Question 1

This question asks you to prove the classic binomial theorem.

For integers $n \geq i \geq 0$, define

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

where $n! = 1 \cdot 2 \cdot 3 \cdots n$. Notice $0! = 1$, following our usual convention.

(a) Prove that (when $n > i$)

$$\binom{n}{i} + \binom{n}{i+1} = \binom{n+1}{i+1}$$

(b) Prove that $\binom{n}{i}$ is an integer.

(c) Prove the binomial theorem: for $a, b, n \in \mathbb{Z}$ with $n \geq 0$, the following holds:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

Question 2

The goal of this problem is to prove Fermat's Little Theorem.

(a) Let p be a positive prime and i a positive integer satisfying $p > i$. Prove p divides $\binom{p}{i}$.

(b) Let p be as above and let a, b be integers. Prove p divides $(a+b)^p - a^p - b^p$.

(c) Deduce Fermat's little theorem: given a positive prime p and an integer a, p divides $a^p - a$.