Haim Brezis, Petru Mironescu

Sobolev Maps to the Circle

From the Perspective of Analysis, Geometry, and Topology

The theory of real-valued Sobolev functions is a classical part of analysis and has a wide range of applications in pure and applied mathematics. By contrast, the study of manifold-valued Sobolev maps is relatively new. The incentive to explore these spaces arose in the last forty years from geometry and physics. This monograph is the first to provide a unified, comprehensive treatment of Sobolev maps to the circle, presenting numerous results obtained by the authors and others. Many surprising connections to other areas of mathematics are explored, including the Monge-Kantorovich theory in optimal transport, items in geometric measure theory, Fourier series, and non-local functionals occurring, for example, as denoising filters in image processing. Numerous digressions provide a glimpse of the theory of sphere-valued Sobolev maps.

Each chapter focuses on a single topic and starts with a detailed overview, followed by the most significant results, and rather complete proofs. The "Complements and Open Problems" sections provide short introductions to various subsequent developments or related topics, and suggest new directions of research. Historical perspectives and a comprehensive list of references close out each chapter. Topics covered include lifting, point and line singularities, minimal connections and minimal surfaces, uniqueness spaces, factorization, density, Dirichlet problems, trace theory, and gap phenomena.

Sobolev Maps to the Circle will appeal to mathematicians working in various areas, such as nonlinear analysis, PDEs, geometric analysis, minimal surfaces, optimal transport, and topology. It will also be of interest to physicists working on liquid crystals and the Ginzburg-Landau theory of superconductors.

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Dedicated to the memory of Jean Bourgain,
with immense admiration and gratitude

Preface

The theory of Sobolev functions from a domain $\Omega \subset \mathbb{R}^N$ with values in $\mathbb{R}$ (or $\mathbb{R}^k$) is by now a classical part of analysis, and has a wide range of applications in pure and applied mathematics. By contrast, the study of Sobolev maps from Ω to $\mathscr{N}$, where $\mathscr{N}$ is a compact manifold embedded in some $\mathbb{R}^\ell$, e.g., $\mathscr{N} = \mathbb{S}^k$, with $k \geq 1$, is relatively new, not yet fully developed, and, up to now, reserved for a small "club" of researchers. The incentive to explore these spaces came, in recent years, from two directions: geometry and physics. In geometry, they appear in the study of harmonic maps. In physics, they arise, for example, in the Oseen–Frank theory of nematic liquid crystals, in the Ginzburg–Landau theory of superconductors, in superfluids and in micromagnetics. The reason one is led to work with Sobolev maps rather than smooth maps is to allow *singularities* such as $\dfrac{x}{|x|}$ in 2D, or line singularities in 3D, which occur naturally both in geometry and physics.

In this monograph, we only briefly mention the connections to harmonic maps and to physics, for which we refer the reader to specialized works such as Schoen and Uhlenbeck [317, 318], Bethuel, Brezis, and Hélein [31] and Sandier and Serfaty [315]. Our main goal here is the study of the *intrinsic properties* of the Sobolev spaces

$$W^{1,p}(\Omega;\mathbb{S}^1) := \{u \in W^{1,p}(\Omega;\mathbb{R}^2);\ |u(x)| = 1 \text{ a.e. on } \Omega\},\ 1 \leq p < \infty,$$

and, more generally, the fractional Sobolev spaces $W^{s,p}(\Omega;\mathbb{S}^1)$, $0 < s < \infty$, $1 \leq p < \infty$.

In order to simplify the presentation, we concentrate on $\mathbb{S}^1$, rather than $\mathbb{S}^k$, $k \geq 1$, since the $\mathbb{S}^1$ theory is fairly polished. However, we mention here and there, in Chapter 4 and in sections entitled "Complements and Open Problems," what needs to be modified – or what remains to be proved – when the target space is $\mathbb{S}^k$.

Our decision (a long time ago) to write a book that would include "everything you wanted to know" about Sobolev maps to the circle, prompted us to address many unsettled questions. In turn, this led to a series of publications, whose results have entered the current book. We traveled part of the road jointly with our dearly missed friend Jean Bourgain, a luminous and inspiring collaborator. As a matter of fact, Jean became attracted by the subject when one of us (HB) asked him the seemingly innocuous question whether a map $u \in H^s((0,1);\mathbb{S}^1)$ admits a lifting (i.e., a phase) $\varphi \in H^s((0,1);\mathbb{R})$ when $0 < s < 1/2$. Jean proved that the answer is positive, but the question turned out to be tougher – and the proof much more involved – than expected. This result became the heart of our first joint paper [50]. Incidentally, we still do not have a simple proof of this fact (see Open Problem 7).

As the reader will discover, the classes $W^{s,p}(\Omega;\mathbb{S}^1)$ have an *amazingly rich structure*. Topological and geometrical effects are already visible, even in this simple framework, since $\mathbb{S}^1$ has non-trivial topology. Moreover, the fact that the target space is $\mathbb{S}^1$ (as opposed to $\mathbb{S}^k$) offers the option to consider liftings of maps $u : \Omega \to \mathbb{S}^1$. The search for "optimal liftings" leads to an interesting connection with minimal surfaces spanned by a curve.

The reader will also find throughout this book unexpected connections with a range of topics such as the Monge–Kantorovich theory in optimal transport, items in geometric measure theory, such as the fine theory of BV functions, rectifiable currents, minimal surfaces, etc., Fourier series, non-local functionals occurring, for example, as denoising filters in image processing, and more.

A final word of warning. Anyone familiar with standard properties of the space $W^{1,p}(\Omega;\mathbb{R})$ should expect surprises. Here are, for example, three striking differences, that we discuss in great length in this text:

a) $C^\infty(\overline{\Omega};\mathbb{R})$ is dense in $W^{1,p}(\Omega;\mathbb{R})$; by contrast, $C^\infty(\overline{\Omega};\mathbb{S}^1)$ need *not* be dense in $W^{1,p}(\Omega;\mathbb{S}^1)$.
b) The trace space for $W^{1,p}(\Omega;\mathbb{R})$ is precisely $W^{1-1/p,p}(\partial\Omega;\mathbb{R})$, $1 < p < \infty$. By contrast, the trace space for $W^{1,p}(\Omega;\mathbb{S}^1)$ is contained in $W^{1-1/p,p}(\partial\Omega;\mathbb{S}^1)$, but the inclusion can be *strict*.
c) If $\varphi \in W^{1,p}(\Omega;\mathbb{R})$, then $u := e^{\imath\varphi} \in W^{1,p}(\Omega;\mathbb{S}^1)$ and $|\nabla u| = |\nabla\varphi|$ a.e. on Ω. One might expect (by analogy with the C^k case) that, conversely, any $u \in W^{1,p}(\Omega;\mathbb{S}^1)$ comes from a $\varphi \in W^{1,p}(\Omega;\mathbb{R})$. This is *not* true!

We hope that intrigued readers will "join the club" and tackle some of the open problems scattered throughout the book.

Acknowledgments

HB is indebted to Jerald Ericksen and David Kinderlehrer for introducing him to the enchanted world of sphere-valued maps during the special program on liquid crystals they organized at IMA (the University of Minnesota) in 1984–1985.

The earliest papers of HB on this topic were written (in the 1980s and 1990s) jointly with Jean-Michel Coron, Elliott Lieb, and later with Fabrice Bethuel and Frédéric Hélein. They were followed by several contributions with the late Louis Nirenberg and with Yanyan Li. He thanks his coauthors for these fruitful collaborations that paved the way for numerous subsequent developments.

More recently, we enjoyed working with Luigi Ambrosio, Alessio Figalli, Emmanuel Russ, Itai Shafrir, and Yannick Sire on some topics that are presented in this book.

In the early 2000s, several talented young mathematicians shared our gusto for Sobolev maps to manifolds: Pierre Bousquet, Juan Dávila, Radu Ignat, Benoît Merlet, Vincent Millot, Hoai-Minh Nguyen, Adriano Pisante, Arkady Poliakovsky, Augusto Ponce, and Jean Van Schaftingen. Lately, Eduard Curcă, Xavier Lamy, and Ioana Molnar “joined the club.”

Over the years, HB taught graduate courses at the Université Paris VI (now called Sorbonne Université) and at Rutgers, based on sections of this monograph. He received useful feedback from the attendees, in particular from Vicenţiu Rădulescu and David Herrera.

The *Rencontres d'analyse*, co-organized by PM at the Camille Jordan Institute in Lyon, have been a great environment for discussing some of the results in this book, and for fruitful collaborations.

The presentation of the first chapter owes much to Itai Shafrir, who shared with us an illuminating duality approach. We warmly thank him for his precious input on various other sections, and also for the constructive reading of preliminary versions of parts of this book.

Throughout the years of work on this book, we have received help and advice from many colleagues: Eli Aljadeff, Eric Baer, John Ball, Vincent Borrelli (with special thanks from PM), George Dinca, Lawrence Craig Evans, Michael Frazier, Mariano Giaquinta, Robert Hardt, Bernd Kawohl, Fanghua Lin, Jean Mawhin, Vladimir Maz'ya, Élisabeth Mironescu, Tristan Rivière, Jacob Rubinstein, Étienne Sandier, Ovidiu Savin, Peter Sternberg, and Cédric Villani, to whom we express our gratitude.

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Paris, New Brunswick, Jerusalem, Haifa — Haïm Brezis
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Overview

Our text is structured as follows. Each chapter deals with a single topic. They start with an Introduction, which consists of a rather detailed overview. The first sections of each chapter are written in a textbook style, focusing on results and giving rather complete proofs. The Complements and Open problems sections provide short introductions to various subsequent developments or related topics, and suggest new directions of research. The Comments sections present some historical perspectives on each topic, together with a rather comprehensive list of references.

For the sake of simplicity, we leave aside throughout the monograph (except in Chapter 14) the topological effects that may possibly arise when Ω has non-trivial topology. In order to further simplify the presentation, we assume in the overview of Chapters 1–13 that Ω is a ball.

Chapter 1. We investigate here the question of lifting within the framework of Sobolev spaces $W^{1,p}$. In the process, we are led to the introduction of two *fundamental* tools, Ju and $\Sigma(u)$, which are ubiquitous throughout the entire book.

Given a map $u : \Omega \to \mathbb{S}^1$, we may always write $u = e^{\imath\varphi}$, for some $\varphi : \Omega \to \mathbb{R}$. The function φ is called a *lifting* or a *phase* of u. Some liftings φ can be very rough; for example, $u \equiv 1$ admits non-measurable liftings. However, when u is continuous (or better), we may find a lifting φ as smooth as u: for example, if $u \in C^k$, then we may find $\varphi \in C^k$.

This need not be the case in the context of Sobolev spaces. More specifically, consider, for $1 \le p < \infty$, the space

$$\begin{aligned} W^{1,p}(\Omega; \mathbb{S}^1) :=&\{u \in W^{1,p}(\Omega; \mathbb{C});\ |u(x)| = 1 \text{ for a.e. } x \in \Omega\} \\ \simeq&\{u \in W^{1,p}(\Omega; \mathbb{R}^2);\ |u(x)| = 1 \text{ for a.e. } x \in \Omega\}. \end{aligned}$$

When $N = 1$ and $u \in W^{1,p}(\Omega; \mathbb{S}^1)$, then u is continuous, and any continuous lifting φ of u belongs to $W^{1,p}$. However, when $N \ge 2$, a map $u \in W^{1,p}(\Omega; \mathbb{S}^1)$ *need not have a* $W^{1,p}(\Omega; \mathbb{R})$ *phase*. For example, if $N = 2$, $0 \in \Omega$, and

$$u(x) := \frac{x}{|x|}, \tag{1}$$

then $u \in W^{1,p}$, $\forall\, 1 \le p < 2$, but u has no lifting in $W^{1,p}$, and not even in $W^{1,1}_{loc}$.

This is a "typical" example of non-lifting. By contrast, when $N \ge 2$ and $p \ge 2$, $W^{1,p}(\Omega; \mathbb{S}^1)$ does have the *lifting property*, i.e., *every* $u \in W^{1,p}(\Omega; \mathbb{S}^1)$ can be written as $u = e^{\imath\varphi}$ for some $\varphi \in W^{1,p}\,(\Omega; \mathbb{R})$.

In the "nasty" range $N \ge 2$, $1 \le p < 2$, where the lifting property fails, there are two natural directions to investigate.

Direction 1. *Characterize* the u's in $W^{1,p}(\Omega; \mathbb{S}^1)$ that can be written as $u = e^{\imath\varphi}$ for some $\varphi \in W^{1,p}(\Omega; \mathbb{R})$.

Direction 2. If $u \in W^{1,p}(\Omega; \mathbb{S}^1)$ admits no lifting in $W^{1,p}$, what are the "best possible" alternative classes in which a lifting exists?

In the first direction, a major role is played by the "distributional Jacobian" Ju. To motivate its introduction in our context, we start with the following elementary calculation. Let $u = (u_1, u_2) \in W^{1,p}(\Omega; \mathbb{S}^1)$ and assume that

$$u = e^{\imath\varphi} \text{ for some } \varphi \in W^{1,p}(\Omega; \mathbb{R}). \tag{2}$$

Differentiating the equalities

$$u_1 = \cos\varphi,\ \ u_2 = \sin\varphi,$$

we find that

$$\nabla u_1 = -\sin\varphi\,\nabla\varphi,\ \ \nabla u_2 = \cos\varphi\,\nabla\varphi,$$

and therefore

$$\nabla\varphi = \cos\varphi\,\nabla u_2 - \sin\varphi\,\nabla u_1 = u_1\nabla u_2 - u_2\nabla u_1 := u \wedge \nabla u. \tag{3}$$

Since the left-hand side of (3) is a gradient, it follows that the curl of the right-hand side (3) (considered in the sense of distributions) vanishes. We can express this fact as $Ju = 0$ in $\mathscr{D}'(\Omega)$, where Ju is the $N \times N$ antisymmetric matrix with entries in $\mathscr{D}'(\Omega; \mathbb{R})$, given, $\forall\, 1 \le i, j \le N$, by

$$\begin{aligned}(Ju)_{ij} :=& \frac{1}{2}\left[\frac{\partial}{\partial x_i}(u \wedge \nabla u)_j - \frac{\partial}{\partial x_j}(u \wedge \nabla u)_i\right]\\ =& \frac{1}{2}\left[\frac{\partial}{\partial x_i}\left(u \wedge \frac{\partial u}{\partial x_j}\right) - \frac{\partial}{\partial x_j}\left(u \wedge \nabla\frac{\partial u}{\partial x_i}\right)\right].\end{aligned} \tag{4}$$

For reasons that will be explained in Chapter 1, this object is called the distributional Jacobian of u.

Thus, the condition $Ju = 0$ is a *necessary condition* for the existence of $\varphi \in W^{1,p}$ satisfying (2). Remarkably, this condition is also *sufficient*, and this gives a complete and useful answer to Direction 1.

In particular, this implies that, for the u given by (1), we have $Ju \neq 0$. More specifically, we have the following formula (which will be generalized considerably in Chapters 2 and 3):

$$Ju = \begin{pmatrix} 0 & \pi\,\delta_0 \\ -\pi\,\delta_0 & 0 \end{pmatrix}. \tag{5}$$

As we are going to see, there is a way to "quantify" the failure of the condition $Ju = 0$. The quantity $\Sigma(u)$ defined below measures the "deviation" of $u \wedge \nabla u$ from the space of gradients. Given $u \in W^{1,1}(\Omega; \mathbb{S}^1)$, we set

$$\Sigma(u) := \inf\{\|u \wedge \nabla u - \nabla\eta\|_{L^1};\ \eta \in W^{1,1}(\Omega; \mathbb{R})\}. \tag{6}$$

In Section 1.5, we will prove (via duality) that $\Sigma(u)$ coincides with $\|Ju\|$ for some appropriate seminorm $\|\ \|$ on the space of distributions $(W^{-1,1})^{N\times N}$ (to which Ju belongs).

In summary, we have, $\forall\, u \in W^{1,1}(\Omega; \mathbb{S}^1)$,

$$\Sigma(u) = 0 \iff Ju = 0 \iff u = e^{\imath\varphi} \text{ for some } \varphi \in W^{1,1}(\Omega; \mathbb{R}). \tag{7}$$

Here are three important occurrences of $\Sigma(u)$.

A) From the above discussion, we know that, if $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ is such that $Ju \neq 0$, then there exists no lifting φ of u with $\varphi \in W^{1,1}(\Omega; \mathbb{R})$. However, *any* $u \in W^{1,1}(\Omega; \mathbb{R})$ may be written as

$$u = e^{\imath\varphi} \text{ for some } \varphi \in BV(\Omega; \mathbb{R}), \tag{8}$$

where BV denotes the spaces of functions of bounded variation – a space slightly larger than $W^{1,1}$. For example, the map u in (1) can be written as $u = e^{\imath\varphi}$, where $\varphi := \arg u \in [0, 2\pi)$ is the principal branch of the argument of u. This φ belongs to BV, and jumps by 2π on $\Omega \cap ([0, \infty) \times \{0\})$.

As the above example shows, a major difference between BV and $W^{1,1}$ is that BV allows discontinuities along hypersurfaces (e.g., line discontinuities when $N = 2$).

Another major difference between lifting in the space of continuous functions (or, as we are going to see in Chapter 6, in $W^{1,1}$) and lifting in BV is that φ in (8) is not unique, even (mod 2π): for example, if B is a ball such that $B \subset \Omega$ and φ satisfies (8), then $\varphi + 2\pi\,\chi_B$ still satisfies (8).

We may thus ask whether some BV liftings are "better" than others. It turns out that a natural way of selecting "best" BV liftings involves the "energy"

$$E(u) := \min\{|\varphi|_{BV};\ \varphi \in BV(\Omega;\mathbb{R}) \text{ and } u = e^{\imath\varphi} \text{ in } \Omega\}, \tag{9}$$

where $|\varphi|_{BV}$ is the mass of the measure $|D\varphi|$.

One of the main results in this chapter is that

$$E(u) = \int_\Omega |\nabla u| + \Sigma(u),\ \forall\, u \in W^{1,1}(\Omega;\mathbb{S}^1).$$

B) As we are going to see in Chapters 2 and 3, the distribution $(1/\pi)\, Ju$ is supported by the "singular set" of u and carries information about the location and strength of the topological singularities of u. With this in mind, the quantity

$$S_p(u) := \inf\left\{\int_\Omega |\nabla v|^p;\ v \in W^{1,p}(\Omega;\mathbb{S}^1),\ Jv = Ju\right\},$$

defined, for every $u \in W^{1,p}(\Omega;\mathbb{S}^1)$, represents the *least* energy required to "produce" a map $v \in W^{1,p}(\Omega;\mathbb{S}^1)$ having the same topological singularities as u.

Here again $\Sigma(u)$ pops up:

$$S_1(u) = 2\pi\, \Sigma(u),\ \forall\, u \in W^{1,1}(\Omega;\mathbb{S}^1). \tag{10}$$

C) The *relaxed energy* is defined for every $u \in W^{1,p}(\Omega;\mathbb{S}^1)$ by

$$R_p(u) := \inf\left\{\liminf_{n\to\infty} \int_\Omega |\nabla u_n|^p;\ u_n \in C^\infty(\overline{\Omega};\mathbb{S}^1),\ u_n \to u \text{ a.e. on } \Omega\right\},$$

where the first inf is taken over all sequences (u_n) in $C^\infty(\overline{\Omega};\mathbb{S}^1)$ such that $u_n \to u$ a.e. on Ω. We will prove that

$$R_1(u) = \int_\Omega |\nabla u| + \Sigma(u),\ \forall\, u \in W^{1,1}(\Omega;\mathbb{S}^1). \tag{11}$$

Chapter 2. In this chapter, we give illuminating geometric interpretations for Ju and $\Sigma(u)$ when $N = 2$. In 2D, we naturally identify Ju, given by (4), with the *scalar* distribution $(Ju)_{12}$. Assuming, for example, that $u \in W^{1,1}(\Omega;\mathbb{S}^1)$ is smooth except at some point $a \in \Omega$, and that the winding number, $\deg(u, a)$, of u on small circles around a, equals one, we have the following generalization of (5):

$$Ju = \pi\delta_a \text{ in } \mathscr{D}'(\Omega). \tag{12}$$

More generally, if $u \in W^{1,1}(\Omega;\mathbb{S}^1)$ is a "nice" map, i.e., continuous on Ω except at a finite number of distinct points $a_1, \ldots, a_k$, we have

$$Ju = \pi \sum_{j=1}^{k} \deg(u, a_j)\, \delta_{a_j} \text{ in } \mathscr{D}'(\Omega). \tag{13}$$

Nice maps u play an important role because they are "generic" in $W^{1,1}(\Omega; \mathbb{S}^1)$ (as explained in Chapter 10). This allows results valid for nice maps to be transferred to general u's in $W^{1,1}(\Omega; \mathbb{S}^1)$. A typical result is the following extension of (13): for every $u \in W^{1,1}(\Omega; \mathbb{S}^1)$, there exist points $P_j, N_j \in \overline{\Omega}$ such that

$$\sum_{j=1}^{\infty} |P_j - N_j| < \infty, \tag{14}$$

$$Ju = \pi \sum_{j=1}^{\infty} (\delta_{P_j} - \delta_{N_j}) \text{ in } \mathscr{D}'(\Omega). \tag{15}$$

Conversely, given any points $P_j, N_j \in \overline{\Omega}$ satisfying (14), there exists some $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ such that (15) holds.

Next, we turn to the geometric interpretation of $\Sigma(u)$ as a "minimal length" required to "connect the singularities."

For simplicity, assume again that u is a nice map, continuous except at the distinct points P_j, N_j, $j = 1, \ldots, m$, satisfying $\deg(u, P_j) = 1$, $\deg(u, N_j) = -1$, $\forall j$. Then we have

$$\Sigma(u) = \min_{\sigma \in S_m} 2\pi \sum_{j=1}^{m} d(P_j, N_{\sigma(j)}), \tag{16}$$

where S_m denotes the group of permutations of the integers $\{1, \ldots, m\}$ and $d(P, N)$ is the pseudometric defined on $\overline{\Omega}$ by

$$d(P, N) := \min\{\text{dist}(P, \partial\Omega) + \text{dist}(N, \partial\Omega),\ |P - N|\}.$$

Obviously, the right-hand side of (16) has a flavor of optimal transport, and indeed the Monge–Kantorovich formula plays a role in the proof of (16).

Again, we can adapt formula (16) to general u's in $W^{1,1}(\Omega; \mathbb{S}^1)$; this involves the minimization of the length of 1-rectifiable currents $\mathscr{C}$ (in the sense of geometric measure theory) such that $\partial\mathscr{C} = (1/\pi)\, Ju$.

Chapter 3. This chapter is the 3D (and higher) counterpart of the previous one. Here, geometry enters decisively, the basic objects being curves of singularities, which play in 3D the same role as points in 2D.

We illustrate this analogy when $N = 3$ and u has the simplest possible singular set: a closed curve Γ.

In 3D, we naturally identify Ju given by (4) with a vector field whose entries are scalar distributions:

$$Ju \simeq ((Ju)_{23}, (Ju)_{31}, (Ju)_{12}).$$

Assume that $u \in W^{1,1}(\Omega; \mathbb{S}^1) \cap C(\Omega \setminus \Gamma)$, where $\Gamma \subset \Omega$ is a smooth simple closed oriented curve. If the winding number of u around Γ equals one, then $(1/\pi)\, Ju$ can be identified with the integration along Γ, i.e.,

$$\langle Ju, \zeta \rangle = \pi \int_\Gamma \tau \cdot \zeta, \ \forall \zeta \in C_c^\infty(\Omega; \mathbb{R}^3),$$

where τ denotes the unit tangent to Γ; this is the analogue of (12).

For such maps, we have the following counterpart of (16):

$$\Sigma(u) = 2\pi \ \inf\{|M|;\, M \subset \Omega \text{ is a smooth oriented surface with boundary } \Gamma\}, \tag{17}$$

where $|M|$ is the area of M and the inf in (17) is the quantity that arises in the celebrated Plateau problem.

Incidentally, as a byproduct of our analysis, we obtain the following formula for the least area spanned by a smooth simple closed oriented curve $\Gamma \subset \mathbb{R}^3$:

$$\begin{aligned} &\inf\{|M|;\ M \subset \mathbb{R}^3 \text{ is a smooth oriented surface with boundary } \Gamma\} \\ &= \sup\left\{ \int_\Gamma \tau \cdot \zeta;\ \zeta \in C_c^\infty(\mathbb{R}^3; \mathbb{R}^3),\ \|\operatorname{curl} \zeta\|_{L^\infty} \le 1 \right\}. \end{aligned} \tag{18}$$

Note that assertion (18) does not involve $\mathbb{S}^1$-valued maps. Instead, it provides a new approach to minimal surfaces via a kind of "Monge–Kantorovich formula." Interestingly, the proof of (18) that we present transits via $\mathbb{S}^1$-valued maps.

Chapter 4. The purpose of this brief introduction to the world of sphere-valued maps is to explain how some of the tools and results presented in the previous chapters extend to maps $u : \Omega \to \mathbb{S}^k$, with $1 \le k \le N-1$, and where $W^{1,k}(\Omega; \mathbb{S}^k)$ plays the role of $W^{1,1}(\Omega; \mathbb{S}^1)$.

When $k \ge 2$, lifting is not available, but there are natural counterparts for Ju, $\Sigma(u)$, $S_p(u)$, and $R_p(u)$.

The "historical" case $k = 2$, $N = 3$, which was originally motivated by liquid crystals, is very similar to the case $k = 1$, $N = 2$ discussed in Chapter 2. When u is nice, there are formulas similar to (13) and (16). The counterparts of (10) and (11) are

$$S_2(u) = 2\, \Sigma(u),\ \forall u \in H^1(\Omega; \mathbb{S}^2), \tag{19}$$

and

$$R_2(u) = \int_\Omega |\nabla u|^2 + 2\, \Sigma(u),\ \forall u \in H^1(\Omega; \mathbb{S}^2). \tag{20}$$

At the next levels of the extreme case $k = N - 1$, i.e., $k = 3$, $N = 4$, etc., the situation is still rather satisfactory. Formula (19) has to be replaced by

$$S_k(u) = k^{k/2}\, \Sigma(u),\ \forall\, u \in W^{1,k}(\Omega; \mathbb{S}^k). \tag{21}$$

The analogue of (20) has not been established, but is highly plausible.

However, in the “intermediate” range, $2 \le k \le N - 2$ (starting with $N = 4$, $k = 2$), a new phenomenon occurs. One can still assert that $S_k(u) \ge k^{k/2}\, \Sigma(u)$, but in general one may have strict inequality. However, the geometric flavor of $S_k(u)$ persists. Equality (21) has to be replaced by

$$S_k(u) = k^{k/2}\, \Sigma^*(u),\ \forall\, u \in W^{1,k}(\Omega; \mathbb{S}^k), \tag{22}$$

where $\Sigma^*(u)$ corresponds to the least mass among all $(N-k)$-rectifiable currents spanned by the singular set of u.

Here, it is also expected (but not proved when $3 \le k \le N - 1$) that

$$R_k(u) = \int_\Omega |\nabla u|^k + k^{k/2}\, \Sigma^*(u),\ \forall\, u \in W^{1,k}(\Omega; \mathbb{S}^k).$$

Chapter 5. Starting from this chapter, we introduce the reader into the realm of fractional Sobolev spaces. The first question we ask, in the spirit of Chapter 1, is: does *any* $u \in W^{s,p}(\Omega; \mathbb{S}^1)$ admit a lifting $\varphi \in W^{s,p}(\Omega; \mathbb{R})$? Here, for $0 < s < \infty$ and $1 \le p < \infty$, we set

$$\begin{aligned} W^{s,p}(\Omega; \mathbb{S}^1) :=&\{u \in W^{s,p}(\Omega; \mathbb{C});\ |u(x)| = 1 \text{ for a.e. } x \in \Omega\} \\ \simeq&\{u \in W^{s,p}(\Omega; \mathbb{R}^2);\ |u(x)| = 1 \text{ for a.e. } x \in \Omega\}. \end{aligned}$$

We give a full answer to this question.

When $N = 1$, the answer is positive, $\forall\, s$, $\forall\, p$, but the proof is surprisingly difficult if $sp < 1$. When $N \ge 2$, a significant “phase transition” occurs at $s = 1$. In the case $s < 1$, the answer is positive if and only if $sp < 1$ or $sp \ge N$. In the case $s \ge 1$, the answer is positive if and only if $sp \ge 2$ – a fact that is consistent with the result in Chapter 1 concerning $s = 1$.

Chapter 6. We investigate here the question of *uniqueness* (mod 2π) of a lifting. More precisely, assume that we have two liftings φ_1, φ_2 for the same u. Then, $\varphi_1(x) - \varphi_2(x) = 2\pi\, \psi(x)$ for some $\psi : \Omega \to \mathbb{Z}$. Therefore, we are led to the question of finding minimal assumptions on a measurable function $\psi : \Omega \to \mathbb{Z}$ implying that ψ is constant a.e. Clearly, continuity is sufficient, but, as we are going to see, constancy holds for a surprisingly large class of possibly *discontinuous* functions.

It is not difficult to see that any $\psi \in VMO\,(\Omega; \mathbb{Z})$ must be constant a.e., where VMO denotes the space of functions of vanishing mean oscillation. This is a small step beyond continuity.

It is also well-known that any $\psi \in W^{1,1}(\Omega; \mathbb{Z})$ must be constant a.e., simply because $\nabla\psi = 0$ a.e. on Ω.

A more surprising (and not so well-known) fact is that any $\psi \in W^{1/p,p}(\Omega; \mathbb{Z})$, $1 < p < \infty$, must be constant a.e. For the enjoyment of the reader, we present several proofs of this result.

Our favorite one (but not the shortest one) goes as follows. If $\psi \in W^{1/p,p}$, then (by definition)

$$\int_\Omega \int_\Omega \frac{|\psi(x) - \psi(y)|^p}{|x - y|^{N+1}}\, dxdy < \infty,$$

and thus (using the fact that $\psi(x) - \psi(y) \in \mathbb{Z}$)

$$\int_\Omega \int_\Omega \frac{|\psi(x) - \psi(y)|}{|x - y|^{N+1}}\, dxdy < \infty. \tag{23}$$

At this stage, one can discard the assumption that ψ is integer-valued and prove that any measurable function $\psi : \Omega \to \mathbb{R}$ satisfying (23) must be constant a.e.

Here enters the "BBM formula," which asserts that, for any measurable function $f : \Omega \to \mathbb{R}$,

$$\lim_{\varepsilon\to 0} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|}{|x - y|}\, \rho_\varepsilon(|x - y|)\, dxdy = K_N \int_\Omega |Df|, \tag{24}$$

where (ρ_ε) is a family of radial mollifiers, K_N is a constant depending only on N, and $\int_\Omega |Df|$ is understood to be $+\infty$ if f is not a BV function. Applying (24) to $\rho_\varepsilon(r) := \dfrac{C\varepsilon}{r^{N-\varepsilon}}\, \chi(r)$, where χ is a cutoff function, and using (23), yields $\psi \in BV$ and $D\psi = 0$ in $\mathscr{D}'$, and thus ψ is constant a.e.

We seize this opportunity to "advertise" some of the many avatars of the BBM formula, and its potential applications in a variety of directions, including "non-local minimal surfaces," "s-perimeters," and image processing.

Chapter 7. As explained above, a map $u \in W^{s,p}(\Omega; \mathbb{S}^1)$, where $N \geq 2$, $0 < s < 1$, $1 \leq p < \infty$ and $1 \leq sp < N$, need not have a phase in $W^{s,p}$.

The main (and highly non-trivial) result in this chapter asserts, in particular, that any $u \in W^{s,p}(\Omega; \mathbb{S}^1)$, where $N \geq 1$, $0 < s < 1$, $1 \leq p < \infty$ and $sp \geq 1$, can be "factorized" as

$$u = e^{\imath\varphi} v, \text{ with } \varphi \in W^{s,p}(\Omega; \mathbb{R}) \text{ and } v \in W^{1,sp}(\Omega; \mathbb{S}^1),$$

with corresponding estimates.

Note that v has much better regularity than u – it admits a "full" gradient, as opposed to u, which has only "fractional" derivatives – moreover, $W^{1,sp}(\Omega; \mathbb{S}^1) \hookrightarrow$

$W^{s,p}(\Omega; \mathbb{S}^1)$ when $sp > 1$. Consequently, we can apply to v all the results in Chapters 1–3, including the analysis of its topological singularities. As we are going to see in Chapter 8, this basic tool allows us to give a proper meaning to the topological singularities of any $u \in W^{s,p}(\Omega; \mathbb{S}^1)$ with $N \geq 2$, $0 < s < 1$, $1 \leq p < \infty$, and $sp \geq 1$ – for example $u \in H^{1/2}(\Omega; \mathbb{S}^1)$.

Chapter 8. We present various applications of the factorization. Arguably the most spectacular one is the possibility of giving a "robust" definition to $u \wedge \nabla u$ when $u \in W^{1/p,p}(\Omega; \mathbb{S}^1)$, with $1 < p < \infty$. As a byproduct, we also obtain a robust definition of the distribution Ju for such u. For the sake of completeness, we mention that there exists a totally different technique for defining Ju when $u \in W^{1/p,p}$, that does not rely on factorization and can be adapted to $\mathbb{S}^k$-valued maps.

We explain how the factorization allows us to complement the results on lifting obtained in Chapter 3, by providing alternative lifting spaces in the cases where the answer to the lifting problem is negative.

Chapter 9. While the previous chapters were concerned with the *existence* and *uniqueness* of a lifting, this chapter investigates the matter of *estimates* for the phase φ of u. We exhibit two unusual aspects: "existence without estimates," and "estimates without existence."

Chapter 10. We investigate here density questions. For *real-valued* Sobolev spaces, $C^\infty(\overline{\Omega}; \mathbb{R})$ is dense in $W^{s,p}(\Omega; \mathbb{R})$, for any $s > 0$ and $1 \leq p < \infty$. This need not be true for the Sobolev spaces $W^{s,p}(\Omega; \mathscr{N})$, where $\mathscr{N}$ is a manifold. In particular, this is not always the case when $\mathscr{N} = \mathbb{S}^1$.

We present the optimal conditions on s and p so that $C^\infty(\overline{\Omega}; \mathbb{S}^1)$ is dense in $W^{s,p}(\Omega; \mathbb{S}^1)$. More precisely, we prove that density holds except in the "nasty" range $N \geq 2$, $s > 0$, $1 \leq p < \infty$, and $1 \leq sp < 2$.

We prove that, in the nasty range, maps in $W^{s,p}(\Omega; \mathbb{S}^1)$ that are smooth except on "small sets" $\Sigma \subset \Omega$ are dense in $W^{s,p}(\Omega; \mathbb{S}^1)$. In particular, when $N = 2$, Σ can be a finite collection of points.

In the same range, we characterize the closure of $C^\infty(\overline{\Omega}; \mathbb{S}^1)$ in $W^{s,p}(\Omega; \mathbb{S}^1)$. More specifically, we prove that

$$\overline{C^\infty(\overline{\Omega}; \mathbb{S}^1)}^{W^{s,p}} = \{u \in W^{s,p}(\Omega; \mathbb{S}^1);\ Ju = 0\}.$$

When $0 < s < 1$, we use here the definition of Ju introduced in Chapter 8.

Chapter 11. We present a complete trace theory for $\mathbb{S}^1$-valued maps. When $s > 0$ is not an integer and $1 \leq p < \infty$, standard trace theory for *real-valued* maps asserts that

$$\operatorname{tr}_\Omega W^{s+1/p,p}(\Omega \times (0,1); \mathbb{R}) = W^{s,p}(\Omega; \mathbb{R}),$$

where we identify Ω with $\Omega \times \{0\}$. When $\mathbb{R}$ is replaced by a manifold $\mathscr{N}$, in general we only have the inclusion

$$\operatorname{tr}_\Omega W^{s+1/p,p}(\Omega \times (0,1); \mathscr{N}) \subset W^{s,p}(\Omega; \mathscr{N}), \tag{25}$$

and equality may fail.

In the special case where $\mathscr{N} = \mathbb{S}^1$, we prove that equality occurs in (25) if and only if $sp < 1$ or $sp \geq N$.

In the remaining cases, where $1 \leq sp < N$, we characterize the range of the mapping $v \mapsto \operatorname{tr}_\Omega v$, with $v \in W^{s+1/p,p}(\Omega \times (0,1); \mathbb{S}^1)$.

Chapter 12. We revisit the notion of topological degree $\deg f$ (aka index or winding number) for maps $f : \mathbb{S}^1 \to \mathbb{S}^1$. This is a classical concept when f is continuous: $\deg f$ counts "how many times $f(\mathbb{S}^1)$ covers $\mathbb{S}^1$, taking into account algebraic multiplicity." One can still give a robust definition for $\deg f$ when f belongs merely to $VMO(\mathbb{S}^1; \mathbb{S}^1)$, and thus, by the Sobolev embeddings, for maps in the critical spaces $W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$, with $1 < p < \infty$. We establish some basic properties of this degree.

We also derive an integral formula for the degree of VMO maps, generalizing the Kronecker formula

$$\deg f = \frac{1}{\pi} \int_{\mathbb{D}} \operatorname{Jac} u, \ \forall f \in C^1(\mathbb{S}^1; \mathbb{S}^1), \ \forall u \in C^1(\overline{\mathbb{D}}; \mathbb{R}^2) \text{ with } u_{|\partial\mathbb{D}} = f.$$

Note that, when $f \in VMO(\mathbb{S}^1; \mathbb{S}^1)$ – or even $f \in C(\mathbb{S}^1; \mathbb{S}^1)$ – standard trace theory does not yield the existence of a map $u : \mathbb{D} \to \mathbb{R}^2$ such that $u_{|\partial\mathbb{D}} = f$ and $u \in H^1(\mathbb{D})$ – and thereby $\operatorname{Jac} u \in L^1(\mathbb{D})$. We will explain how to bypass this difficulty.

We investigate bounds of $|\deg f|$ in terms of some natural norms; a typical estimate is

$$|\deg f| \leq C_p |f|^p_{W^{1/p,p}}, \ \forall f \in W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1), \ \forall\, 1 < p < \infty.$$

A full section is dedicated to the formula

$$\deg f = \sum_{n \in \mathbb{Z}} n\, |a_n|^2, \ \forall f \in H^{1/2}(\mathbb{S}^1; \mathbb{S}^1),$$

where the a_n's are the Fourier coefficients of f.

This striking formula is the starting point of the intriguing saga "Can one hear the degree?," which involves deep interactions with Fourier analysis and still raises challenging open problems.

Chapter 13. We investigate minimization problems of the form

$$\min \left\{ \int_\Omega |\nabla u|^p; \ u \in W^{1,p}_g(\Omega; \mathbb{S}^1) \right\}, \tag{26}$$

where $1 \leq p < \infty$ and $g \in C^\infty(\partial\Omega; \mathbb{S}^1)$ is a given boundary condition (satisfying also $\deg g = 0$ when $N = 2$).

We prove that a "phase transition" appears at $p = 2$. When $p \geq 2$, the minimizer in (26) is unique and smooth; lifting theory from Chapter 1 enters in the proof. By contrast, when $1 < p < 2$, minimizers may have singularities. This is a consequence of the following "gap phenomenon," which occurs for some g's:

$$\min\left\{\int_\Omega |\nabla u|^p;\ u \in W_g^{1,p}(\Omega;\mathbb{S}^1)\right\} < \inf\left\{\int_\Omega |\nabla u|^p;\ u \in C_g^1(\overline{\Omega};\mathbb{S}^1)\right\} < \infty.$$

We discuss the nature of these singularities when $N = 2$.

We also survey what happens to (26) when the class $W_g^{1,p}(\Omega;\mathbb{S}^1)$ is empty. This occurs, for example, when $N = 2$, $p \geq 2$, $g \in C^\infty(\partial\Omega;\mathbb{S}^1)$, and $\deg g \neq 0$. Alternatively, this may occur when $N \geq 3$ and $2 \leq p < N$, for some g's such that $g \in W^{1-1/p,p}(\partial\Omega;\mathbb{S}^1)$, whereas $W_g^{1,p}(\Omega;\mathbb{S}^1) = \emptyset$ (see Chapter 11). These are typical examples of "infinite energy" minimization problems, and they can be tackled via a Ginzburg–Landau-type approximation

$$\min\left\{\int_\Omega |\nabla u|^p + \frac{1}{\varepsilon^2}\int_\Omega (|u|^2-1)^2;\ u \in W_g^{1,p}(\Omega;\mathbb{R}^2)\right\},$$

analyzing the asymptotic behavior of minimizers as $\varepsilon \to 0$. This analysis can be extremely delicate, and we only sketch some of the highlights, whose details are beyond the scope of this book.

Chapter 14. We investigate the impact of the topology of Ω on the main topics presented above (existence of lifting, relaxed energy, density of smooth maps, etc.).

Appendices. In the final appendices, we recall standard properties of Sobolev spaces used throughout the book, and we gather the most technical parts of some of the proofs presented in the main text.

Contents