]$. As $M_n(A)$ is a decreasing function of both u and y while $M_{n\pm1}(A)$ are increasing functions, the minimum value of $M^+_n(A)$ occurs on one of the solid curves of the figure. Consider the typical are $\{(u, N + 1/u): 1 \leq u \leq [(N)]\}$. Here $M^+_n(A) = x + 1/x$ with $x = N + 1/u$. As this is an increasing function of x, the minimum occurs for the smallest x, for which $u = [(N)]$. Similar analysis of the other arcs proves the proposition.

Corollary : If any $a_n \geq N$, then $M(A) \geq \sqrt{N^2 + 4}$.

Remark : The function $z \pm 1/z$ occur prominantly in these proofs. The critical fact is that they are increasing functions of z for $z > 1$.

These two propositions are typical of Perron's method. The next result, independently discovered by Gbur [9] and myself [3] allows Perron's method to be applied in greater generality than had been earlier realized.

Theorem 6 : Suppose that $k > 0$ and write $u = [a_0, a_{-1}, a_{-2}, ...]$ and $y = [a_k, a_{k+1}, a_{k+2}, ...]$. Define m, n, r, and s by

$$\begin{bmatrix} m & n \\ r & s \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_2 & 1 \\ 1 & 0 \end{bmatrix} .... \begin{bmatrix} a_{k-1} & 1 \\ 1 & 0 \end{bmatrix} \quad (9)$$

Then $M_0(A) \geq M_k(A)$ if and only if $(u - y) \geq (n - r)/m$.

Proof : Since the factors on the right side of (9) are ali symmetric matrices, the effect of transposing is only to invert the order of the factors. Thus

$$\begin{bmatrix} m & r \\ n & s \end{bmatrix} = \begin{bmatrix} a_{k-1} & 1 \\ 1 & 0 \end{bmatrix} ..... \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \quad (9)$$

It then follows that

$M_0(A) = u + (ry + s) / (my + n)$, $M_k(A) = y + (nu + s) / (mu + r)$ and

$m(M_0(A) - M_n(A)) = ((mu + r) - (ms - rn)/(mu+r)) - ((my + n) - (ms-rn)/(my + n))$.

Here $(ms - nr) = (-1)^{k-1}$, so that the expression on the right is the difference of two values of one of the functions $z \pm 1/z$, which are increasing functions for all $z > 1$. As $(mu + r)$ and $(my + n)$ are certainly greater than 1, this implies that $M_0(A) > M_k(A)$ if and only if $mu + r > my + n$, as required.

Corollary : If $a_i = a_{k-i}$ for $1 \leq i \leq k-1$, then $M_0(A) \geq M_k(A)$ if and only if $u \geq y$.

Corresponding results for $k < 0$ are clear and will be used without further comment.

Theorem 6 has been useful in determining the extreme values of M(A) under the assumptions that $M(A) = M_0(A)$ and that certain a_i are fixed. (These are the "local" extrema considered by Davis and Kinney [6].)

The cases of interest in this analysis have all $a_i \leq 4$, because of the phenomenon know as "Hall's ray". It has been shown by Freiman [8] building on earlier results (see [10], [16]) that all numbers greater than 4.52783 belong to the Markoff spectrum, and that 4.52782955 does not. Thus, only with small values of a_i will this local analysis have consequences about global properties of the Markoff spectrum.

Our notation thus need only represent the elements of the chain A by single digits, so a minimum of punctuation will be required. Out with the commas and spaces separating terms! On the other hand, a_0 plays a special role, so need to be located easily. I propose delimiting it with dots. Thus the chains with $a_{-5} = a_0 = a_2 = a_5 = 2$ and $a_{-4} = a_{-3} = a_{-2} = a_{-1} = a_1 = a_3 = a_4 = 1$ would be said to "contain the string 21111.2.12112". In working with the continued fraction representation of numbers, the fact that this representation is alternately an increasing and decreasing function of the partial quotients makes it desirable to separate pairs of integers by spaces. Thus, if all partial quotients are bounded by 2, the pairs 12, 11, 22, 21 in this order should be considered as the basic "digits" of the representation. They must be separated by spaces so one can see where each symbol begins. Periodicity will be indicated by enclosing the period in parantheses.

So far we have shown:

$$\mathrm{Max}(.1.) = M(<(1)>) = \sqrt{5}$$

$$\mathrm{Max}(.2.) = M(<(2)>) = \sqrt{8}$$

$$\mathrm{Max}(.2.) = M(<(21)>) = \sqrt{12}$$

some further examples

Example 1 : Max (2.2.2)

Introduce $x = [a_{-2}, a_{-3},...]$ and $y = [a_2, a_3, ...]$. M(A) is an increasing function of both of these quantities so they should be as large as possible. The corollary to Theorem 6 for k = -2 and k = 2 gives

$$x \leq [22\ \ y] \qquad\qquad y \leq [22\ \ x]$$

Each quantity is bounded by an increasing function of the other. Composing these gives

$$x \leq [22\ \ 22\ \ x]$$

which requires $x \le [(2)]$. This, in turn gives $y \le [(2)]$. These values give the desired maximum. Notice that this is the same as Min(2.2.2), so this constaraint determined only a single point.

Example 2: Min(.2.1)

Take $x = [a_{-1}, a_{-2},...]$ and $u = [a_0, a_1,...]$. Then $x \le u$ and $M_0(A) \ge u + 1/u$. Thus small values of $M_0(A)$ must have small values of u. Assume $a_2 = 1$. (The result will justify the assumption.) Introduce $y = [a_3, a_4, ...]$ Apply the corollary of Theorem 6 with $k = 3$ to get $y \le [2x]$. Combine with the previous result $(k = -1)$ $x \le [211y]$. Thus each of x and y is bounded by a decreasing function of the other. As in the analysis of the figure , the "vertex" $x = [(21\ \ 12)]$ $y = [(22\ \ 11)]$ actually minimizes $\max(M_0(A), M_{-1}(A), M_3(A))$. The periodic chain <(2211)> then gives the desired minimum (which is $\frac{\sqrt{221}}{5}$).

Remark : Notice that this gives a lower bound on $\max(M_0(A), M_{-1}(A), M_3(A))$ if $a_0 = 2, a_1 = a_2 = 1$. Any chain A containing the string '211', would have $M(A) \ge \sqrt{221}/5$.

Example 3: Max (21111.2.12112)

Start with $x = [a_{-6}, a_{-7},...]$ and $y = [a_6, a_7,...]$. The desired maximum is given by choosing x and y as large as possible subject to the infinitely many conditions of Theorem 6. I will only sketch the calculation. This calcuiation would only be done only after simpler strings had been studied. It would then be known that certain strings force larger values of M(A) than known bounds on $M_0(A)$, hence these strings may be excluded. Likewise, some applications of Theorem 6 would be redundant since the analysis of max(.1.) and max(2.2.2), for example, leads to nearby locations in the chain where $M_k(A)$ is larger unless $M(A) \le \sqrt{5}$ or $\sqrt{8}$, respectively. The same will be true of every such upper bound result. It is thus not much work to show that the continued fraction for x begins

$$x = [21\ 21\ 11\ 22\ 11\ 21\ 21\ 11\ 12\ x_1].$$

The bound on x_1 produced by Theorem 6 is very close to the bound on x. This suggests that $x_1 = x$, giving periodicity, for the desired maximum. This can actually be proved (see Bumby [2]). Similarly it can be shown that

$$y = [(21\ 21\ 11\ 21\ 22)]$$

This was the first endpoint of a gap in the Markoff Spectrum which required two different periods.

In general suppose $j > 0$ and a_i is known for $i < j$. Theorem 6 gives all restrictions on the remaining a_i in order to have $M(A) = M_0(A)$. Let $u = [a_0, a_{-1}, a_{-2}, \ldots]$ and $y = [a_j, a_{j+1}, \ldots]$. Increasing u would relax all of these conditions, so max(y) will not decrease and min(y) will not increase. In finding the maximum for chains containing a given finite string, the upper bounds on each of $u = [a_0, a_{-1}, a_{-2}, \ldots]$ and $v = [a_0, a_1, a_2, \ldots]$ will be increasing functions of the other. Thus, it has always been easy to give convincing calculations of maximum values. In the case of minima, however, u and v are bounded below by decreasing functions which may be discontinuous. It is more difficult to describe the values of u and v which minimize M(A), but it has not been too difficult to find minima in practice.

4. Conclusion : Much of this paper is an elaboration of themes of Cassels [4]. In particular, the emphasis on the infimum of a function restricted to a lattice can be found in chapter II, and the approach to continued fractions is sketched in section X.8. I wish to call special attention to the concluding remarks of [4, X.8] where several problems are mentioned for which analogs of the continued fraction could lead to new results.

There is no difficulty generalizing definitions 1 and 2 and proving an analog of Lemma 1 for any of these examples. There is considerable difficulty going any farther, since analogs of (5) as a characterization of reduced forms do not seem to exists and no generalization of Definition 3 has been found.

Although the work on the Markoff Spectrum has relied heavily on Definition 3, it should be noted that Theorem 6 uses only thematrices relating a pair of reductins and not the factorization of this matrix given by the continued fraction. This suggests that continued fraction methods might be available where continued fractions themselves have no analog.

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