In Memoriam: ABBAS BAHRI

Our colleague Abbas Bahri passed away on January 10, 2016 after four years of heroic fight against two forms of cancer. Until his last days he worked frantically, and with a sense of urgency, on a legacy he was eager to leave to the mathematical community. In recent months Abbas expressed often his love and concern for his wife Diana and his four children. I am sure that everyone in the Department joins me in sending them our affection and sympathy.

Abbas was born on January 1, 1955 in a Tunisian family practicing an enlightened form of Muslim traditions. The top priority of Abbas' father was to offer his children access to the best possible Western education. At age 17 Abbas left Tunis for Paris where he spent two years preparing for the highly competitive "Concours des Grandes Ecoles". He was admitted in 1974 to the most exclusive of them - the ENS (= Ecole Normale Supérieure, Ulm). Nowadays the doors of the ENS are widely open to foreign students, but at the time this was quite unusual.

Here are some landmarks of Abbas' career:

1974-78	Student at ENS,
1977-81	Thèse d'Etat at the Université Paris VI,
1979-81	Research fellow at the CNRS,
1981-82	Maître de Conférences at Faculté des Sciences, Tunis,
1982-84	Dickson Instructor at the University of Chicago,
1984-92	Maître de Conférences at Polytechnique, full-time (84-86) and part-time (86-92),
1986-87	Visitor at Courant Institute, NYU,
1987-2016	Professor, Rutgers University.

Starting in 1990 he was also a part-time Professor at the Ecole Nationale d'Ingénieurs de Tunis. He resigned in 1994 due to some conflicts. But he continued until last year to spend much time in Tunisia, lecturing on a voluntary basis to a selected group of disciples at his family home in a seaside neighborhood of Tunis.

Abbas had a rich and complex personality. It is beyond my capacity to make a fair portrait of his passions and wide range of interests. I leave it to others, and I will concentrate on his mathematical life. Let me start with one anecdote related to his arrival at Rutgers, and which says a lot about Abbas. I vividly remember a conversation I had with Abbas in 1987. He informed me abruptly that he had decided to leave France. "The French bureaucracy is requesting a document emanating from the Tunisian authorities, and I am unable to get it. They are giving me a hard time because I am a Arab. I am disgusted". In order to quiet him down I told Abbas that I had encountered myself similar difficulties. I was born during WWII under a false identity because my parents were hiding in France as Jewish refugees. Twenty years later, it was discovered by a French bureaucrat that in some obscure register I was still named Jean-Jacques Vienne. In order to regain my real identity I had to hire a lawyer and prove to a judge that I was indeed the son of my parents! Abbas would not budge, and I tried another argument: "Abbas, you should not complain too much about France. You received here a first-class education, with generous fellowships, and you have now a position in an elite French school". This argument did not convince him either. I switched strategy. I had recently been offered an exceptional arrangement by Rutgers through the initiative of Felix Browder (who was then VP for Research). I mentioned to him that Abbas wanted to leave France. Felix knew Abbas well from his years at Chicago, and Abbas promptly received a tenured offer from the Rutgers Math. Dept.

Typeset by $\mathcal{AMS}$-TEX

During his formative years at the ENS Abbas was fascinated by Algebraic Topology and he read many books on this subject. Given his taste for this field I do not quite understand why he asked me, during his last year at the ENS, to be his PhD supervisor. He might just have been attracted by the galaxy of brilliant students already in my team, some coming from the ENS. They included e.g. Henri Berestycki, Jean-Michel Coron, Pierre-Louis Lions and Jean-Michel Morel, with all of whom Abbas co-authored papers. This group made for an ebullient mathematical life at Paris VI, in particular with the "Friday seminar" on nonlinear PDEs and applied mathematics that Jacques-Louis Lions and myself organized there and later at Collège de France, a seminar that attracted leading experts from all over the world.

Abbas was a kind of "singular point" in this group because his background was not Functional Analysis and PDEs. Thanks to his deep knowledge of Topology he was able to infuse fresh and innovative perspectives into nonlinear PDEs. Thus, Abbas was naturally drawn to the novel methods of Calculus of Variations. These were concerned with the use of critical point theory to study nonlinear PDEs and Hamiltonian systems. In this respect he was also influenced by the approach of Antonio Ambosetti and Paul Rabinowitz. Paul had taught a basic course on this topic at Paris VI and Orsay a few years earlier and planted this seed, as it were, in my team.

In order to simplify the presentation I will divide the work of Abbas into four parts.

I. Nonlinear elliptic PDEs and periodic orbits of Hamiltonian systems with a forcing term

Consider the simple looking nonlinear elliptic equation

$$\begin{cases} -\Delta u = |u|^{p-1}u + f & \text{in } \Omega \subset \mathbb{R}^N \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N, N \geq 2$ (e.g. a ball) and p is a real number satisfying

$$1 < p < \frac{N+2}{N-2}; \tag{2}$$

u is the unknown and $f(x)$ is a given "forcing" term. The solutions of (1) correspond to critical points of the functional

$$F(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{(p+1)}\int_\Omega |u|^{p+1} - \int_\Omega fu \tag{3}$$

well defined on the Sobolev space $H_0^1(\Omega)$ (because of the restriction (2) on p).

The Ljusternik-Schnirelman critical point theory (Moscow, 1930) provides the existence of infinitely many solutions for (1) when $f \equiv 0$. The invariance of F under symmetry is essential for this type of theory to apply. Nothing was known (except for dimension $N = 1$ by classical ODE techniques) when departing from symmetry. It was an open problem whether even one solution persisted when adding a forcing term f in (1). This was the problem I gave Abbas as a first step towards his PhD.

Within a short time Abbas came up with a remarkable partial result and innovative strategies involving a combination of Algebraic Topology and Analysis. He proved [CRAS, 1980 and JFA 1981] that problem (1) has infinitely many solutions for a dense set of $f's$. Subsequently A. Bahri -

H. Berestycki [Trans. AMS, 1981] proved that there exists some $1 < p_N << (N+2)/(N-2)$, such that for every $p \in (1, p_N)$ and *every* f, equation (1) admits infinitely many solutions. Later on A. Bahri- P.L. Lions [CPAM, 1988] improved this result and established that the same conclusion holds when $p < N/(N-2)$. To this day it is not known whether problem (1) admits *even one* solution for every f under the natural assumption (2). A. Bahri and H. Berestycki [CPAM, 1984 and Acta Math, 1984] also successfully applied their method to study second order systems of ODEs and periodic orbits of non autonomous Hamiltonian systems.

II. Equations involving the limiting exponent $p = (N+2)/(N-2)$. Critical points at infinity.

This section includes some of the most fundamental and influential works of Abbas. They are concerned with nonlinearities involving the limiting exponent $p = (N+2)/(N-2)$. Here is a basic example. Consider the equation

$$\tag{4} \begin{cases} -\Delta u = u^p + a(x)u & \text{in } \Omega, \\ \quad u > 0 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N, N \geq 3$, and $a(x) \in C(\overline{\Omega})$ is such that the linear operator $-\Delta - a$ is coercive. A superb result of A. Bahri - J.-M. Coron [CRAS, 1985 and CPAM, 1988] asserts that if Ω has "nontrivial topology" (when $N = 3$, Ω not contractible suffices, when $N \geq 4$ the assumption is more technical), and if $a(x) \equiv 0$, then (4) admits a solution.

In order to appreciate this result, its beautiful proof, and its impact (almost 300 citations according to MathSciNet), it is important to put things in perspective. In the mid 60s Pohozaev established that if Ω is starshaped and $a(x) \equiv 0$, problem (4) admits no solution. On the other hand Yamabe claimed the existence of a solution for an equation resembling (4) with roots in Geometry. Trudinger (1967) discovered a serious flaw in Yamabe's proof; the heart of the error came from the non compactness of the Sobolev embedding $H_0^1 \subset L^{p+1}$. These two "negative" facts triggered much interest for this kind of problem. The first breakthrough came from Th. Aubin [J. Math. Pures Appl. 1976] who produced a correct proof of Yamabe's claim where $N \geq 6$. Inspired by Aubin's success we tackled problem (4) in [Brezis-Nirenberg, CPAM, 1983]. The solutions of (4) correspond to critical points of the functional $F(u) = \|u\|_{p+1}^{-2}$ subject to the constraint

$$u \in \Sigma_+ = \left\{ u \in H_0^1; u \geq 0 \text{ and } \int_\Omega (|\nabla u|^2 - au^2) = 1 \right\}.$$

One of our key discoveries was that $F_{|\Sigma_+}$ satisfies the $(PS)_c$ condition for every $c < S$, where S is the best constant for the Sobolev embedding $H_0^1 \subset L^{p+1}$, i.e., $S = \operatorname*{Inf}_{v \in H_0^1} \left\{ \int |\nabla v|^2 / \|v\|_{p+1}^2 \right\}$. The $(PS)_c$ condition was introduced in [H. Brezis - J.M. Coron- L. Nirenberg, CPAM, 1980], and is a "localized" refinement of the classical (PS) condition; one says that Φ satisfies $(PS)_c$ if every sequence (v_n) such that $\Phi(v_n) \to c$ and $\Phi'(v_n) \to 0$, is relatively compact. Returning to F, the first natural strategy is to consider

$$\tag{5} J = \operatorname*{Inf}_{u \in \Sigma_+} F(u).$$

We established in [H. Brezis - L. Nirenberg CPAM, 1983] that $J \leq S$ (always); moreover from the above we deduced that if $J < S$, then the Inf in (5) is achieved. In fact, a much sharper result holds:

$$\text{(6)}\quad \text{The Inf in (5) is achieved } \Leftrightarrow J < S \Leftrightarrow \begin{cases} a(x) > 0 & \text{somewhere in } \Omega \text{ when } N \geq 4 \\ g(x,x) > 0 & \text{somewhere in } \Omega \text{ when } N = 3 \end{cases}.$$

Note that when $N = 3$ the condition "$a > 0$ somewhere" needs to be replaced by the stronger condition "$g > 0$ somewhere", where $g(x,y)$ is the regular part of the Green's function of the linear operator $-\Delta - a$. The last equivalence is due to H. Brezis - L. Nirenberg [CPAM, 1983] when $N \geq 4$, to H. Brezis [CPAM, 1986] for the sufficient condition when $N = 3$, and to O. Druet [Ann IHP, 2002] for the necessary condition when $N = 3$. The same strategy i.e., via minimization, has been used successfully for the Yamabe problem by Th. Aubin [J. Math. Pures Appl., 1976] and by R. Schoen [J. Diff. Geom., 1984], where the positive mass theorem plays a role similar to the condition "$g > 0$ somewhere". In view of (6) one can say that the minimization approach to (4) is by now *completely* understood.

By contrast there might be situations where the minimization approach fails, but (4) admits *non minimizing* solutions. A typical case occurs when $a(x) \equiv 0$, because $J = S$ and the Inf in (5) is *never achieved*. A major breakthrough was made by J. M. Coron [CRAS, 1984]. He proved that if $a(x) \equiv 0$ and Ω has a small hole, then (4) admits a solution. The proof is short but extremely clever and original. It consists of 4 steps.

Step 1. Show that F satisfies $(PS)_c$ for every $c \in (S, 2^{2/N}S)$. This relies heavily on an argument of M. Struwe [Math. Z., 1984].

Step 2. Assuming by contradiction that F has no critical value in the interval $(S, 2^{2/N}S)$, then the topology of the level set $F^c = \{u; F(u) \leq c\}$ remains unchanged as c varies in the interval $(S, 2^{2/N}S)$. More precisely F^a is a deformation retract of F^b for every $a, b \in (S, 2^{2/N}S)$. This is a standard fact in Morse theory.

Step 3. For $\varepsilon > 0$ sufficiently small the level set $F^{S+\varepsilon}$ "inherits" the topology of Ω. This relies on a result of P. L. Lions [CRAS, 1983 and Rev. Mat. Iberoamericana, 1985] which asserts that if v_n is a minimizing sequence for (5), then $|\nabla v_{n_k}|^2$ converges to a Dirac mass δ_a for some a.

Step 4. Assuming that the hole in Ω is sufficiently small, construct a map $f \in C^0(\mathbb{S}^{N-1}, F^{S+\varepsilon})$ which is not homotopic to a constant in $C^0(\mathbb{S}^{N-1}, F^{S+\varepsilon})$, but is homotopic to a constant in $C^0(\mathbb{S}^{N-1}, F^{\alpha})$ for some $\alpha < 2^{2/N}S$. A contradiction with Step 2.

We now return to the result of Bahri-Coron mentioned at the beginning of this section. Its proof is long (40p) and arduous; I can only describe the basic ingredients.

Assume by contradiction that (4) admits no solution. Then F has no critical value. Using again Struwe's result one can show that F satisfies $(PS)_c$ for every $c \neq k^{2/N}S$, where $k = 1, 2, ...$ Consequently the topology of F^t remains unchanged as t varies in the intervals $(k^{2/N}S, (k+1)^{2/N}S), k = 1, 2, ...$

At each level $c = k^{2/N}S, k = 1, 2, ...$, there exists sequences (v_n) such that $F(v_n) \to c, F'(v_n) \to 0$ and (v_n) admits no convergent subsequence (in the strong H_0^1 sense). Following A. Bahri one says that F admits a "critical point at infinity" at level c. Such a situation *may* or *may not* produce a change in topology of the level sets F^t as t crosses the value c. (Abbas loved to illustrate this phenomenon by drawing functions $F : \mathbb{R} \to \mathbb{R}$ with a horizontal asymptote at level c). The bulk of the Bahri-Coron proof is a *very precise computation* of the change of topology between $F^{c+\varepsilon}$ and $F^{c-\varepsilon}$ at each level $c = k^{2/N}S$. These informations combined with the topological properties of $F^{S+\varepsilon}$ inherited from the assumptions about the topology of Ω yield a contradiction. A. Bahri and J. M. Coron made a tremendous team. Abbas came with a deep knowledge of Topology, while Jean-Michel had a considerable experience with the failure of the $(PS)_c$ condition; in particular the sharp description in [H. Brezis - J.M. Coron, ARMA, 1985] of all non compact sequences (v_n) such $F(v_n) \to c$ and $F'(v_n) \to 0$ (in a different, but related setting) turned out to be enormously useful. The Bahri-Coron collaboration produced another very important paper concerning the so-called Kazdan-Warner problem, see [A. Bahri -Coron, JFA, 1991]. We also used this approach in [A. Bahri-H. Brezis, volume in memory of J. d'Atri, Birkhäuser, 1996] to obtain solutions of the Yamabe problem without referring to the positive mass theorem or variations thereof.

III. Periodic orbits of singular Hamiltonian systems with roots in Celestial Mechanics.

The n-body problem in Celestial Mechanics is modeled by the following Hamiltonian system

$$m_i\ddot{q}_i + V_{q_i}(q) = 0 \quad i = 1, 2, ..., n, \tag{7}$$

where $m_i > 0$, $q_i \in \mathbb{R}^\ell$ and

$$V(q) = -\sum_{\substack{i,j=1\\ i\neq j}}^{n} \frac{m_i m_j}{|q_i - q_j|^\alpha}, \alpha > 0. \tag{8}$$

The classical 3-body problem corresponds to $n = \ell = 3$ and $\alpha = 1$. One question of interest for (7) is to find periodic solutions of prescribed period T. Equation (7) has a variational structure; its solutions correspond to critical points of the functional

$$I(q) = \int_0^T \left(\frac{1}{2}\sum_i m_i|\dot{q}_i|^2 - V(q)\right) dt \tag{9}$$

defined on the class of T-periodic functions. A. Bahri and P. Rabinowitz [CRAS, 1990 and Ann. IHP, 1991] investigated in great detail the case $n = \ell = 3$. They assumed that $\alpha \geq 2$, making V a strong force potential, a notion already used by H. Poincaré [CRAS, 1896], although not under that terminology. They established the existence of infinitely many T-periodic solutions of (7) corresponding to distinct critical values of I. The proof of this remarkable result involves a careful analysis of the failure of (PS) condition in the spirit of critical points at infinity. They also established, under weaker conditions on V— which cover e.g. the case $\alpha > 0$ in (8) — the existence of generalized solutions allowing collisions.

IV. Contact form geometry.

Abbas started thinking about problems in contact geometry early in his career and he revisited constantly this subject until his last days; he wrote no less than three monographs on this topic. He was fascinated by the Weinstein conjecture which claims the existence of at least one periodic orbit for the Reeb flow of a contact form. This problem admits a variational formulation: the periodic orbits are critical points of a functional which fails to satisfy any form of compactness condition

(Palais-Smale or variations thereof). It is in this framework that Abbas [CRAS 1984] first coined the expression "critical points at infinity". Even though the Weinstein conjecture has been proved in a number of cases by other methods, Abbas always returned to this problem with his original approach, aiming not only at proving the conjecture, but also obtaining an algebraic count of the number of orbits. He had his own opinion on what should be the "right approach"—quite different from the "standard methods" of symplectic geometry. Abbas' approach is an uncompromising mix of originality, insight and technical achievements. Although it has not attracted much attention in the symplectic community, some experts believe it contains hidden gems yet to be found.

Towards the end of the 80s I estimated that Abbas' remarkable achievements deserved a public recognition. Two opportunities presented themselves simultaneously. The Fermat prize had just been established, and independently, as a freshly elected member of the French Academy, I had my first opportunity to nominate candidates for the Academy awards. Since I was totally uncertain of the outcome, I presented Abbas for both. To my surprise and delight Abbas won the Fermat and the Langevin prizes!

At Rutgers, Abbas received in 1990 the Board of Trustees Award for Excellence, the University's highest honor for outstanding Research. He was the Director of the Center for Nonlinear Analysis from 1988 to 2009, and played a significant role in inviting visitors, organizing conferences and the weekly Nonlinear Analysis Seminar. He listened to all lectures with great interest, always raising questions and making suggestions. Abbas was a great speaker. His students were charmed by his talent to "create" mathematics in front of their eyes. He had been very encouraging, caring, and supporting of young researchers from Rutgers and from all over the world.

Abbas came for a final time to our seminar on December 8, 2015. He had made tremendous efforts — despite his physical condition— to attend a lecture by Paul Yang. At some point, Paul mentioned a work by one of Abbas' Tunisian disciples. A glimpse of fondness illuminated Abbas' face. It is the last image I keep of him.

Haïm Brezis

(With the help of Diana Bahri Nunziante, Henri Berestycki, Jean-Michel Coron, Helmut Hofer, Yanyan Li, Paul Rabinowitz and Claude Viterbo).