Positive Solutions of Nonlinear Elliptic Equations Involving Critical Sobolev Exponents

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0. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with $n \geqq 3$. We are concerned with the problem of existence of a function u satisfying the nonlinear elliptic equation

$$
\begin{aligned}
-\Delta u &= u^p + f(x,u) && \text{on } \Omega,\\
u &> 0 && \text{on } \Omega,\\
u &= 0 && \text{on } \partial\Omega,
\end{aligned}
\tag{0.1}
$$

where $p = (n+2)/(n-2)$, $f(x,0) = 0$ and $f(x,u)$ is a lower-order perturbation of u^p in the sense that $\lim_{u\to+\infty} f(x,u)/u^p = 0$. A typical example is $f(x,u) = \lambda u$, where λ is a real constant. The exponent $p = (n+2)/(n-2)$ is critical from the viewpoint of Sobolev embedding. Indeed solutions of (0.1) correspond to critical points of the functional

$$\Phi(u) = \tfrac{1}{2}\int |\nabla u|^2 - \frac{1}{p+1}\int |u|^{p+1} - \int F(x,u),$$

where $F(x,u) = \int_0^u f(x,t)\,dt$. Note that $p+1 = 2n/(n-2)$ is the limiting Sobolev exponent for the embedding $H_0^1(\Omega) \subset L^{p+1}(\Omega)$. Since this embedding is *not compact*, the functional Φ *does not satisfy the* (PS) *condition*. Hence there are serious difficulties when trying to find critical points by standard variational methods. In fact, there is a sharp contrast between the case $p < (n+2)/(n-2)$ for which the Sobolev embedding is compact, and the case $p = (n+2)/(n-2)$. Many *existence* results for problem (0.1) are known when $p < (n+2)/(n-2)$ (see the review article by P. L. Lions [20] and the abundant list of references in [20]). On the other hand, a well-known *nonexistence* result of Pohozaev [24]

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asserts that if Ω is starshaped there is *no* solution of the problem

$$\begin{aligned} -\Delta u &= u^{(n+2)/(n-2)} && \text{on } \Omega, \\ u &> 0 && \text{on } \Omega, \\ u &= 0 && \text{on } \partial\Omega; \end{aligned}$$

see (1.4). But, as we shall see, *lower-order terms can reverse this situation.*

Our motivation for investigating (0.1) comes from the fact that it resembles some variational problems in geometry and physics where lack of compactness also occurs. The most notorious example is *Yamabe's problem*: find a function u satisfying

$$\begin{aligned} -4\frac{(n-1)}{(n-2)}\Delta u &= R' u^{(n+2)/(n-2)} - R(x)u && \text{on } M, \\ u &> 0 && \text{on } M, \end{aligned}$$

for some constant R'. Here M is an n-dimensional Riemannian manifold, Δ its Laplacian, and $R(x)$ is the scalar curvature.

But there are many other examples:

(a) Existence of extremal functions for isoperimetric inequalities, Hardy–Littlewood–Sobolev inequalities, trace inequalities, etc.; see Jacobs [14][1], Lieb [19], P. L. Lions [21].

(b) Existence of non-minimal solutions for Yang–Mills functionals; see C. Taubes [29].[2]

(c) Existence of non-minimal solutions for H-systems[3] (Rellich's conjecture concerning the existence of "large" surfaces of constant prescribed mean curvature spanned by a given curve in $\mathbb{R}^3$); see [5].

(d) See K. K. Uhlenbeck [31] for still more.

Our paper is organized as follows. In Section 1, we investigate the model problem

$$\text{(0.2)} \qquad \begin{aligned} -\Delta u &= u^p + \lambda u && \text{on } \Omega, \\ u &> 0 && \text{on } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where $p = (n+2)/(n-2)$ and λ is a real constant. Surprisingly, the cases where $n = 3$ and $n \geqq 4$ turn out to be quite different:

(a) *when* $n \geqq 4$, problem (0.2) has a solution for every $\lambda \in (0, \lambda_1)$, where λ_1 denotes the first eigenvalue of $-\Delta$; moreover it has no solution if $\lambda \notin (0, \lambda_1)$ and Ω is starshaped (see Theorem 1.1),

[1] This reference was brought to our attention by L. Carleson.

[2] This reference was brought to our attention by M. Atiyah.

[3] This problem was mentioned to us by S. Hildebrandt.

(b) *when* $n=3$, problem (0.2) is much more *delicate* and we have a complete solution only when Ω is a *ball.* In that case, problem (0.2) has a solution *if and only if* $\lambda \in (\frac{1}{4}\lambda_1, \lambda_1)$ (see Theorem 1.2).

This unexpected phenomenon can perhaps shed some light on Yamabe's problem which was solved by Th. Aubin [3] in high dimensions, namely $n \geqq 6$, in case the Weyl curvature tensor of the Riemannian metric is not identically zero. (In case it is identically zero, and the manifold has finite Poincaré group, the problem is also solved in [3].)

Our approach for proving the above results is the following. The solutions of (0.2) correspond to nontrivial critical points of the functional

$$\Phi(u)=\frac{1}{2}\int |\nabla u|^2-\frac{1}{p+1}\int |u|^{p+1}-\frac{1}{2}\lambda\int u^2.$$

Another viewpoint—which we shall use—is to look for critical points of the functional $\int |\nabla u|^2-\lambda\int u^2$ on the sphere $\|u\|_{p+1}=1$. Such a critical point u satisfies the equation

$$-\Delta u-\lambda u=\mu u^p,$$

where μ is a Lagrange multiplier. After "stretching" the Lagrange multiplier we obtain a solution of (0.2). We prove indeed that for suitable λ's we have:

(0.3) $$\inf_{\substack{u\in H_0^1\\ \|u\|_{p+1}=1}}\left\{\int |\nabla u|^2-\lambda\int u^2\right\} \text{ is achieved.}$$

The major difficulty in proving (0.3) stems from the fact that the function $u\mapsto\|u\|_{p+1}$ is *not* continuous under weak convergence in $H_0^1(\Omega)$. The decisive device in order to overcome this lack of compactness is to establish that for suitable λ's we have

(0.4) $$\inf_{\substack{u\in H_0^1\\ \|u\|_{p+1}=1}}\left\{\int |\nabla u|^2-\lambda\int u^2\right\}<\inf_{\substack{u\in H_0^1\\ \|u\|_{p+1}=1}}\int |\nabla u|^2\equiv S,$$

where S corresponds to the best constant for the Sobolev embedding $H_0^1(\Omega)\subset L^{p+1}(\Omega)$.

Our arguments are inspired by the work [3] of Aubin. The main point of the proof consists in evaluating the ratio

$$Q_\lambda(u)=\frac{\|\nabla u\|_2^2-\lambda\|u\|_2^2}{\|u\|_{p+1}^2}$$

for

(0.5) $$u(x)=\frac{\varphi(x)}{(\varepsilon+|x|^2)^{(n-2)/2}},\qquad \varepsilon>0,$$

where φ is a cut-off function. The functions $(\varepsilon+|x|^2)^{-(n-2)/2}$ play a natural role because they are extremal functions for the Sobolev inequality in $\mathbb{R}^n$. This approach has served as a source of inspiration in [5] where a similar method is used; in [5] it is not the Sobolev inequality but a certain isoperimetric inequality that plays the key role.

Finally, for the nonexistence part of Theorem 1.2 (i.e., $\lambda \leqq \frac{1}{4}\lambda_1$) we use an argument "à la Pohozaev" with more complicated multipliers.

In Section 2, we turn to the general form of problem (0.1). Once more there is a difference between the cases $n=3$ and $n \geqq 4$. We summarize our result on the following simple example:

$$
\begin{aligned}
-\Delta u &= u^p + \mu u^q \quad &&\text{on} \quad \Omega,\\
u &> 0 &&\text{on} \quad \Omega,\\
u &= 0 &&\text{on} \quad \partial\Omega,
\end{aligned}
\tag{0.6}
$$

where $p=(n+2)/(n-2)$, $1<q<p$, and $\mu>0$ is a constant. When $n \geqq 4$, problem (0.6) has a solution for every $\mu>0$. When $n=3$ $(p=5)$, problem (0.6) is again much more delicate:

(a) if $3<q<5$, problem (0.6) has a solution for *every* $\mu>0$;

(b) if $1<q\leqq 3$, it is only for *large values* of μ that (0.6) possesses a solution.

The proofs involve a combination of various ingredients. We start with a geometrical result which is an expression of the Ambrosetti–Rabinowitz [1] mountain pass theorem without the (PS) condition:

THEOREM 2.2. *Let Φ be a C^1 function on a Banach space E. Suppose*

(0.7) *there exists a neighborhood U of 0 in E and a constant ρ such that $\Phi(u) \geqq \rho$ for every u in the boundary of U,*

$$\Phi(0)<\rho \text{ and } \Phi(v)<\rho \text{ for some } v \notin U. \tag{0.8}$$

Set

$$c=\inf_{P\in\mathcal{P}} \max_{w\in P} \Phi(w) \geqq \rho,$$

where $\mathcal{P}$ denotes the class of paths joining 0 to v.

Conclusion:

there is a sequence (u_j) in E such that
$\Phi(u_j)\to c$ and $\Phi'(u_j)\to 0$ in E^*.

When applying Theorem 2.2 to (0.6) we choose $E=H_0^1(\Omega)$ and

$$\Phi(u)=\tfrac{1}{2}\int|\nabla u|^2-\frac{1}{p+1}\int|u|^{p+1}-\frac{\mu}{q+1}\int|u|^{q+1}.$$

Condition (0.7) is clearly satisfied (U being a small ball). The major difficulty lies in using the conclusion of Theorem 2.2. For this purpose we prove (see Theorem 2.1) that if

$$c < \frac{1}{n} S^{n/2}, \tag{0.9}$$

then one can pass to the limit in the sequence (u_j) and obtain a nontrivial critical point of Φ. Thus we are left with the question: can one find a v such that the corresponding c satisfies (0.9)?[4] This last step is rather technical; it is achieved by choosing some special v's, for example of the form (0.5). We believe that this method can be useful in solving other problems where one is in a borderline situation for the (PS) condition—so that the standard approach fails.

Our thanks to E. Lieb for his kind help (see Lemma 1.2), to F. Browder and P. Rabinowitz for stimulating discussions, and to O. Bristeau (at INRIA) for suggestive numerical computations at stages where we could not guess the answer.

1. Existence of Positive Solutions for $-\Delta u = u^p + \lambda u$ on Ω, $u = 0$ on $\partial\Omega$ with $p = (n+2)/(n-2)$

Let $\Omega \subset \mathbb{R}^n$, $n \geqq 3$, be a bounded domain. We are concerned with the problem of existence of a function u satisfying:

$$\begin{aligned} -\Delta u &= u^p + \lambda u \quad &&\text{on} \quad \Omega, \\ u &> 0 &&\text{on} \quad \Omega, \\ u &= 0 &&\text{on} \quad \partial\Omega, \end{aligned} \tag{1.1}$$

where $p = (n+2)/(n-2)$ and λ is a real constant. As we have indicated, the cases $n = 3$ and $n \geqq 4$ are different and will be treated separately.

In subsections 1.1 and 1.2 we consider the cases $n \geqq 4$ and $n = 3$, respectively.

In subsection 1.3 we have collected a number of additional properties and open problems. We denote by λ_1 the first eigenvalue of $-\Delta$ with zero Dirichlet condition on Ω.

1.1. The case $n \geqq 4$.

Our main result is the following:

THEOREM 1.1. *Assume $n \geqq 4$. Then for every $\lambda \in (0, \lambda_1)$ there exists a solution of* (1.1).

Remark 1.1. There is *no* solution of (1.1) when $\lambda \geqq \lambda_1$. Indeed, let φ_1 be the eigenfunction of $-\Delta$ corresponding to λ_1 with $\varphi_1 > 0$ on Ω. Suppose u is a

[4] Note that c depends on v.

solution of (1.1). We have

$$-\int (\Delta u)\varphi_1 = \lambda_1 \int u\varphi_1 = \int u^p \varphi_1 + \lambda \int u\varphi_1 > \lambda \int u\varphi_1$$

and thus $\lambda < \lambda_1$.

Remark 1.2. There is *no* solution of (1.1) when $\lambda \leqq 0$ and Ω is a (smooth) *starshaped* domain. This follows from *Pohozaev's identity* (see Pohozaev [24]) which we now recall. Suppose u is a (smooth) function satisfying

$$\begin{aligned} -\Delta u &= g(u) \quad \text{on} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.2}$$

where g is a continuous function on $\mathbb{R}$. Then we have

$$(1-\tfrac{1}{2}n)\int_\Omega g(u)\cdot u + n\int_\Omega G(u) = \tfrac{1}{2}\int_{\partial\Omega} (x\cdot\nu)\left(\frac{\partial u}{\partial \nu}\right)^2, \tag{1.3}$$

where

$$G(u) = \int_0^u g(t)\, dt$$

and ν denotes the outward normal to $\partial\Omega$. In particular, when $g(u) = u^p + \lambda u$ we deduce from (1.3) that

$$\lambda \int_\Omega u^2 = \tfrac{1}{2}\int_{\partial\Omega} (x\cdot\nu)\left(\frac{\partial u}{\partial \nu}\right)^2. \tag{1.4}$$

If Ω is starshaped about the origin we have $(x\cdot\nu)>0$ a.e. on $\partial\Omega$. When $\lambda<0$ it follows immediately from (1.4) that $u\equiv 0$. When $\lambda = 0$ we deduce from (1.4) that $\partial u/\partial\nu = 0$ on $\partial\Omega$ and then by (1.1) we have

$$0 = -\int_\Omega \Delta u = \int_\Omega u^p;$$

thus $u \equiv 0$.

The situation can be quite different when Ω is *not starshaped.* For example if Ω is an annulus, there exists a radial solution of (1.1) for every $\lambda \in (-\infty, \lambda_1)$; this fact was first pointed out by Kazdan and Warner [16]; see also subsection 1.3 (point 3) below.

Set

$$S_\lambda = \inf_{\substack{u\in H_0^1 \\ \|u\|_{p+1}=1}} \{\|\nabla u\|_2^2 - \lambda\|u\|_2^2\} \quad \text{with} \quad \lambda\in\mathbb{R}, \tag{1.5}$$

so that

$$S_0 = S = \inf_{\substack{u \in H_0^1 \\ \|u\|_{p+1}=1}} \|\nabla u\|_2^2 \tag{1.6}$$

corresponds to the best constant for the Sobolev embedding $H_0^1(\Omega) \subset L^{p+1}(\Omega)$, $p+1 = 2n/(n-2)$. We start with some remarks concerning the best Sobolev constant S:

(a) S is independent of Ω and depends only on n. This follows from the fact that the ratio $\|\nabla u\|_2/\|u\|_{p+1}$ with $p+1 = 2n/(n-2)$ is invariant under scaling; in other words, the ratio $\|\nabla u_k\|_2/\|u_k\|_{p+1}$ is independent of k, where $u_k(x) = u(kx)$.

(b) The infimum in (1.6) is *never* achieved when Ω is a *bounded* domain. Indeed, suppose that S were attained by some function $u \in H_0^1(\Omega)$. We may assume that $u \geqq 0$ on Ω (otherwise replace u by $|u|$). Fix a ball $\tilde{\Omega} \supset \Omega$ and set

$$\tilde{u} = \begin{cases} u & \text{on} \quad \Omega, \\ 0 & \text{on} \quad \tilde{\Omega}\backslash\Omega. \end{cases}$$

Thus S is also achieved on $\tilde{\Omega}$ by $\tilde{u}$ and $\tilde{u}$ satisfies $-\Delta\tilde{u} = \mu\tilde{u}^p$ for some constant $\mu > 0$; this contradicts Pohozaev's result.

(c) When $\Omega = \mathbb{R}^n$, the infimum in (1.6) is achieved by the function

$$U(x) = C(1+|x|^2)^{-(n-2)/2} \tag{1.7}$$

or (after scaling) by any of the functions

$$U_\varepsilon(x) = C_\varepsilon(\varepsilon + |x|^2)^{-(n-2)/2}, \qquad \varepsilon > 0, \tag{1.8}$$

where C and C_ε are normalization constants; see Th. Aubin [2], G. Talenti [28] (both are based on some earlier work of G. A. Bliss [4]) and also E. Lieb [19].

Our first lemma plays a crucial rôle in the proof of Theorem 1.1; it is an adaptation of an original argument due to Th. Aubin [3] in the context of Yamabe's conjecture.

LEMMA 1.1. *We have*

$$S_\lambda < S \quad \textit{for all} \quad \lambda > 0. \tag{1.9}$$

Proof: Without loss of generality we may assume that $0 \in \Omega$. We shall estimate the ratio

$$Q_\lambda(u) = \frac{\|\nabla u\|_2^2 - \lambda\|u\|_2^2}{\|u\|_{p+1}^2}$$

with

$$u(x) = u_\varepsilon(x) = \frac{\varphi(x)}{(\varepsilon + |x|^2)^{(n-2)/2}}, \qquad \varepsilon > 0, \tag{1.10}$$

where $\varphi \in \mathcal{D}_+(\Omega)$ is a *fixed* function such that $\varphi(x) \equiv 1$ for x in some neighborhood of 0. We claim that, as $\varepsilon \to 0$, we have

$$\|\nabla u_\varepsilon\|_2^2 = \frac{K_1}{\varepsilon^{(n-2)/2}} + O(1), \tag{1.11}$$

$$\|u_\varepsilon\|_{p+1}^2 = \frac{K_2}{\varepsilon^{(n-2)/2}} + O(\varepsilon), \tag{1.12}$$

$$\|u_\varepsilon\|_2^2 = \begin{cases} \dfrac{K_3}{\varepsilon^{(n-4)/2}} + O(1) & \text{if } n \geqq 5, \\ K_3|\log \varepsilon| + O(1) & \text{if } n = 4, \end{cases} \tag{1.13}$$

where K_1, K_2 and K_3 denote positive constants which depend only on n and such that $K_1/K_2 = S$.

VERIFICATION OF (1.11): We have

$$\nabla u_\varepsilon(x) = \frac{\nabla\varphi(x)}{(\varepsilon + |x|^2)^{(n-2)/2}} - \frac{(n-2)\varphi(x)x}{(\varepsilon + |x|^2)^{n/2}}.$$

Since $\varphi \equiv 1$ near 0, it follows that

$$\begin{aligned}\int_\Omega |\nabla u_\varepsilon|^2 &= (n-2)^2 \int_\Omega \frac{|x|^2\,dx}{(\varepsilon + |x|^2)^n} + O(1) \\ &= (n-2)^2 \int_{\mathbb{R}^n} \frac{|x|^2\,dx}{(\varepsilon + |x|^2)^n} + O(1) \\ &= \frac{K_1}{\varepsilon^{(n-2)/2}} + O(1),\end{aligned}$$

where

$$K_1 = (n-2)^2 \int_{\mathbb{R}^n} \frac{|x|^2\,dx}{(1+|x|^2)^n} = \|\nabla U\|_2^2.$$

VERIFICATION OF (1.12): We have

$$\begin{aligned}\int_\Omega |u_\varepsilon|^{p+1} &= \int_\Omega \frac{\varphi^{p+1}(x)\,dx}{(\varepsilon + |x|^2)^n} = \int_\Omega \frac{[\varphi^{p+1}(x) - 1]\,dx}{(\varepsilon + |x|^2)^n} + \int_\Omega \frac{dx}{(\varepsilon + |x|^2)^n} \\ &= O(1) + \int_{\mathbb{R}^n} \frac{dx}{(\varepsilon + |x|^2)^n} = \frac{K_2'}{\varepsilon^{n/2}} + O(1),\end{aligned}$$

where

$$K_2' = \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^n} = \|U\|_{p+1}^{p+1}.$$

Thus (1.12) follows with $K_2=\|U\|_{p+1}^2$, and $K_1/K_2=S$.

VERIFICATION OF (1.13): We have

$$\int_\Omega |u_\varepsilon|^2=\int_\Omega \frac{[\varphi^2(x)-1]\,dx}{(\varepsilon+|x|^2)^{n-2}}+\int_\Omega \frac{dx}{(\varepsilon+|x|^2)^{n-2}}=O(1)+\int_\Omega \frac{dx}{(\varepsilon+|x|^2)^{n-2}}.$$

When $n\geqq 5$, we have

$$\int_\Omega \frac{dx}{(\varepsilon+|x|^2)^{n-2}}=\int_{\mathbf{R}^n} \frac{dx}{(\varepsilon+|x|^2)^{n-2}}+O(1)$$

and (1.13) follows with

$$K_3=\int_{\mathbf{R}^n}\frac{dx}{(1+|x|^2)^{n-2}}.$$

When $n=4$, we have, for some constants R_1 and R_2,

$$\int_{|x|\leqq R_1}\frac{dx}{(\varepsilon+|x|^2)^2}\leqq\int_\Omega \frac{dx}{(\varepsilon+|x|^2)^2}\leqq\int_{|x|\leqq R_2}\frac{dx}{(\varepsilon+|x|^2)^2}$$

and

$$\int_{|x|\leqq R}\frac{dx}{(\varepsilon+|x|^2)^2}=\omega\int_0^R\frac{r^3\,dr}{(\varepsilon+r^2)^2}=\tfrac{1}{2}\omega\,|\log\varepsilon|+O(1),$$

where ω is the area of S^3; thus (1.13) follows with $K_3=\frac{1}{2}\omega$. Combining (1.11), (1.12) and (1.13), we obtain

$$Q_\lambda(u_\varepsilon)=\begin{cases} S+O(\varepsilon^{(n-2)/2})-\lambda\dfrac{K_3}{K_2}\varepsilon & \text{if } n\geqq 5,\\[2ex] S+O(\varepsilon)-\lambda\dfrac{K_3}{K_2}\varepsilon\,|\log\varepsilon| & \text{if } n=4.\end{cases}$$

In all cases we deduce that $Q_\lambda(u_\varepsilon)<S$ provided $\varepsilon>0$ is small enough.

LEMMA 1.2. (E. Lieb) *If $S_\lambda<S$, the infimum in* (1.5) *is achieved.*

Proof: Let $(u_j)\subset H_0^1$ be a minimizing sequence for (1.5), that is,

$$\|u_j\|_{p+1}=1, \tag{1.14}$$

$$\|\nabla u_j\|_2^2-\lambda\|u_j\|_2^2=S_\lambda+o(1)\quad\text{as}\quad j\to\infty. \tag{1.15}$$

Since u_j is bounded in H_0^1 we may extract a subsequence—still denoted by u_j—such that

$$\begin{aligned} u_j &\rightharpoonup u \quad \text{weakly in} \quad H_0^1,\\ u_j &\to u \quad \text{strongly in} \quad L^2,\\ u_j &\to u \quad \text{a.e. on} \quad \Omega,\end{aligned}$$

with $\|u\|_{p+1} \leqq 1$. Set $v_j = u_j - u$, so that

$$v_j \rightharpoonup 0 \quad \text{weakly in} \quad H_0^1$$

$$v_j \to 0 \quad \text{a.e. on} \quad \Omega.$$

By (1.6) and (1.14) we have $\|\nabla u_j\|_2 \geqq S$. From (1.15) it follows that $\lambda \|u\|_2^2 \geqq S - S_\lambda > \infty$ and therefore $u \not\equiv 0$. Using (1.15) we obtain

$$\|\nabla u\|_2^2 + \|\nabla v_j\|_2^2 - \lambda \|u\|_2^2 = S_\lambda + o(1) \tag{1.16}$$

since $v_j \rightharpoonup 0$ weakly in H_0^1. On the other hand, we deduce from a result of Brezis and Lieb [8] that

$$\|u + v_j\|_{p+1}^{p+1} = \|u\|_{p+1}^{p+1} + \|v_j\|_{p+1}^{p+1} + o(1)$$

(which holds since v_j is bounded in L^{p+1} and $v_j \to 0$ a.e.). Thus (by (1.14)) we have

$$1 = \|u\|_{p+1}^{p+1} + \|v_j\|_{p+1}^{p+1} + o(1)$$

and therefore

$$1 \leqq \|u\|_{p+1}^2 + \|v_j\|_{p+1}^2 + o(1)$$

which leads to

$$1 \leqq \|u\|_{p+1}^2 + \frac{1}{S} \|\nabla v_j\|_2^2 + o(1). \tag{1.17}$$

We claim that

$$\|\nabla u\|_2^2 - \lambda \|u\|_2^2 \leqq S_\lambda \|u\|_{p+1}^2; \tag{1.18}$$

this will conclude the proof of Lemma 1.2 since $u \not\equiv 0$.

We distinguish two cases:

(a) $S_\lambda > 0$ (i.e., $0 < \lambda < \lambda_1$),

(b) $S_\lambda \leqq 0$ (i.e., $\lambda \geqq \lambda_1$).

In case (a) we deduce from (1.17) that

$$S_\lambda \leqq S_\lambda \|u\|_{p+1}^2 + (S_\lambda / S) \|\nabla v_j\|_2^2 + o(1). \tag{1.19}$$

Combining (1.16) and (1.19) we obtain (1.18).

In case (b) we have $S_\lambda \leqq S_\lambda \|u\|_{p+1}^2$ since $\|u\|_{p+1} \leqq 1$. We deduce, again, (1.18) from (1.16).

F. Browder has pointed out that this argument proves more: in fact, $v_j \to 0$ *strongly* in H_0^1; in other words, every minimizing sequence for (1.5) is relatively compact in H_0^1 for the strong H_0^1 topology.

Proof of Theorem 1.1 concluded: Let $u \in H_0^1$ be given by Lemma 1.2, that is,

$$\|u\|_{p+1} = 1 \quad \text{and} \quad \|\nabla u\|_2^2 - \lambda \|u\|_2^2 = S_\lambda.$$

We may as well assume that $u \geqq 0$ on Ω (otherwise we replace u by $|u|$). Since u is a minimizer for (1.5) we obtain a Lagrange multiplier $\mu \in \mathbb{R}$ such that

$$-\Delta u - \lambda u = \mu u^p \quad \text{on} \quad \Omega.$$

In fact, $\mu = S_\lambda$, and $S_\lambda > 0$ since $\lambda < \lambda_1$. It follows that ku satisfies (1.1) for some appropriate constant $k > 0$ $(k = S_\lambda^{1/(p-1)})$; note that $u > 0$ on Ω by the strong maximum principle.

Remark 1.3. Our first proof of Theorem 1.1 did not involve Lemma 1.2. Instead, we considered, as in the works of N. Trudinger [30] and Th. Aubin [3]:

$$\mu_q = \inf_{\substack{u \in H_0^1 \\ \|u\|_{q+1}=1}} \{\|\nabla u\|_2^2 - \lambda \|u\|_2^2\} \quad \text{for} \quad q < p. \tag{1.20}$$

It is easy to check that $\lim_{q \to p} \mu_q = S_\lambda$. Moreover since the embedding $H_0^1 \subset L^{q+1}$ is compact, the infimum in (1.20) is achieved by some $u_q \in H_0^1$ such that $u_q \geqq 0$ on Ω, $\|u_q\|_{q+1} = 1$ and

$$-\Delta u_q - \lambda u_q = \mu_q u_q^q. \tag{1.21}$$

It follows that

$$S\|u_q\|_{p+1}^2 - \lambda \|u_q\|_2^2 \leqq \|\nabla u_q\|_2^2 - \lambda \|u_q\|_2^2 = \mu_q. \tag{1.22}$$

As $q \to p$ (through a subsequence), $u_q \rightharpoonup u$ weakly in H_0^1. Passing to the limit in (1.22) we obtain

$$S - \lambda \|u\|_2^2 \leqq S_\lambda$$

and thus (by Lemma 1.1), $u \not\equiv 0$. Finally, we deduce from (1.21) that u satisfies

$$-\Delta u - \lambda u = S_\lambda u^p.$$

Stretching S_λ, as above, we obtain a solution of (1.1).

1.2. The case $n = 3$. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. We are concerned with the problem of existence of a function u satisfying

$$\begin{aligned} -\Delta u &= u^5 + \lambda u && \text{on} \quad \Omega, \\ u &> 0 && \text{on} \quad \Omega, \\ u &= 0 && \text{on} \quad \partial\Omega, \end{aligned} \tag{1.23}$$

where λ is a real constant. This problem turns out to be rather delicate and we have a complete solution only when Ω is a *ball* (see subsection 1.3 for more general domains). Our main result is the following:

THEOREM 1.2. *Assume Ω is a ball. There exists a solution of* (1.23) *if and only if* $\lambda \in (\frac{1}{4}\lambda_1, \lambda_1)$.

For simplicity we take

$$\Omega = \{x \in \mathbb{R}^3;\ |x| < 1\}$$

so that $\lambda_1 = \pi^2$ (the corresponding eigenfunction is $|x|^{-1} \sin(\pi|x|)$.

We already know that (1.23) has no solution for $\lambda \geqq \lambda_1$ and for $\lambda \leqq 0$ (see subsection 1.1). As in subsection 1.1 we set

$$S_\lambda = \inf_{\substack{u \in H_0^1 \\ \|u\|_6 = 1}} \{\|\nabla u\|_2^2 - \lambda \|u\|_2^2\} \quad \text{with} \quad \lambda \in \mathbb{R}, \tag{1.24}$$

and $S = S_0$.

The counterpart of Lemma 1.1 is

LEMMA 1.3. *We have*

$$S_\lambda < S \quad \textit{for all} \quad \lambda > \tfrac{1}{4}\lambda_1. \tag{1.25}$$

Proof: We shall estimate the ratio

$$Q_\lambda(u) = \frac{\|\nabla u\|_2^2 - \lambda \|u\|_2^2}{\|u\|_6^2}$$

with

$$u(x) = u_\varepsilon(r) = \frac{\varphi(r)}{(\varepsilon + r^2)^{1/2}}, \qquad r = |x|,\ \varepsilon > 0, \tag{1.26}$$

where φ is a *fixed* smooth function such that $\varphi(0) = 1$, $\varphi'(0) = 0$ and $\varphi(1) = 0$. We claim that, as $\varepsilon \to 0$, we have

$$\|\nabla u_\varepsilon\|_2^2 = \frac{K_1}{\varepsilon^{1/2}} + \omega \int_0^1 |\varphi'(r)|^2\, dr + O(\varepsilon^{1/2}), \tag{1.27}$$

$$\|u_\varepsilon\|_6^2 = \frac{K_2}{\varepsilon^{1/2}} + O(\varepsilon^{1/2}), \tag{1.28}$$

$$\|u_\varepsilon\|_2^2 = \omega \int_0^1 \varphi^2(r)\, dr + O(\varepsilon^{1/2}), \tag{1.29}$$

where K_1 and K_2 are positive constants such that $K_1/K_2 = S$ and ω is the area of S^2.

VERIFICATION OF (1.27): We have

$$u_\varepsilon'(r) = \frac{\varphi'(r)}{(\varepsilon + r^2)^{1/2}} - \frac{r\varphi(r)}{(\varepsilon + r^2)^{3/2}}$$

and thus

$$\|\nabla u_\varepsilon\|_2^2 = \omega \int_0^1 \left[\frac{|\varphi'(r)|^2}{(\varepsilon+r^2)} - \frac{2r\varphi(r)\varphi'(r)}{(\varepsilon+r^2)^2} + \frac{r^2\varphi^2(r)}{(\varepsilon+r^2)^3}\right] r^2\,dr.$$

Integrating by parts we find

$$-2\int_0^1 \frac{\varphi(r)\varphi'(r)r^3}{(\varepsilon+r^2)^2}\,dr = \int_0^1 \varphi^2(r)\left[\frac{3r^2}{(\varepsilon+r^2)^2} - \frac{4r^4}{(\varepsilon+r^2)^3}\right] dr,$$

and therefore

$$\|\nabla u_\varepsilon\|_2^2 = \omega \int_0^1 \frac{|\varphi'(r)|^2}{(\varepsilon+r^2)} r^2\,dr + 3\omega\varepsilon \int_0^1 \frac{\varphi^2(r)r^2}{(\varepsilon+r^2)^3}\,dr. \tag{1.30}$$

Using the fact that $\varphi(0)=1$ and $\varphi'(0)=0$ we obtain

$$\int_0^1 \frac{|\varphi'(r)|^2 r^2}{(\varepsilon+r^2)}\,dr = \int_0^1 |\varphi'(r)|^2\,dr + O(\varepsilon), \tag{1.31}$$

$$\int_0^1 \frac{\varphi^2(r)r^2}{(\varepsilon+r^2)^3}\,dr = \int_0^1 \frac{r^2}{(\varepsilon+r^2)^3}\,dr + O(\varepsilon^{-1/2}). \tag{1.32}$$

Also, we have

$$\int_0^1 \frac{r^2}{(\varepsilon+r^2)^3}\,dr = \frac{1}{\varepsilon^{3/2}} \int_0^{\varepsilon^{-1/2}} \frac{s^2}{(1+s^2)^3}\,ds = \frac{1}{\varepsilon^{3/2}} \int_0^\infty \frac{s^2}{(1+s^2)^3}\,ds + O(1). \tag{1.33}$$

Combining (1.30)–(1.32) and (1.33) we obtain (1.27) with

$$K_1 = 3\omega \int_0^\infty \frac{s^2}{(1+s^2)^3}\,ds.$$

Finally we note that $K_1 = \int_{\mathbf{R}^3} |\nabla U|^2\,dx$, where $U(x) = 1/(1+|x|^2)^{1/2}$; here we use the fact that

$$\int_0^\infty \frac{s^2}{(1+s^2)^3}\,ds = \tfrac{1}{16}\pi \quad \text{and} \quad \int_0^\infty \frac{s^4}{(1+s^2)^3}\,ds = \tfrac{3}{16}\pi.$$

VERIFICATION OF (1.28): We have

$$\|u_\varepsilon\|_6^6 = \omega \int_0^1 \frac{\varphi^6(r)r^2}{(\varepsilon+r^2)^3}\,dr = \omega \int_0^1 \frac{(\varphi^6(r)-1)r^2}{(\varepsilon+r^2)^3}\,dr + \omega \int_0^1 \frac{r^2}{(\varepsilon+r^2)^3}\,dr$$
$$= I_1 + I_2.$$

Since $\varphi(0)=1$ and $\varphi'(0)=0$ we obtain

$$|I_1| \le C \int_0^1 \frac{r^4}{(\varepsilon+r^2)^3}\,dr = O(\varepsilon^{-1/2}).$$

Next we have

$$I_2=\frac{\omega}{\varepsilon^{3/2}}\int_0^{\varepsilon^{-1/2}}\frac{s^2}{(1+s^2)^3}\,ds=\frac{\omega}{\varepsilon^{3/2}}\int_0^{\infty}\frac{s^2}{(1+s^2)^3}\,ds+O(1).$$

Therefore we find

$$\|u_\varepsilon\|_6^6=\frac{1}{\varepsilon^{3/2}}\left[\omega\int_0^{\infty}\frac{s^2}{(1+s^2)^3}\,ds+O(\varepsilon)\right]$$

and (1.28) follows with

$$K_2=\left[\omega\int_0^{\infty}\frac{s^2}{(1+s^2)^3}\,ds\right]^{1/3}=\|U\|_6^2.$$

VERIFICATION OF (1.29): We have

$$\|u_\varepsilon\|_2^2=\omega\int_0^1\frac{\varphi^2(r)r^2}{(\varepsilon+r^2)}\,dr=\omega\int_0^1\varphi^2(r)\,dr+O(\varepsilon^{1/2}).$$

Combining (1.27), (1.28) and (1.29) we obtain

$$(1.34)\qquad Q_\lambda(u_\varepsilon)=S+\varepsilon^{1/2}\frac{\omega}{K_2}\left[\int_0^1|\varphi'(r)|^2\,dr-\lambda\int_0^1\varphi^2(r)\,dr\right]+O(\varepsilon).$$

Choosing $\varphi(r)=\cos(\frac{1}{2}\pi r)$ we have

$$\int_0^1|\varphi'(r)|^2\,dr=\tfrac{1}{4}\pi^2\int_0^1\varphi^2(r)\,dr$$

and thus

$$Q_\lambda(u_\varepsilon)=S+(\tfrac{1}{4}\pi^2-\lambda)C\varepsilon^{1/2}+O(\varepsilon)$$

for some positive constant C. The conclusion of Lemma 1.3 follows by choosing $\varepsilon>0$ small enough.

The next Lemma is a crucial step in the proof of Theorem 1.2:

LEMMA 1.4. *There is no solution of* (1.23) *for* $\lambda\leqq\frac{1}{4}\lambda_1$.

Proof: Suppose u is a solution of (1.23); by a result of Gidas–Ni–Nirenberg [13] we know that u must be spherically symmetric. We write $u(x)=u(r)$, where $r=|x|$, and thus u satisfies

$$(1.35)\qquad -u''-\frac{2}{r}u'=u^5+\lambda u\quad\text{on}\quad(0,1),$$

$$(1.36)\qquad u'(0)=u(1)=0.$$

We claim that

$$(1.37)\qquad \int_0^1 u^2(\lambda\psi'+\tfrac{1}{4}\psi''')r^2\,dr=\tfrac{2}{3}\int_0^1 u^6(r\psi-r^2\psi')\,dr+\tfrac{1}{2}|u'(1)|^2\psi(1)$$

for every smooth function ψ such that $\psi(0)=0$.[5] Indeed, we first multiply (1.35) by $r^2\psi u'$ and obtain

$$(1.38)\qquad \begin{aligned}&\int_0^1 |u'|^2(\tfrac{1}{2}r^2\psi'-r\psi)\,dr-\tfrac{1}{2}|u'(1)|^2\psi(1)\\ &\quad=-\tfrac{1}{6}\int_0^1 u^6(2r\psi+r^2\psi')\,dr-\tfrac{1}{2}\lambda\int_0^1 u^2(2r\psi+r^2\psi')\,dr.\end{aligned}$$

Next we multiply (1.35) by $(\frac{1}{2}r^2\psi'-r\psi)u$ and obtain

$$(1.39)\qquad \begin{aligned}&\int_0^1 |u'|^2(\tfrac{1}{2}r^2\psi'-r\psi)\,dr-\tfrac{1}{4}\int_0^1 u^2r^2\psi'''\,dr\\ &\quad=\int_0^1 u^6(\tfrac{1}{2}r^2\psi'-r\psi)\,dr+\lambda\int_0^1 u^2(\tfrac{1}{2}r^2\psi'-r\psi)\,dr.\end{aligned}$$

Combining (1.38) and (1.39) we obtain (1.37). We already know that there is no solution of (1.23) for $\lambda\leqq 0$; thus we may assume that $0<\lambda\leqq\frac{1}{4}\pi^2$. In (1.37) we choose $\psi(r)=\sin((4\lambda)^{1/2}r)$ so that $\psi(1)\geqq 0$,

$$\lambda\psi'+\tfrac{1}{4}\psi'''=0,$$

and

$$r\psi-r^2\psi'=r\sin((4\lambda)^{1/2}r)-r^2(4\lambda)^{1/2}\cos((4\lambda)^{1/2}r)>0\quad\text{on}\quad(0,1]$$

(since $\sin\theta-\theta\cos\theta>0$ for all $\theta\in(0,\pi]$) and we obtain a contradiction.

Proof of Theorem 1.2 concluded: If $\lambda>\frac{1}{4}\lambda_1$ we know that $S_\lambda<S$ (see Lemma 1.3). We may proceed exactly as in the proof of Theorem 1.1 (Lemma 1.2) and conclude that the infimum in (1.24) is achieved. Thus we obtain some $u\in H_0^1$ with $u\geqq 0$ on Ω, $\|u\|_6=1$ and

$$-\Delta u-\lambda u=S_\lambda u^5.$$

If, in addition, $\lambda<\lambda_1$, then $S_\lambda>0$ and after stretching, we obtain a solution of (1.23).

1.3. Additional properties, miscellaneous remarks and open problems.

(1). REGULARITY OF SOLUTIONS. The solution u of (1.1) given by Theorem 1.1 (respectively Theorem 1.2) lies in $H_0^1(\Omega)$. In fact, u belongs to

[5] Note that Pohozaev's identity corresponds to the case where $\psi(r)=r$.

$L^\infty(\Omega)$. This is proved by Trudinger [30] for Yamabe's problem (on a manifold without boundary) but the same argument applies here. Alternatively, one could also invoke the following Lemma which is essentially contained in Brezis-Kato [7]:

LEMMA 1.5. *Assume $u \in H_0^1(\Omega)$ satisfies*

$$-\Delta u = au \quad in \quad \mathcal{D}'(\Omega),$$

where $a(x) \in L^{n/2}(\Omega)$ and $n \geqq 3$. Then $u \in L^t(\Omega)$ for all $t < \infty$.

For our purpose we use Lemma 1.5 with $a = \lambda + u^{p-1} \in L^{n/2}$ (since $u \in L^{p+1}$).

Finally we note that $u \in C^\infty(\Omega)$ (since $u > 0$ in Ω) and, up to the boundary, u is as smooth as $\partial\Omega$ and p permit.

(2). THE CASE $p > (n+2)/(n-2)$ WITH $n \geqq 3$. It follows from general bifurcation theory—see e.g. Rabinowitz [25]—that for any $p > 1$ (even $p > (n+2)/(n-2)$) problem (1.1) possesses a *component* $\mathscr{C}$ of solutions (λ, u) which meets $(\lambda_1, 0)$ and which is unbounded in $\mathbb{R} \times L^\infty(\Omega)$. Theorem 1.1 *suggests* that, when $p = (n+2)/(n-2)$ and $n \geqq 4$, the projection of $\mathscr{C}$ on the λ-axis contains the interval $(0, \lambda_1)$ (with the obvious modification when $n = 3$ and $p = 5$).

On the other hand *when* $p > (n+2)/(n-2)$ and Ω *is starshaped,* problem (1.1) has *no* solution if $\lambda \leqq \lambda^*$, where λ^* is some *positive* constant which depends on Ω and p. This was pointed out by Rabinowitz [26] in the case $n = 3$ and $p = 7$, but the same argument works in the general case: suppose u satisfies (1.1); Pohozaev's identity leads to (assuming star-shapedness about the origin)

$$(1 - \tfrac{1}{2}n) \int_\Omega (u^{p+1} + \lambda u^2) + n \int_\Omega \left(\frac{u^{p+1}}{p+1} + \tfrac{1}{2}\lambda u^2\right) = \tfrac{1}{2} \int_{\partial\Omega} (x \cdot \nu)\left(\frac{\partial u}{\partial \nu}\right)^2 > 0$$

and thus we find

$$\left(-1 + \tfrac{1}{2}n - \frac{n}{p+1}\right) \int_\Omega u^{p+1} < \lambda \int_\Omega u^2. \tag{1.40}$$

We deduce from (1.1) and (1.40) that

$$\begin{aligned} \lambda_1 \int_\Omega u^2 \leqq \int_\Omega |\nabla u|^2 &= \int_\Omega u^{p+1} + \lambda \int_\Omega u^2 \\ &< \lambda \left(-1 + \tfrac{1}{2}n - \frac{n}{p+1}\right)^{-1} \int_\Omega u^2 + \lambda \int_\Omega u^2, \end{aligned}$$

that is,

$$\lambda > \lambda_1 \cdot \frac{n-2}{n} \cdot \frac{p - (n+2)/(n-2)}{p-1}.$$

(3). UNIQUENESS-NONUNIQUENESS When Ω is a *ball*, every solution of (1.1) is spherically symmetric (see [13]). Even in this case we do not know whether (1.1) has a unique solution. Uniqueness results for some semilinear elliptic equations in *all of* $\mathbb{R}^n$ have been obtained by Coffman [9], L. A. Peletier and J. Serrin [23], and K. McLeod and J. Serrin [22].[6] On the other hand, if Ω is *an annulus*, say $\Omega=\{x\in\mathbb{R}^n;\,1<|x|<2\}$ with $n\geqq 4$, then (1.1) admits both radial and nonradial solutions for all $\lambda>0$ sufficiently small.[7] Indeed, set

$$\Sigma_\lambda=\inf_{\substack{u\in H_r\\ \|u\|_{p+1}=1}}\{\|\nabla u\|_2^2-\lambda\|u\|_2^2\}, \tag{1.41}$$

where $H_r=\{u\in H_0^1;\ u \text{ is radial}\}$. Since the injection $H_r\subset L^{p+1}$ is *compact*, the infimum in (1.41) is achieved (for any $\lambda\in\mathbb{R}$) by some $u_\lambda\in H_r$ such that

$$u_\lambda\geqq 0 \quad\text{on}\quad \Omega,\qquad \|u_\lambda\|_{p+1}=1$$

and $-\Delta u_\lambda-\lambda u_\lambda=\Sigma_\lambda u_\lambda^p$ on Ω. If $\lambda<\lambda_1$, then $\Sigma_\lambda>0$ and, after stretching Σ_λ, we obtain a solution of (1.1). Next we consider S_λ defined by (1.5). It is easy to check that the functions $\lambda\mapsto\Sigma_\lambda$ and $\lambda\mapsto S_\lambda$ are continuous (even Lipschitz continuous). We have $S=S_0<\Sigma_0$ (otherwise the best Sobolev constant would be achieved—which is impossible; see subsection 1.1). Thus for $\lambda>0$ sufficiently small, $S_\lambda<\Sigma_\lambda$, and the infimum in (1.5) is achieved (see Lemma 1.2) by some *nonradial* function; in this way we obtain a nonradial solution of (1.1). We do not know whether the nonradial solutions occur by secondary bifurcation from the branch of radial solutions

A similar argument shows that the problem

$$\begin{aligned} -\Delta u&=u^q \quad\text{on}\quad \text{the annulus}\quad \Omega,\\ u&>0 \quad\text{on}\quad \Omega,\\ u&=0 \quad\text{on}\quad \partial\Omega, \end{aligned} \tag{1.42}$$

admits both radial and nonradial solutions for all $q<(n+2)/(n-2)$ sufficiently close to $(n+2)/(n-2)$.

(4). EQUATIONS WITH VARIABLE COEFFICIENTS. Let $\Omega\subset\mathbb{R}^n$, $n\geqq 4$, be a bounded domain. Assume $a(x)\in L^\infty(\Omega)$ is given such that

$$a(x)\geqq\delta \quad\text{on some open subset of}\quad \Omega, \tag{1.43}$$

[6] Other uniqueness results have been obtained by W. M. Ni: *Uniqueness of solutions of nonlinear Dirichlet problems*, J. Diff. Eqns., to appear, and in a paper by Ni and R. Nussbaum (in preparation).

[7] Of course this fact does not contradict the result on spherical symmetry of [13] which holds only on balls. Nonradial solutions for some semilinear equations on the annulus have also been investigated by D. Schaeffer [27] and C. Coffman [10].

$$\int |\nabla v|^2 - av^2 \geqq \delta \int v^2 \quad \text{for all} \quad v \in H_0^1, \tag{1.44}$$

and for some constant $\delta > 0$. Then there exists a solution of the following problem:

$$\begin{aligned} -\Delta u &= u^p + a(x)u \quad &&\text{on} \quad \Omega \quad \text{with} \quad p = (n+2)/(n-2), \\ u &> 0 &&\text{on} \quad \Omega, \\ u &= 0 &&\text{on} \quad \partial\Omega. \end{aligned} \tag{1.45}$$

By the same argument as in subsection 1.1 we first prove that

$$J = \inf_{\substack{u \in H_0^1 \\ \|u\|_{p+1}=1}} \left\{ \int |\nabla u|^2 - au^2 \right\} < S, \tag{1.46}$$

(here we use (1.43)). Then, we show that the infimum in (1.46) is achieved. Therefore we obtain some $u \in H_0^1$ satisfying

$$u \geqq 0 \quad \text{on} \quad \Omega, \qquad \|u\|_{p+1} = 1,$$
$$-\Delta u - a(x)u = Ju^p.$$

Since $J > 0$ (by (1.44)) we obtain, after stretching, a solution of (1.45).

On the other hand, the problem

$$\begin{aligned} -\Delta u &= a(x)u^p + \lambda u \quad &&\text{on} \quad \Omega \quad \text{with} \quad p = (n+2)/(n-2), \\ u &> 0 &&\text{on} \quad \Omega, \\ u &= 0 &&\text{on} \quad \partial\Omega, \end{aligned} \tag{1.47}$$

where $a(x)$ is a smooth function on $\bar{\Omega}$ with $a(x) \geqq \delta > 0$, seems more delicate and we have only partial results.

(5). SHARP SOBOLEV INEQUALITIES. As a by-product of the proof of Theorem 1.2 we obtain the following surprising inequality:

COROLLARY 1.1. *Assume $\Omega \subset \mathbb{R}^3$ is a bounded domain. Then there exists a constant λ^* with $0 < \lambda^* < \lambda_1$ (λ^* depends on Ω) such that*

$$\|\nabla u\|_2^2 \geqq S\|u\|_6^2 + \lambda^* \|u\|_2^2 \quad \textit{for all} \quad u \in H_0^1. \tag{1.48}$$

We may take $\lambda^* = \frac{1}{4}\pi^2 (3 \text{ meas } \Omega/4\pi)^{-2/3}$ (this value is sharp when Ω is a ball).

Proof: Let Ω^* be the ball such that meas $\Omega^* =$ meas Ω. Let u^* denote the symmetric decreasing rearrangement of u. It is known (see e.g. the Appendix in Lieb [18] or Talenti [28]) that if $u \in H_0^1(\Omega)$, then $u^* \in H_0^1(\Omega^*)$ and

$$\|\nabla u^*\|_{L^2(\Omega^*)}^2 \leqq \|\nabla u\|_{L^2(\Omega)}^2. \tag{1.49}$$

On the other hand, for every $u^* \in H_0^1(\Omega^*)$,

$$\|\nabla u^*\|_{L^2(\Omega^*)}^2 \geqq S\|u^*\|_{L^6(\Omega^*)}^2 + \tfrac{1}{4}\lambda_1(\Omega^*)\|u^*\|_{L^2(\Omega^*)}^2. \tag{1.50}$$

Indeed (1.50) just says that $S_\lambda \geqq S$ when $\lambda = \frac{1}{4}\lambda_1(\Omega^*)$; and indeed we have $S_\lambda \geqq S$, because the strict inequality $S_\lambda < S$ would imply (through Lemma 1.2) the existence of a solution of (1.23) on Ω^* with $\lambda = \frac{1}{4}\lambda_1(\Omega^*)$ contradicting Theorem 1.2.

Finally we note that $\lambda_1(\Omega^*) = \pi^2/R^2$, where R is given by $\frac{4}{3}\pi R^3 = \text{meas}\,\Omega$. Combining (1.49), (1.50) and the fact that $\|u^*\|_{L^q(\Omega^*)} = \|u\|_{L^q(\Omega)}$ for all q, we obtain (1.48).

In other words, Corollary 1.1 asserts that given a bounded domain $\Omega \subset \mathbb{R}^3$ there is a number λ^* attached to Ω, with $0 < \lambda^* < \lambda_1$, such that

$$\begin{aligned} S_\lambda &< S \quad \text{for} \quad \lambda > \lambda^*, \\ S_\lambda &= S \quad \text{for} \quad 0 \leqq \lambda \leqq \lambda^*. \end{aligned} \tag{1.51}$$

When Ω is a ball we have $\lambda^* = \frac{1}{4}\lambda_1$.

Remark 1.4. When $n \geqq 4$, there is no inequality of the type:

$$\|\nabla u\|_2^2 \geqq S\|u\|_{2n/(n-2)}^2 + \lambda^*\|u\|_2^2 \quad \text{for all} \quad u \in H_0^1, \tag{1.52}$$

with $\lambda^* > 0$.

Indeed, (1.52) would imply that $S_{\lambda^*} \geqq S$ and we know (see Lemma 1.1) that $S_{\lambda^*} < S$. On the other hand, the following inequality holds:

$$\|\nabla u\|_2^2 \geqq S\|u\|_{2n/(n-2)}^2 + \lambda_q\|u\|_q^2 \quad \text{for all} \quad u \in H_0^1, \tag{1.53}$$

for each $n \geqq 3$ and each $q < n/(n-2)$, where $\lambda_q > 0$ is a constant depending on q and Ω. See appendix for the proof.

Remark 1.5. Assume $\Omega \subset \mathbb{R}^n$, $n \geqq 3$, is a bounded domain and let $\lambda \leqq 0$. Then $S_\lambda = S$ and the infimum in (1.5) is not attained. Indeed, the proofs of Lemma 1.1 and Lemma 1.3 lead to

$$Q_\lambda(u_\varepsilon) = \begin{cases} S + O(\varepsilon) & \text{if} \quad n \geqq 5, \\ S + O(\varepsilon|\log \varepsilon|) & \text{if} \quad n = 4, \\ S + O(\varepsilon^{1/2}) & \text{if} \quad n = 3, \end{cases}$$

and therefore $S_\lambda \leqq S$. On the other hand, the function $\lambda \mapsto S_\lambda$ is nonincreasing and thus $S_\lambda \geqq S$ for $\lambda \leqq 0$. We already know that the infimum in (1.5) is not attained when $\lambda = 0$; *a fortiori* it cannot be attained when $\lambda < 0$ (since $S_\lambda = S$).

(6) FURTHER RESULTS AND OPEN PROBLEMS CONCERNING THE CASE $n=3$. Let $\Omega\subset\mathbb{R}^3$ be a bounded domain.

(a) Assume Ω contains a ball of radius R. Then we have the following positive result:

THEOREM 1.2′. *Problem* (1.23) *possesses a solution for each* λ *such that*

$$\frac{\pi^2}{4R^2}<\lambda<\lambda_1(\Omega)$$

(this set may be vacuous). This is a consequence of the arguments used in the proof of Theorem 1.2.

(b) Consider Ω strictly starshaped about the origin, meaning that $x\cdot\nu>0$ on $\partial\Omega$ (assumed smooth).

THEOREM 1.2″. *If* (1.23) *has a solution, then*

$$\lambda\geqq\lambda_0(\Omega)>0.$$

Proof: By Pohozaev's identity, (1.4),

$$\begin{aligned}\lambda\int_\Omega u^2&=\tfrac12\int_{\partial\Omega}(x\cdot\nu)\left(\frac{\partial u}{\partial\nu}\right)^2\\&\geqq a\int_{\partial\Omega}\left(\frac{\partial u}{\partial\nu}\right)^2\geqq b\left(\int_{\partial\Omega}\left(\frac{\partial u}{\partial\nu}\right)\right)^2\quad\text{for}\quad a,b>0\\&=b\left(\int_\Omega\Delta u\right)^2=b\left(\int_\Omega|\Delta u|\right)^2\\&\geqq c\int_\Omega u^2,\qquad c>0,\end{aligned}$$

since Δ^{-1} is a bounded operator from L^1 to L^2 (by duality, from standard elliptic estimates for $n=3$).

(c) Let λ^* be defined by (1.51). Is the infimum in (1.5) attained when $\lambda=\lambda^*$? (In view of Theorem 1.2 we suspect that the answer is negative.)

(d) Set $\underset{\bar{}}{\lambda}=\inf\{\lambda\in\mathbb{R};$ problem (1.23) possesses a solution$\}$ so that $\underset{\bar{}}{\lambda}\leqq\lambda^*$. When is $\underset{\bar{}}{\lambda}=\lambda^*$?

2. Existence of Positive Solutions for $-\Delta u=u^p+f(x,u)$ on Ω, $u=0$ on Ω, where $p=(n+2)/(n-2)$ and $f(x,u)$ is a Lower-Order Perturbation

Let $\Omega\subset\mathbb{R}^n$, $n\geqq3$, be a bounded domain. We assume that $f(x,u):\Omega\times[0,+\infty)\to\mathbb{R}$ is measurable in x, continuous in u and that $\sup_{x\in\Omega,0\leqq u\leqq M}|f(x,u)|<\infty$ for every $M>0$.

Let $p=(n+2)/(n-2)$. We assume that $f(x,0)=0$ and that f is a lower-order perturbation of u^p, that is,

$$\lim_{u\to+\infty}\frac{f(x,u)}{u^p}=0.$$

We are concerned with the problem of existence of a function u satisfying

$$\begin{aligned} -\Delta u &= u^p+f(x,u) && \text{on } \Omega,\\ u&>0 && \text{on } \Omega,\\ u&=0 && \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

In subsection 2.1, we present a general tool for the study of (2.1), which is based on a variant of the mountain pass theorem.

In subsection 2.2, we discuss some technical (but concrete) assumption under which the above tool may be used.

In subsections 2.3, 2.4 and 2.5, we present some examples. The cases $n\geqq 5$, $n=4$, and $n=3$ turn out to be different and we treat them separately. In subsection 6, we consider the problem

$$\begin{aligned} -\Delta u &= \lambda(1+u)^p && \text{on } \Omega,\\ u&>0 && \text{on } \Omega,\\ u&=0 && \text{on } \partial\Omega. \end{aligned}$$

Using our previous results we establish the existence of at least *two* positive solutions for each $\lambda>0$ small enough.

2.1. A general tool. We assume that $f(x,u)$ can be written as

$$f(x,u)=a(x)u+g(x,u), \tag{2.2}$$

with

$$a(x)\in L^\infty(\Omega), \tag{2.3}$$

$$g(x,u)=o(u) \quad \text{as} \quad u\to 0^+, \quad \text{uniformly in} \quad x, \tag{2.4}$$

$$g(x,u)=o(u^p) \quad \text{as} \quad u\to +\infty, \quad \text{uniformly in} \quad x. \tag{2.5}$$

Moreover we assume that the operator $-\Delta-a(x)$ has its least eigenvalue positive, that is

$$\int |\nabla\phi|^2-a\phi^2\geqq\alpha\int\phi^2 \quad \text{for all} \quad \phi\in H_0^1, \qquad \alpha>0, \tag{2.6}$$

or equivalently

$$\int |\nabla\phi|^2-a\phi^2\geqq\alpha'\int|\nabla\phi|^2 \quad \text{for all} \quad \phi\in H_0^1, \qquad \alpha'>0. \tag{2.6'}$$

The values of $f(x,u)$ for $u<0$ are irrelevant and we may define

$$f(x,u)=0 \quad \text{for} \quad x\in\Omega, \qquad u\leqq 0.$$

Set

$$F(x,u)=\int_0^u f(x,t)\,dt \quad \text{for} \quad x\in\Omega, \qquad u\in\mathbb{R},$$

and

$$\Psi(u)=\int\left\{\tfrac{1}{2}|\nabla u|^2-\frac{1}{p+1}|u|^{p+1}-F(x,u)\right\} \quad \text{for} \quad u\in H_0^1. \tag{2.7}$$

The main result of subsection 2.1 is the following:

THEOREM 2.1. *Assume* (2.2)–(2.6) *and suppose, moreover, that*

(2.8) *there exists some* $v_0\in H_0^1$, $v_0\geqq 0$ *on* Ω, $v_0\not\equiv 0$, *such that*

$$\sup_{t\geqq 0}\Psi(tv_0)<\frac{1}{n}S^{n/2}.$$

Then, problem (2.1) *possesses a solution.*

Remark 2.1. In case $f(x,u)=\lambda u$, assumption (2.6) corresponds to $\lambda<\lambda_1$ while assumption (2.8) is equivalent to $S_\lambda<S$.

Indeed we have $\Psi(tv_0)=\frac{1}{2}At^2-(B/(p+1))t^{p+1}$ with $A=\|\nabla v_0\|_2^2-\lambda\|v_0\|_2^2$ and $B=\|v_0\|_{p+1}^{p+1}$; thus

$$\sup_{t\geqq 0}\Psi(tv_0)=\frac{1}{n}\left[\frac{A}{B^{2/(p+1)}}\right]^{n/2}.$$

Therefore Theorem 2.1 implies Theorem 1.1 and the positive part of Theorem 1.2 once we know that $S_\lambda<S$.

The proof of Theorem 2.1 relies on the following variant of the mountain pass theorem of Ambrosetti and Rabinowitz *without the* (PS) *condition*:

THEOREM 2.2. *Let* Φ *be a* C^1 *function on a Banach space* E. *Suppose*

(2.9) *there exists a neighborhood* U *of* 0 *in* E *and a constant* ρ *such that* $\Phi(u)\geqq\rho$ *for every* u *in the boundary of* U,

$$\Phi(0)<\rho \quad \textit{and} \quad \Phi(v)<\rho \quad \textit{for some} \quad v\notin U. \tag{2.10}$$

Set

$$c=\inf_{P\in\mathcal{P}}\max_{w\in P}\Phi(w)\geqq\rho, \tag{2.11}$$

where $\mathcal{P}$ *denotes the class of continuous paths joining* 0 *to* v.

Conclusion:

$$\text{there is a sequence } (u_j) \text{ in } E \text{ such that} \quad \Phi(u_j)\to c \quad \text{and} \quad \Phi'(u_j)\to 0 \quad \text{in} \quad E^*. \tag{2.12}$$

The proof of Theorem 2.2 is exactly the same as in Ambrosetti and Rabinowitz [1] (see also the Appendix in Brezis, Coron and Nirenberg [6]), and we shall omit it. Note that here we do *not* assume condition (PS). If condition (PS) (or condition $(\mathrm{PS})_c$ from [6]) is satisfied, then we deduce from (2.2) that c is a critical value. Theorem 2.2 is reminiscent, in some ways, of I. Ekeland's theorem in [12] (which asserts that if Φ is a C^1 function on a Banach space E and Φ is bounded below on E, there exists a sequence (u_j) in E such that $\Phi(u_j)\to\inf_E \Phi$ and $\Phi'(u_j)\to 0$ in E^*).

Proof of Theorem 2.1. Using (2.2)–(2.5) we may fix a constant $\mu \geqq 0$ large enough so that

$$-f(x,u)\leqq \mu u+u^p \quad \text{for a.e.} \quad x\in\Omega, \quad \text{and for all} \quad u\geqq 0 \tag{2.13}$$

(in case $f(x,u)\geqq 0$ for all $u\geqq 0$ we may, of course, choose $\mu=0$). On $E=H_0^1$ we define

$$\Phi(u)=\int\left\{\tfrac{1}{2}|\nabla u|^2+\tfrac{1}{2}\mu u^2-\frac{1}{p+1}(u^+)^{p+1}-F(x,u^+)-\tfrac{1}{2}\mu(u^+)^2\right\}.$$

Clearly Φ is C^1 on E; we shall verify the assumptions of Theorem 2.2.

VERIFICATION OF (2.9): By (2.4) we have, for any $\varepsilon>0$, there is a $\delta>0$ such that

$$g(x,u)\leqq \varepsilon u \quad \text{for a.e.} \quad x\in\Omega, \quad \text{and for all} \quad 0\leqq u\leqq\delta,$$

thus, by (2.5), we obtain

$$g(x,u)\leqq \varepsilon u+Cu^p \quad \text{for a.e.} \quad x\in\Omega \quad \text{for all} \quad u\geqq 0,$$

and for some constant C (depending on ε). Therefore we have

$$F(x,u)\leqq \tfrac{1}{2}a(x)u^2+\tfrac{1}{2}\varepsilon u^2+\frac{C}{p+1}u^{p+1} \quad \text{for a.e.} \quad x\in\Omega, \quad \text{and for all} \quad u\geqq 0. \tag{2.14}$$

Hence we find, for all $u\in H_0^1$,

$$\Phi(u)\geqq\int\left\{\tfrac{1}{2}|\nabla u|^2-\tfrac{1}{2}a(x)(u^+)^2-\tfrac{1}{2}\varepsilon(u^+)^2-\frac{C+1}{p+1}(u^+)^{p+1}\right\}.$$

Using (2.6′) and the fact that $\int |\nabla u|^2 = \int |\nabla u^+|^2 + |\nabla u^-|^2$ we conclude that (with ε small enough) there exist constants $k > 0$ and C' such that

$$\Phi(u) \geqq k\|u\|^2_{H_0^1} - C'\|u\|^{p+1}_{H_0^1} \quad \text{for all} \quad u \in H_0^1,$$

which implies (2.9) with some $\rho > 0$ (and U a small ball in H_0^1).

VERIFICATION OF (2.10): For any $u \in H_0^1$, $u \geqq 0$, $u \not\equiv 0$, we have by (2.5) $\lim_{t\to+\infty} \Phi(tu) = -\infty$. Thus, there are many v's satisfying (2.10). However, it will be important for later purposes to use Theorem 2.2 with a *special* v, namely $v = t_0 v_0$, where v_0 is given by (2.8) and $t_0 > 0$ is chosen large enough so that $v \notin U$ and $\Phi(v) \leqq 0$.

It follows from (2.8) that

$$\sup_{t\geqq 0} \Phi(tv) < \frac{1}{n} S^{n/2}$$

and therefore we have

$$c < \frac{1}{n} S^{n/2}. \tag{2.15}$$

Applying Theorem 2.2 we obtain a sequence (u_j) in H_0^1 such that $\Phi(u_j) \to c$ and $\Phi'(u_j) \to 0$ in H^{-1}, that is,

$$\int \left\{ \tfrac{1}{2}|\nabla u_j|^2 + \tfrac{1}{2}\mu u_j^2 - \frac{1}{p+1}(u_j^+)^{p+1} - F(x, u_j^+) - \tfrac{1}{2}\mu (u_j^+)^2 \right\} = c + o(1), \tag{2.16}$$

and

$$-\Delta u_j + \mu u_j - (u_j^+)^p - f(x, u_j^+) - \mu u_j^+ = \zeta_j \tag{2.17}$$

with $\zeta_j \to 0$ in H^{-1}. We claim that

$$\|u_j\|_{H_0^1} \leqq C. \tag{2.18}$$

Indeed, multiplying (2.17) by u_j we obtain

$$\int \{|\nabla u_j|^2 + \mu u_j^2 - (u_j^+)^{p+1} - f(x, u_j^+)u_j^+ - \mu (u_j^+)^2\} = \langle \zeta_j, u_j \rangle. \tag{2.19}$$

Taking (2.16) $-\frac{1}{2}$ (2.19) yields

$$\frac{1}{n} \int (u_j^+)^{p+1} \leqq \int \{F(x, u_j^+) - \tfrac{1}{2} f(x, u_j^+)u_j^+\} + c + o(1) + \|\zeta_j\|_{H^{-1}} \|u_j\|_{H_0^1}. \tag{2.20}$$

On the other hand, from (2.5) we have

$$\text{for all} \quad \varepsilon > 0 \quad \text{there is} \quad C \quad \text{such that}$$

$$|f(x, u)| \leqq \varepsilon u^p + C \quad \text{for} \quad \text{a.e.} \quad x \in \Omega, \quad \text{and for all} \quad u \geqq 0, \tag{2.21}$$

so

$$(2.22)\quad |F(x,u)| \leqq \frac{\varepsilon}{p+1} u^{p+1} + C \quad \text{for a.e. } x \in \Omega, \text{ and for all } u \geqq 0.$$

We deduce from (2.20)–(2.22) (with ε small enough) that

$$(2.23)\qquad \int (u_j^+)^{p+1} \leqq C + C\|u_j\|_{H_0^1}.$$

Combining (2.16) and (2.23) we obtain (2.18). Extract a subsequence, still denoted by u_j, so that

$$\begin{aligned} u_j &\rightharpoonup u && \text{weakly in } H_0^1, \\ u_j &\to u && \text{strongly in } L^q \text{ for all } q < p+1, \\ u_j &\to u && \text{a.e. on } \Omega, \\ (u_j^+)^p &\rightharpoonup (u^+)^p && \text{weakly in } (L^{p+1})^* \\ f(x,u_j^+) &\rightharpoonup f(x,u^+) && \text{weakly in } (L^{p+1})^*. \end{aligned}$$

Passing to the limit in (2.17) we obtain

$$(2.24)\qquad -\Delta u + \mu u = (u^+)^p + f(x,u^+) + \mu u^+ \quad \text{in } H^{-1}.$$

We deduce from (2.13) and (2.24), in which the right-hand side is greater than or equal to 0, and the Stampacchia maximum principle, that $u \geqq 0$ on Ω and therefore u satisfies

$$(2.25)\qquad -\Delta u = u^p + f(x,u).$$

We shall now verify that $u \not\equiv 0$ (and consequently $u > 0$ on Ω by the strong maximum principle).

Indeed, suppose that $u \equiv 0$. We claim that

$$(2.26)\qquad \int f(x,u_j^+)u_j^+ \to 0,$$

$$(2.27)\qquad \int F(x,u_j^+) \to 0.$$

From (2.21) and (2.22) we deduce that

$$\left|\int f(x,u_j^+)u_j^+\right| \leqq \varepsilon \int (u_j^+)^{p+1} + C \int u_j^+,$$

$$\left|\int F(x,u_j^+)\right| \leqq \frac{\varepsilon}{p+1} \int (u_j^+)^{p+1} + C \int u_j^+.$$

Since u_j remains bounded in L^{p+1} and $u_j \to 0$ in L^2 we obtain (2.26) and (2.27).
Extracting still another sequence we may assume that

$$\int |\nabla u_j|^2 \to l \tag{2.28}$$

for some constant $l \geqq 0$. Passing to the limit in (2.19) and then in (2.16) we obtain

$$\int (u_j^+)^{p+1} \to l \tag{2.29}$$

and

$$\frac{1}{n} l = c. \tag{2.30}$$

On the other hand, we have

$$\|\nabla u_j\|_2^2 \geqq S\|u_j\|_{p+1}^2 \geqq S\|u_j^+\|_{p+1}^2$$

and (using (2.28) and (2.29)) we find in the limit

$$l \geqq S l^{2/(p+1)}. \tag{2.31}$$

From (2.30) and (2.31) we deduce that

$$c \geqq \frac{1}{n} S^{n/2}, \tag{2.32}$$

a contradiction to (2.15). Thus $u \not\equiv 0$.

Remark 2.2. The solution u of problem (2.1) which we have obtained has an additional property.

Namely, we claim that:
either

$$\Phi(u) = c, \tag{2.33$_1$}$$

or

$$\Phi(u) \leqq c - \frac{1}{n} S^{n/2} < 0. \tag{2.33$_2$}$$

In some cases, (2.33$_2$) is excluded for instance if we assume that

$$F(x, u) \leqq \tfrac{1}{2} f(x, u)u + \frac{1}{n} u^{p+1} \quad \text{for a.e. } x \in \Omega, \quad \text{and for all } u \geqq 0. \tag{2.34}$$

(Assumption (2.34) holds for example when $f(x, u) = a(x)u + \mu u^q$ with $a \in L^\infty$, $\mu \geqq 0$ and $1 \leqq q < p$.) Indeed if u is a solution of (2.1) we have

$$\int |\nabla u|^2 = \int \{u^{p+1} + f(x, u)u\} \tag{2.35}$$

and thus

$$\Phi(u)=\int\left\{\tfrac{1}{2}|\nabla u|^2-\frac{1}{p+1}u^{p+1}-F(x,u)\right\}=\int\left\{\frac{1}{n}u^{p+1}+\tfrac{1}{2}f(x,u)u-F(x,u)\right\}.$$

Therefore, using (2.34), we have $\Phi(u)\geqq 0$. In fact, when we assume (2.34), the argument below shows that Φ *satisfies the condition* $(\mathrm{PS})_c$ (introduced in [6]) *for every* $c<(1/n)S^{n/2}$.

VERIFICATION OF (2.33_1)–(2.33_2): We consider again the sequence (u_j) as in the proof of Theorem 2.1. It is easy to check that

$$\int f(x,u_j^+)u_j^+\to\int f(x,u^+)u^+ \quad\text{and}\quad \int F(x,u_j^+)\to\int F(x,u) \tag{2.36}$$

(this follows from the fact that $f(x,u)$ is of lower order than u^p, that (u_j) is bounded in L^{p+1} and that $u_j\to u$ a.e. on Ω). We set $v_j=u_j-u$, so that

$$\int|\nabla u_j|^2=\int|\nabla u|^2+\int|\nabla v_j|^2+o(1), \tag{2.37}$$

and from [8] we deduce that

$$\int(u_j^+)^{p+1}=\int u^{p+1}+\int(v_j^+)^{p+1}+o(1). \tag{2.38}$$

Combining (2.16) and (2.19) with (2.36), (2.37) and (2.38) leads to

$$\begin{aligned}&\Psi(u)+\int\left\{\tfrac{1}{2}|\nabla v_j|^2-\frac{1}{p+1}(v_j^+)^{p+1}\right\}=c+o(1),\\&\int\{|\nabla u|^2-u^{p+1}-f(x,u)u\}+\int\{|\nabla v_j|^2-(v_j^+)^{p+1}\}=o(1),\end{aligned} \tag{2.39}$$

which reduces (with the help of (2.35)) to

$$\int|\nabla v_j|^2=\int(v_j^+)^{p+1}+o(1). \tag{2.40}$$

Therefore (2.39) becomes

$$\Psi(u)+\frac{1}{n}\int|\nabla v_j|^2=c+o(1). \tag{2.41}$$

Finally, we may assume (for a subsequence) that

$$\int|\nabla v_j|^2\to k\geqq 0 \quad\text{and}\quad \int(v_j^+)^{p+1}\to k. \tag{2.42}$$

Sobolev's inequality leads to $k\geqq Sk^{2/(p+1)}$. Thus we have, either $k=0$, or $k\geqq S^{n/2}$, which (together with (2.41) and (2.42)) proves (2.33_1) and (2.33_2).

2.2. Towards the verification of condition (2.8). Lemma 2.1 below furnishes a general, though awkward, assumption under which the crucial condition (2.8) of Theorem 2.1 holds. With its aid we shall present some applications in subsections 2.3–2.6.

LEMMA 2.1. *Assume $f(x, u)$ satisfies* (2.2)–(2.5). *Suppose also that there is some function $f(u)$ such that*

$$(2.43)\qquad f(x,u)\geqq f(u)\geqq 0 \quad \textit{for} \quad \textit{a.e.} \quad x\in\omega, \quad \textit{and for all} \quad u\geqq 0,$$

where ω is some nonempty open set in Ω and the primitive $F(u)=\int_0^u f(t)\,dt$ satisfies

$$(2.44)\qquad \lim_{\varepsilon\to 0}\varepsilon\int_0^{\varepsilon^{-1/2}} F\left[\left(\frac{\varepsilon^{-1/2}}{1+s^2}\right)^{(n-2)/2}\right]s^{n-1}\,ds=\infty.$$

Then condition (2.8) *holds.*

Proof of Lemma 2.1: First we recall that if $f(x, u)=\lambda u$, then condition (2.8) amounts to

$$\frac{\|\nabla v_0\|_2^2-\lambda\|v_0\|_2^2}{\|v_0\|_{p+1}^2}<S.$$

Therefore it is natural to use for v_0 the same type of function as in Lemma 1.1.

Assume $0\in\omega$ and fix a function $\phi\in\mathscr{D}_+(\omega)$ such that $\phi(x)\equiv 1$ for $|x|<R$ ($R>0$). Set

$$(2.45)\qquad u_\varepsilon(x)=\frac{\phi(x)}{(\varepsilon+|x|^2)^{(n-2)/2}},\qquad \varepsilon>0,$$

and

$$(2.46)\qquad v_\varepsilon(x)=\frac{u_\varepsilon(x)}{\|u_\varepsilon\|_{p+1}}\qquad (\text{so that } \|v_\varepsilon\|_{p+1}=1).$$

We claim that v_ε satisfies condition (2.8) for $\varepsilon>0$ sufficiently small. The computations in the proofs of Lemma 1.1 and Lemma 1.3 show that

$$(2.47)\qquad \|u_\varepsilon\|_{p+1}=\frac{K}{\varepsilon^{(n-2)/4}}+o(1),\qquad n\geqq 3,$$

where K depends only on n,

$$(2.48)\qquad \|\nabla v_\varepsilon\|_2^2=S+O(\varepsilon^{(n-2)/2}),\qquad n\geqq 3,$$

$$(2.49)\qquad \|v_\varepsilon\|_2^2=\begin{cases}O(\varepsilon) & \text{if}\quad n\geqq 5,\\ O(\varepsilon|\log\varepsilon|) & \text{if}\quad n=4,\\ O(\varepsilon^{1/2}) & \text{if}\quad n=3.\end{cases}$$

We set $X_\varepsilon = \|\nabla v_\varepsilon\|_2^2$ and so we have

$$\Psi(tv_\varepsilon) = \tfrac{1}{2}t^2 X_\varepsilon - \frac{t^{p+1}}{p+1} - \int F(x, tv_\varepsilon).$$

Note that $\Psi(tv_\varepsilon) \leqq \frac{1}{2}t^2 X_\varepsilon - t^{p+1}/(p+1)$ and thus $\lim_{t\to+\infty} \Psi(tv_\varepsilon) = -\infty$. Therefore, $\sup_{t\geqq 0} \Psi(tv_\varepsilon)$ is achieved at some $t_\varepsilon > 0$ (if $t_\varepsilon = 0$, then $\sup_{t\geqq 0} \Psi(tv_\varepsilon) = 0$ and there is nothing to prove). Since the derivative of the function $t \mapsto \Psi(tv_\varepsilon)$ vanishes at $t = t_\varepsilon$, we have

(2.50)
$$t_\varepsilon X_\varepsilon - t_\varepsilon^p - \int f(x, t_\varepsilon v_\varepsilon) v_\varepsilon = 0$$

and therefore

(2.51)
$$t_\varepsilon \leqq X_\varepsilon^{1/(p-1)}.$$

Set

$$Y_\varepsilon = \sup_{t\geqq 0} \Psi(tv_\varepsilon) = \Psi(t_\varepsilon v_\varepsilon).$$

Since the function $t \mapsto (\frac{1}{2}t^2 X_\varepsilon - t^{p+1}/(p+1))$ is increasing on the interval $[0, X_\varepsilon^{1/(p-1)}]$ we have, by (2.51),

$$Y_\varepsilon = \tfrac{1}{2}t_\varepsilon^2 X_\varepsilon - \frac{t_\varepsilon^{p+1}}{p+1} - \int F(x, t_\varepsilon v_\varepsilon) \leqq \frac{1}{n} X_\varepsilon^{(p+1)/(p-1)} - \int F(x, t_\varepsilon v_\varepsilon).$$

Using (2.48) we obtain

(2.52)
$$Y_\varepsilon \leqq \frac{1}{n} S^{n/2} + O(\varepsilon^{(n-2)/2}) - \int F(x, t_\varepsilon v_\varepsilon).$$

On the other hand, we claim that

(2.53)
$$t_\varepsilon \to S^{1/(p-1)} \quad \text{as} \quad \varepsilon \to 0.$$

Indeed, by (2.50) we have

$$X_\varepsilon - t_\varepsilon^{p-1} - \int \frac{f(x, t_\varepsilon v_\varepsilon) v_\varepsilon}{t_\varepsilon} = 0.$$

Thus, it suffices to verify that

(2.54)
$$\int \frac{f(x, t_\varepsilon v_\varepsilon) v_\varepsilon}{t_\varepsilon} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Using (2.2)–(2.5) we see that for all $\delta > 0$, there is C such that

$$|f(x, u)| \leqq \delta u^p + Cu \quad \text{for a.e.} \quad x \in \Omega, \quad \text{and for all} \quad u \geqq 0.$$

Therefore we have

$$\left|\int \frac{f(x, t_\varepsilon v_\varepsilon)v_\varepsilon}{t_\varepsilon}\right| \leqq \delta t_\varepsilon^{p-1}\|v_\varepsilon\|_{p+1}^{p+1} + C\|v_\varepsilon\|_2^2 = \delta t_\varepsilon^{p-1} + C\|v_\varepsilon\|_2^2,$$

which implies (2.54) and thereby (2.53).

It follows from (2.53), (2.45)–(2.47) that, for $\varepsilon > 0$ sufficiently small,

$$\int F(x, t_\varepsilon v_\varepsilon) \geqq \int_{|x|<R} F\left(\frac{A\varepsilon^{(n-2)/4}}{(\varepsilon + |x|^2)^{(n-2)/2}}\right) dx \tag{2.55}$$

for some constant $A > 0$. From (2.52) and (2.55) we deduce that

$$Y_\varepsilon \leqq \frac{1}{n} S^{n/2} + O(\varepsilon^{(n-2)/2}) - \int_{|x|<R} F\left(\frac{A\varepsilon^{(n-2)/4}}{(\varepsilon + |x|^2)^{(n-2)/2}}\right) dx. \tag{2.56}$$

Finally we claim that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{(n-2)/2}} \int_{|x|<R} F\left(\frac{A\varepsilon^{(n-2)/4}}{(\varepsilon + |x|^2)^{(n-2)/2}}\right) dx = \infty \tag{2.57}$$

which implies, together with (2.56), that $Y_\varepsilon < (1/n)S^{n/2}$ for $\varepsilon > 0$ sufficiently small.

VERIFICATION OF (2.57): We have

$$\begin{aligned}\frac{1}{\varepsilon^{(n-2)/2}} \int_{|x|<R} F\left(\frac{A\varepsilon^{(n-2)/4}}{(\varepsilon + |x|^2)^{(n-2)/2}}\right) dx &= \frac{\omega}{\varepsilon^{(n-2)/2}} \int_0^R F\left(\frac{A\varepsilon^{(n-2)/4}}{(\varepsilon + r^2)^{(n-2)/2}}\right) r^{n-1}\, dr \\ &= \omega\varepsilon \int_0^{R\varepsilon^{-1/2}} F\left[A\left(\frac{\varepsilon^{-1/2}}{1+s^2}\right)^{(n-2)/2}\right] s^{n-1}\, ds,\end{aligned}$$

where ω is the area of S^{n-1} and $r = \varepsilon^{1/2}s$. After rescaling ε we see that (2.57) is equivalent to

$$\lim_{\varepsilon\to 0} \varepsilon \int_0^{R'\varepsilon^{-1/2}} F\left[\left(\frac{\varepsilon^{-1/2}}{1+s^2}\right)^{(n-2)/2}\right] s^{n-1}\, ds = \infty \tag{2.58}$$

for some constant $R' > 0$. When $R' \geqq 1$, (2.58) is a consequence of (2.44). Otherwise, when $R' < 1$, consider

$$Z_\varepsilon = \varepsilon \int_{R'\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} F\left[\left(\frac{\varepsilon^{-1/2}}{1+s^2}\right)^{(n-2)/2}\right] s^{n-1}\, ds$$

and note that (for some constant C)

$$|Z_\varepsilon| \leqq C\varepsilon F(C\varepsilon^{(n-2)/4})\varepsilon^{-n/2}$$

which is bounded as $\varepsilon \to 0$ (because of (2.2)–(2.4)); and thus (2.58) is again a consequence of (2.44).

2.3. The case $n \geqq 5$. We assume here that $n \geqq 5$ and, moreover,

(2.59) $$f(x, u) \geqq 0 \quad \text{for a.e. } x \in \omega, \quad \text{and for all } u \geqq 0,$$

(2.60) $$f(x, u) \geqq \mu > 0 \quad \text{for a.e. } x \in \omega, \quad \text{and for all } u \in I,$$

where ω is some nonempty open subset of Ω, $I \subset (0, +\infty)$ is some nonempty open interval and $\mu > 0$ is some constant.

COROLLARY 2.1. *Assume that* (2.2)–(2.6), (2.59), (2.60) *hold. Then problem* (2.1) *possesses a solution.*

EXAMPLE 2.1. All the assumptions of Corollary 2.1 are satisfied if $f(x, u) = f(u)$, where $f(u)$ is a C^1 function on $[0, +\infty)$ such that

(2.61) $$f(0) = 0, \quad f(u) \geqq 0 \quad \text{for all } u \geqq 0, \quad f \not\equiv 0,$$
$$f'(0) < \lambda_1 \quad \text{and} \quad \lim_{u \to +\infty} \frac{f(u)}{u^p} = 0,$$

(for instance we may take $f(u) = \lambda u$ with $0 < \lambda < \lambda_1$ or $f(u) = \mu u^q$ with $\mu > 0$ and $1 < q < p$).

Proof of Corollary 2.1: We shall use Theorem 2.1 with Lemma 2.1. Applying (2.59) and (2.60) we see that

$$f(x, u) \geqq \mu \chi_I(u) \equiv f(u) \quad \text{for a.e. } x \in \omega, \quad \text{and for all } u \geqq 0$$

(χ_I is the characteristic function of I). Thus we have

$$F(u) \geqq \beta > 0 \quad \text{for all } u \geqq B,$$

for some constants $\beta > 0$ and $B > 0$.

VERIFICATION OF (2.44): We have

$$F\left[\left(\frac{\varepsilon^{-1/2}}{1+s^2}\right)^{(n-2)/2}\right] \geqq \beta \quad \text{for all } s \quad \text{such that } \frac{\varepsilon^{-1/2}}{1+s^2} \geqq B^{2/(n-2)},$$

and in particular for all $s \leqq C\varepsilon^{-1/4}$, where C is some constant and ε is small. Thus we have for ε small

$$\varepsilon \int_0^{\varepsilon^{-1/2}} F\left[\left(\frac{\varepsilon^{-1/2}}{1+s^2}\right)^{(n-2)/2}\right] s^{n-1}\, ds \geqq \beta\varepsilon \int_0^{C\varepsilon^{-1/4}} s^{n-1}\, ds = C'\varepsilon^{1-(n/4)}$$

and the right-hand side tends to $+\infty$ as $\varepsilon \to 0$ (since $n \geqq 5$).

2.4. The case $n = 4$. We assume here that $n = 4$ and

(2.62) $$f(x, u) \geqq 0 \quad \text{for a.e. } x \in \omega, \quad \text{and for all } u \geqq 0$$

together with *one* of the following conditions:
either

$$(2.63_1) \qquad f(x,u) \geqq \mu u \quad \text{for a.e. } x \in \omega, \text{ and for all } u \in [0, A],$$

or

$$(2.63_2) \qquad f(x,u) \geqq \mu u \quad \text{for a.e. } x \in \omega, \text{ and for all } u \in [A, +\infty),$$

where ω is some nonempty open subset of Ω and $\mu > 0$, $A > 0$ are some constants.

COROLLARY 2.2. *Assume that* (2.2)–(2.6), (2.62), (2.63_1) *or* (2.63_2) *hold. Then problem* (2.1) *possesses a solution.*

EXAMPLE 2.2. All the assumptions of Corollary 2.2. are satisfied if $f(x, u) = f(u)$, where $f(u)$ is a C^1 function on $[0, +\infty)$ such that

$$(2.64) \qquad \begin{gathered} f(0)=0, \quad f(u) \geqq 0 \quad \text{for all} \quad u \geqq 0, \quad f'(0) < \lambda_1, \quad \lim_{u\to+\infty} f(u)/u^3 = 0, \\ \text{and } \textit{either} \quad f'(0) > 0 \quad \textit{or} \quad \liminf_{u\to+\infty} f(u)/u > 0, \end{gathered}$$

(for instance we may take $f(u) = \lambda u$ with $0 < \lambda < \lambda_1$ or $f(u) = \mu u^q$ with $\mu > 0$ and $1 < q < 3$).

Proof of Corollary 2.2. Again we use Theorem 2.1 with Lemma 2.1. We have

$$f(x,u) \geqq \mu u \chi_I(u) \equiv f(u) \quad \text{for a.e. } x \in \omega, \text{ and for all } u \geqq 0,$$

where I is either $[0, A]$ or $[A, +\infty)$. Thus we obtain:
either

$$(2.65_1) \qquad F(u) = \tfrac{1}{2}\mu u^2 \quad \text{for} \quad 0 \leqq u \leqq A,$$

or

$$(2.65_2) \qquad F(u) = \tfrac{1}{2}\mu (u^2 - A^2) \quad \text{for} \quad u \geqq A.$$

VERIFICATION OF (2.44): In case (2.65_1) holds we have, for ε small,

$$\varepsilon \int_0^{\varepsilon^{-1/2}} F\left(\frac{\varepsilon^{-1/2}}{1+s^2}\right) s^3\, ds \geqq \tfrac{1}{2}\mu\varepsilon \int_{A^{-1/2}\varepsilon^{-1/4}}^{\varepsilon^{-1/2}} \frac{\varepsilon^{-1}}{(1+s^2)^2} s^3\, ds \sim C\,|\log \varepsilon|$$

as $\varepsilon \to 0$.

In case (2.65_2) holds we have, for some positive constant B and ε small,

$$\varepsilon \int_0^{\varepsilon^{-1/2}} F\left(\frac{\varepsilon^{-1/2}}{1+s^2}\right) s^3\, ds \geqq \tfrac{1}{4}\mu\varepsilon \int_0^{B\varepsilon^{-1/4}} \frac{\varepsilon^{-1}}{(1+s^2)^2} s^3\, ds \sim C|\log \varepsilon|,$$

as $\varepsilon \to 0$.

A CURIOUS EXAMPLE. Let g be a smooth function which is positive on the interval $(1, 2)$ and zero elsewhere and consider the problem:

$$\begin{aligned} -\Delta u &= u^3 + \mu g(u) && \text{in} \quad \Omega \subset R^4, \\ u &> 0 && \text{in} \quad \Omega, \\ u &= 0 && \text{on} \quad \partial\Omega. \end{aligned}$$

THEOREM 2.3. *For μ large there is a solution, while for $\mu > 0$ and small and Ω strictly starshaped, there is none.*

Proof: The proof of existence for μ large is similar to the proof of Corollary 2.4 below and will be omitted. We shall simply show that there is no solution for μ small and Ω starshaped about the origin. Indeed, if u is a solution, then, by Pohozaev's identity (1.3) (c will denote various positive constants),

$$\begin{aligned} \mu \int 4G(u) - ug(u) &= \tfrac{1}{2} \int (x \cdot \nu)\left(\frac{\partial u}{\partial \nu}\right)^2 \\ &\geqq c\left(\int |\Delta u|\right)^2 \\ &\geqq c[u]_{2,w}^2, \end{aligned}$$

where

$$[u]_{p,w} = \sup_{\lambda > 0} \lambda\ [\text{meas}\,\{u > \lambda\}]^{1/p}$$

which is equivalent to the L^p-weak norm.

The last inequality follows from the fact that

$$u \leqq v = \frac{c}{|x|^2} * |\Delta u|$$

and the fact that $|x|^{-2} \in L^2$-weak. In particular,

$$[u]_{2,w}^2 \geqq \text{meas}\,\{u > 1\}.$$

On the other hand, since g has support in $(1, 2)$,

$$\int 4G(u) - ug(u) \leqq C\ \text{meas}\,\{u > 1\}.$$

If meas $\{u>1\}=0$, then u would be a solution of $\Delta u+u^3=0$, which is impossible. Thus it follows that

$$\mu \geqq \mu_0(\Omega).$$

2.5. The case $n=3$. The case $n=3$ is rather delicate and we have two different results, depending upon the behavior of $f(x,u)$ as $u\to+\infty$.

For the first result we assume that

$$f(x,u)\geqq 0 \quad \text{for} \quad \text{a.e.} \quad x\in\omega, \quad \text{and for all} \quad u\geqq 0 \tag{2.66}$$

$$\lim_{u\to+\infty} f(x,u)/u^3=+\infty \quad \text{uniformly as} \quad x\in\omega, \tag{2.67}$$

where ω is some nonempty open subset of Ω.

COROLLARY 2.3. *Assume that* (2.2)–(2.6), (2.66), (2.67) *hold. Then problem* (2.1) *possesses a solution.*

EXAMPLE 2.3. All the assumptions of Corollary 2.3 are satisfied if $f(x,u)=f(u)$, where $f(u)$ is a C^1 function on $[0,+\infty)$ such that

$$\begin{gathered} f(0)=0, \quad f(u)\geqq 0 \quad \text{for all} \quad u\geqq 0, \quad f'(0)<\lambda_1, \\ \lim_{u\to+\infty} f(u)/u^5=0, \quad \lim_{u\to+\infty} f(u)/u^3=+\infty, \end{gathered} \tag{2.68}$$

(for instance we may take $f(u)=\mu u^q$ with $\mu>0$ and $3<q<5$).

Proof of Corollary 2.3: Once more we apply Theorem 2.1 together with Lemma 2.1. We shall verify (2.44). We set $f(u)=\inf_{x\in\omega} f(x,u)$, so that $\lim_{u\to+\infty} f(u)/u^3=+\infty$. Therefore we have

$$\text{for all} \quad \mu>0 \quad \text{there is an} \quad A>0 \quad \text{such that} \quad F(u)\geqq \mu u^4 \quad \text{for all} \quad u\geqq A.$$

It follows that, for some constant $B>0$ and ε small,

$$\varepsilon\int_0^{\varepsilon^{-1/2}} F\left[\left(\frac{\varepsilon^{-1/2}}{1+s^2}\right)^{1/2}\right]s^2\,ds \geqq \mu\varepsilon\int_0^{B\varepsilon^{-1/4}} \frac{\varepsilon^{-1}}{(1+s^2)^2}s^2\,ds.$$

Hence we obtain

$$\liminf_{\varepsilon\to 0}\varepsilon\int_0^{\varepsilon^{-1/2}} F\left[\left(\frac{\varepsilon^{-1/2}}{1+s^2}\right)^{1/2}\right]s^2\,ds \geqq \mu\int_0^{\infty}\frac{s^2}{(1+s^2)^2}\,ds$$

for all $\mu>0$; this implies (2.44).

We discuss now a second type of result. It is convenient to introduce a parameter $\mu > 0$ and to consider the problem

$$(2.69)\qquad \begin{aligned} -\Delta u &= u^5 + a(x)u + \mu g(x,u) && \text{on } \Omega,\\ u &> 0 && \text{on } \Omega,\\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

We shall assume that

$$(2.70)\qquad \begin{aligned} g(x,u) &\geqq 0 \quad \text{for a.e. } x \in \omega \quad \text{and for all } u \geqq 0,\\ g(x,u) &> 0 \quad \text{for a.e. } x \in \omega \quad \text{and for all } u \in I, \end{aligned}$$

where ω (respectively I) is some nonempty open subset of Ω (respectively $(0, +\infty)$).

COROLLARY 2.4. *Assume that* (2.3)–(2.6) *and* (2.70) *hold. Then there is some* $\mu_0 \geqq 0$ *such that problem* (2.69) *is solvable for each* $\mu \geqq \mu_0$.

EXAMPLE 2.4. Corollary 2.4 applies to the problem

$$(2.71)\qquad \begin{aligned} -\Delta u &= u^5 + \mu u^q && \text{on } \Omega \text{ with } 1 < q \leqq 3,\\ u &> 0 && \text{on } \Omega,\\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

It says that there is some μ_0 (which depends on q and Ω) such that problem (2.71) is solvable for each $\mu \geqq \mu_0$.

Remark 2.3. Numerical computations due to O. Bristeau (at INRIA) concerning the problem (2.71) when Ω is a ball suggest that the following holds:

(a) *If* $q = 3$, there is some $\mu_0 > 0$ such that
  (i) for $\mu > \mu_0$ there is a *unique* solution of (2.71),
  (ii) for $\mu \leqq \mu_0$ there is *no* solution of (2.71).

(b) *If* $1 < q < 3$, there is some $\mu_0 > 0$ such that
  (i) for $\mu > \mu_0$ there are *two* solutions of (2.71),
  (ii) for $\mu = \mu_0$ there is a *unique* solution of (2.71),
  (iii) for $\mu < \mu_0$ there is *no* solution of (2.71).

Here is a related result in $\mathbb{R}^3$.

THEOREM 2.4. *Let* Ω *be strictly starshaped about the origin*; *in* Ω, u *is a solution of* (2.71) *with* $1 < q \leqq 3$. *Then*

$$\mu \geqq \mu_0(q, \Omega) > 0.$$

Proof: We carry out the proof only for the case $q=3$ — if $q<3$, the argument is a bit simpler. By Pohozaev's identity (1.3),

$$\mu\int_\Omega u^4 = 2\int_{\partial\Omega}(x\cdot\nu)\left(\frac{\partial u}{\partial\nu}\right)^2$$
$$\geqq c\left(\int_\Omega |\Delta u|\right)^2$$
$$\geqq c[u]^2_{3,w}.$$

From the equation, we also have $|\Delta u|\geqq u^5$ and hence

$$\mu\int_\Omega u^4 \geqq c\|u\|_5^{10}.$$

Using the interpolation inequality:

$$\|u\|_4^4 \leqq K[u]^{3/2}_{3,w}\|u\|_5^{5/2},$$

and combining with the previous inequalities we find

$$\int_\Omega u^4 \leqq C\left(\mu\int u^4\right)^{3/4}\left(\mu\int u^4\right)^{1/4}$$

and the claim follows.

Proof of Corollary 2.4: We use again Theorem 2.1. But here we shall verify condition (2.8), directly, without invoking Lemma 2.1. We fix $v_0(x)=\phi(x)|x|^{-k}$ (provided $0\in\omega$) with $0<k<\frac{1}{2}$, $\phi\in\mathcal{D}_+(\omega)$, $\phi(0)=1$ and $\|v_0\|_6=1$. We have

$$\Psi_\mu(tv_0)=\tfrac{1}{2}At^2-\tfrac{1}{6}t^6-\mu\int G(x,tv_0),$$

where $A=\int\{|\nabla v_0|^2-av_0^2\}$.

We claim that

$$\lim_{\mu\to+\infty}\sup_{t\geqq 0}\Psi_\mu(tv_0)=0, \tag{2.72}$$

and therefore condition (2.8) is satisfied when μ is large enough. First note that $\lim_{t\to+\infty}\Psi_\mu(tv_0)=-\infty$ and thus $\sup_{t\geqq 0}\Psi_\mu(tv_0)$ is achieved at some t_μ for which we have

$$t_\mu A - t_\mu^5 - \mu\int g(x,t_\mu v_0)v_0 = 0 \tag{2.73}$$

and therefore $t_\mu \leqq A^{1/4}$. It follows that

$$\lim_{\mu \to +\infty} t_\mu = 0 \tag{2.74}$$

(if not we could find some sequence $t_{\mu_j} \to l > 0$ with $\mu_j \to \infty$, and by (2.73) we would have $\int g(x, v_0)v_0 = 0$—a contradiction with (2.70) and the choice of v_0).

Finally we observe that

$$\sup_{t \geqq 0} \Psi_\mu(tv_0) \leqq \tfrac{1}{2}At_\mu^2 - \tfrac{1}{6}t_\mu^6$$

and we deduce (2.72) from (2.74).

2.6. Existence of positive solutions for $-\Delta u = \lambda(1+u)^p$ on Ω, $u = 0$ on $\partial\Omega$, where $p = (n+2)/(n-2)$ and $\lambda > 0$. We conclude with another application answering a question mentioned to us by P. Rabinowitz. Let $\Omega \subset \mathbb{R}^n$, $n \geqq 3$, be a bounded domain. We consider the following problem:

$$\begin{aligned} -\Delta u &= \lambda(1+u)^p \quad &&\text{on} \quad \Omega, \\ u &> 0 &&\text{on} \quad \Omega, \\ u &= 0 &&\text{on} \quad \partial\Omega, \end{aligned} \tag{2.75}$$

where $p = (n+2)/(n-2)$ and $\lambda > 0$ is a constant.

COROLLARY 2.5. *There is a constant $\bar\lambda > 0$ such that for every $\lambda \in (0, \bar\lambda)$ there exist at least two solutions of* (2.75); *there is a unique solution of* (2.75) *when $\lambda = \bar\lambda$ and no solution for $\lambda > \bar\lambda$.*

Proof: It is known (see J. Keener and H. Keller [17] and M. Crandall and P. Rabinowitz [11]) that there is a $\bar\lambda$ satisfying:

(a) for every $\lambda \in (0, \bar\lambda)$ there is a minimal solution $\underline{u}$ of (2.75) with the property that

(2.76) the least eigenvalue of $-\Delta - \lambda p(1+\underline{u}(x))^{p-1}$ is positive,

(b) for $\lambda = \bar\lambda$ there is a unique solution of (2.75),

(c) for $\lambda > \bar\lambda$ there is no solution of (2.75).

In fact, for this result, no restriction on $p > 1$ is needed.

We look for a second solution u of (2.75) of the form

$$u = \underline{u} + v$$

with $v > 0$ on Ω. Thus we have

$$\begin{aligned} -\Delta v &= \lambda(1+\underline{u}+v)^p - \lambda(1+\underline{u})^p \quad &&\text{on} \quad \Omega, \\ v &> 0 &&\text{on} \quad \Omega, \\ v &= 0 &&\text{on} \quad \partial\Omega. \end{aligned}$$

In other words, we have to find v such that

$$\begin{aligned} -\Delta v &= \lambda v^p + h(x,v) && \text{on } \Omega, \\ v &> 0 && \text{on } \Omega, \\ v &= 0 && \text{on } \partial\Omega, \end{aligned}$$

with $h(x,v) = \lambda(1+\underline{u}(x)+v)^p - \lambda(1+\underline{u}(x))^p - \lambda v^p$. Finally, by stretching, we are reduced to solving

$$\begin{aligned} -\Delta w &= w^p + f(x,w) && \text{on } \Omega, \\ w &> 0 && \text{on } \Omega, \\ w &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{2.77}$$

with $f(x,w) = (1/k)h(x,kw)$ and $\lambda k^{p-1} = 1$.

Clearly, $f(x,w)$ satisfies the assumptions (2.2)–(2.5) with $a(x) = \lambda p(1+\underline{u}(x))^{p-1}$ and (2.6) holds in view of (2.76). We examine now separately the cases $n \geqq 5$, $n = 4$, and $n = 3$.

(i) *Case* $n \geqq 5$. Since $\underline{u} \in L^\infty(\Omega)$, there is some $\mu > 0$ such that

$$f(x,w) \geqq \mu \quad \text{for all} \quad x \in \Omega, \quad \text{for all} \quad w \in [1,2].^8 \tag{2.78}$$

Therefore we may use Corollary 2.1 and obtain a solution of (2.77).

(ii) *Case* $n = 4$ $(p = 3)$. Here we have

$$h(x,v) = 3\lambda(1+\underline{u}(x))v^2 + 3\lambda(1+\underline{u}(x))^2 v$$

and so

$$f(x,w) \geqq 3\lambda w \quad \text{for all} \quad x \in \Omega, \quad \text{and for all } w \geqq 0.$$

We use now Corollary 2.2 and obtain a solution of (2.77).

(iii) *Case* $n = 3$ $(p = 5)$. Here we have

$$\begin{aligned} h(x,v) = {} & 5\lambda(1+\underline{u}(x))v^4 + 10\lambda(1+\underline{u}(x))^2 v^3 + 10\lambda(1+\underline{u}(x))^3 v^2 \\ & + 5\lambda(1+\underline{u}(x))^4 v \end{aligned}$$

and so

$$f(x,w) \geqq 5\lambda^{1/4} w^4 \quad \text{for all} \quad x \in \Omega, \quad \text{and for all} \quad w \geqq 0.$$

We may use Corollary 2.3 and obtain a solution of (2.77).

[8] Note that since $p > 1$ we have $(a+b)^p - a^p - b^p > 0$ for all $a > 0$, and for all $b > 0$; we deduce (2.78) from a standard continuity and compactness argument.

Remark 2.4. When $p<(n+2)/(n-2)$ a result similar to Corollary 2.5 has been proved by M. Crandall and P. Rabinowitz [11]. When $1<p\leqq(n+2)/(n-2)$ and Ω is a ball, it is known that (2.75) has exactly two solutions for $\lambda<\bar{\lambda}$ (see D. Joseph and T. Lundgren [15]).

Appendix

We shall prove inequality (1.53). Using $u=\phi(x)(\varepsilon+|x|^2)^{1-n/2}$ and the expansion in ε of subsections 1.1 and 1.2, one sees that (1.53) cannot hold for $q=n/(n-2)$. However, E. Lieb, by a quite different approach, recently derived an improved version, in which $\|u\|_q$ is replaced by $[u]_{n/(n-2),w}$, and derived even more general forms.

Proof of (1.53): By symmetrization we may assume that Ω is a ball. Let

$$\tilde{S}_\lambda=\inf_{\substack{u\in H_0^1\\ \|u\|_{2n/(n-2)}=1}}\left\{\int|\nabla u|^2-\lambda\|u\|_q^2\right\}.$$

Inequality (1.53) asserts that $\tilde{S}_\lambda=S$ for some positive $\lambda=\lambda_q$. Suppose not; i.e., suppose that

$$\tilde{S}_\lambda<S \quad \text{for } \textit{all} \quad \lambda>0.$$

As in Lemma 1.2, it follows that $\tilde{S}_\lambda$ is achieved by some u. After stretching we obtain a solution of

$$\text{(A.1)}\qquad \begin{aligned} -\Delta u&=u^{(n+2)/(n-2)}+\lambda\frac{u^{q-1}}{\|u\|_q^{q-2}} &&\text{in } \Omega,\\ u&>0 &&\text{in } \Omega,\\ u&=0 &&\text{on } \partial\Omega.\end{aligned}$$

By Pohozaev's identity (1.3), we find

$$\begin{aligned}\lambda\left(\frac{n}{q}+1-\tfrac{1}{2}n\right)\|u\|_q^2&=\tfrac{1}{2}\int_{\partial\Omega}(x\cdot\nu)\left(\frac{\partial u}{\partial\nu}\right)^2\\ &\geqq C\left(\int_\Omega|\Delta u|\right)^2\\ &\geqq C[u]^2_{n/(n-2),w}\geqq C\|u\|_q^2.\end{aligned}$$

Thus $\lambda\geqq\lambda_0>0$—a contradiction.

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Bibliography

[1] Ambrosetti, A., and Rabinowitz, P., *Dual variational methods in critical point theory and applications*, J. Funct. Anal. 14, 1973, pp. 349–381.

[2] Aubin, Th., *Problèmes isopérimétriques et espaces de Sobolev*, J. Diff. Geom. 11, 1976, pp. 573–598.

[3] Aubin, Th., *Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*, J. Math. Pures et Appl. 55, 1976, pp. 269–293.

[4] Bliss, G. A., *An integral inequality*, J. London Math. Soc. 5, 1930, pp. 40–46.

[5] Brezis, H., and Coron, J. M., *Multiple solutions of H-systems and Rellich's conjecture*, Comm. Pure Appl. Math., to appear.

[6] Brezis, H., Coron, J. M., and Nirenberg, L., *Free vibrations for a nonlinear wave equation and a theorem of P. Rabinowitz*, Comm. Pure Appl. Math. 33, 1980, pp. 667–689.

[7] Brezis, H., and Kato, T., *Remarks on the Schrödinger operator with singular complex potential*, J. Math. Pures et Appl. 58, 1979, pp. 137–151.

[8] Brezis, H., and Lieb, E., *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc., to appear.

[9] Coffman, C. V., *Uniqueness of the ground state solution for $\Delta u - u + u^3 = 0$ and a variational characterization of other solutions*, Arch. Rat. Mech. Anal. 46, 1972, pp. 81–95.

[10] Coffman, C. V., *A non-linear boundary value problem with many positive solutions*, to appear.

[11] Crandall, M. G., and Rabinowitz, P., *Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems*, Arch. Rat. Mech. Anal. 58, 1975, pp. 207–218.

[12] Ekeland, I., *On the variational principle*, J. Math. Anal. Appl. 47, 1974, pp. 324–353.

[13] Gidas, B., Ni, W. M., and Nirenberg, L., *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. 68, 1979, pp. 209–243.

[14] Jacobs, S., *An isoperimetric inequality for functions analytic in multiply connected domains*, Report Mittag-Leffler Institut, 1970.

[15] Joseph, D., and Lundgren, T., *Quasilinear Dirichlet problems driven by positive sources*, Arch. Rat. Mech. Anal. 49, 1973, pp. 241–269.

[16] Kazdan, J., and Warner, F., *Remarks on some quasilinear elliptic equations*, Comm. Pure Appl. Math. 28, 1975, pp. 567–597.

[17] Keener, J., and Keller, H., *Positive solutions of convex nonlinear eigenvalue problems*, J. Diff. Eq. 16, 1974, pp. 103–125.

[18] Lieb, E., *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation*, Studies in Appl. Math. 57, 1977, pp. 93–105.

[19] Lieb, E., *Sharp constants in the Hardy-Littlewood-Sobolev inequality and related inequalities*, in Annals of Math., to appear.

[20] Lions, P. L., *On the existence of positive solutions of semilinear elliptic equations*, SIAM Review 24, 1982, pp. 441–467.

[21] Lions, P. L., *The method of concentration—compactness and applications to the best constants*, in preparation.

[22] McLeod, K., and Serrin, J., *Uniqueness of solutions of semilinear Poisson equations*, Proc. Nat. Acad. Sc. USA, 78, 1981, pp. 6592–6595.

[23] Peletier, L. A., and Serrin, J., *Uniqueness of positive solutions of semilinear equations in $\mathbb{R}^n$*, Arch. Rat. Mech. Anal., 81, 1983, pp. 181–197.

[24] Pohozaev, S. I., *Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$*, Soviet Math. Doklady 6, 1965, pp. 1408–1411 (translated from the Russian Dokl. Akad. Nauk SSSR 165, 1965, pp. 33–36).

[25] Rabinowitz, P., *Some global results for nonlinear eigenvalue problems*, J. Funct. Anal. 7, 1971, pp. 487–513.

[26] Rabinowitz, P., *Variational methods for nonlinear elliptic eigenvalue problems*, Indiana Univ. Math. J. 23, 1974, pp. 729–754.

[27] Schaeffer, D., *An example for the plasma problem with infinitely many solutions*, unpublished note.

[28] Talenti, G., *Best constants in Sobolev inequality*, Annali di Mat. 110, 1976, pp. 353–372.

[29] Taubes, C., *The existence of a non-minimal solution to the $SU(2)$ Yang–Mills–Higgs equations on $\mathbb{R}^3$*, to appear.

[30] Trudinger, N., *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Sc. Norm. Sup. Pisa 22, 1968, pp. 265–274.

[31] Uhlenbeck, K. K., *Variational problems for gauge fields*, Seminar on Differential Geometry, S. T. Yau, Editor, Princeton University Press, 1982, pp. 455–464.

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