Free Vibrations for a Nonlinear Wave Equation and a Theorem of P. Rabinowitz*

Dedicated to Philip Hartman on his 65th birthday

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Abstract

A new and simpler proof is given of the result of P. Rabinowitz for nontrivial time periodic solutions of a vibrating string equation $u_{tt}-u_{xx}+g(u)=0$ and Dirichlet boundary conditions on a finite interval. We assume essentially that g is nondecreasing, and $g(u)/u\to\infty$ as $|u|\to\infty$. The proof uses a modified form $(\mathrm{PS})_c$ of the Palais–Smale condition (PS).

0. Introduction

Let $g:\mathbb{R}\to\mathbb{R}$ be a continuous nondecreasing function such that $g(0)=0$. Set $G(t)=\int_0^t g(s)\,ds$. We seek a nontrivial solution of the equation

$$Au+g(u)\equiv u_{tt}-u_{xx}+g(u)=0\,, \qquad 0<x<\pi,\ t\in\mathbb{R}\,, \tag{1}$$

under the boundary conditions

$$u(0,t)=u(\pi,t)=0 \tag{2}$$

and periodicity condition

$$u(x,t+2\pi)=u(x,t)\,. \tag{3}$$

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We assume

$$\lim_{|t|\to\infty} \frac{g(t)}{t} = \infty, \tag{4}$$

(5) there exist constants $\alpha > 0$ and C such that
$\frac{1}{2}\, tg(t) - G(t) \geqq \alpha\, |g(t)| - C$ for all t.

Our purpose is to provide a new proof of the following theorem which is essentially due to P. Rabinowitz [9].

THEOREM 1. *There exists a nontrivial (weak) solution $u \in L^\infty$ of* (1), (2), (3).

Remarks. 1. We show in fact that there exists a nontrivial solution with any given time period T which is a rational multiple of π. By a nontrivial solution we mean that $g(u(x,t)) \neq 0$ on a set (x, t) of positive measure; in particular, $u(x, t) \neq 0$ on that set. $N(A)$ and $R(A)$ will denote the kernel and range of A.

2. Our proof relies on an elementary but useful variational principle: the "mountain pass" theorem of Ambrosetti and Rabinowitz [1]. Our approach was stimulated by the paper of I. Ekeland [6] and especially by his use of a "duality" argument in conjunction with the mountain pass theorem. The main observation is that problem (1), (2), (3) can be formulated as a variational problem in $R(A)$ "ignoring" the component in $N(A)$. The component of u in $N(A)$—which is usually the most difficult to control—appears here as a "Lagrange multiplier".

3. We have slightly weakened the assumptions of P. Rabinowitz [9]. He also assumes that $\lim_{t\to 0} (g(t)/t) = 0$; or, more generally, he solves $Au + au + g(u) = 0$ with $a \geqq 0$ and g nondecreasing with $\lim_{t\to 0} (g(t)/t) = 0$ (see Theorem 5.13 in [9]). In place of (4), (5) he assumes that there exist $\theta \in [0, \frac{1}{2})$ and C such that

$$G(t) \leqq \theta\, tg(t) \quad \text{for} \quad |t| \geqq c,$$

which is a slightly stronger assumption than (4), (5).

4. In case g is C^∞ and strictly increasing, it was shown in [4] and [9] that any solution is in C^∞.

In order to make the proof more transparent we start with the case where $g(u) = |u|^{p-2}\, u$, $p > 2$, which is especially easy, and prove the existence of a nontrivial 2π-periodic solution.

1. The Case $g(u)=|u|^{p-2}u$, $p>2$

Set $\Omega=(0,\pi)\times(0,2\pi)$.

The kernel N of the operator $Au=u_{tt}-u_{xx}$ acting on functions in L^1 satisfying (2), (3) consists of functions of the form

$$N=\left\{p(t+x)-p(t-x);\ p \text{ is } 2\pi\text{-periodic, } p\in L^1_{\mathrm{loc}}(\mathbb{R}) \quad \text{and} \quad \int_0^{2\pi} p=0\right\}$$

(N is closed in L^1).

Recall that given $f\in L^1(\Omega)$ such that $\int_\Omega f\phi=0$ for all $\phi\in N\cap L^\infty$, there exists a unique function $u\in C(\bar{\Omega})$ such that $Au=f$ and $\int_\Omega u\phi=0$ for all $\phi\in N$.

In fact an explicit formula for u is given by Lovicarová [7]:

$$u(x,t)=\psi(x,t)+p(t+x)-p(t-x)\,, \tag{6}$$

where

$$\psi(x,t)=-\frac{1}{2}\int_x^\pi d\xi\int_{t+x-\xi}^{t-x+\xi} f(\xi,\tau)\,d\tau+c\frac{(\pi-x)}{\pi}\,,$$

$$c=\frac{1}{2}\int_0^\pi d\xi\int_{t-\xi}^{t+\xi} f(\xi,\tau)\,d\tau \qquad (c \text{ is a constant})\,,$$

$$p(y)=\frac{1}{2\pi}\int_0^\pi[\psi(s,y-s)-\psi(s,y+s)]\,ds\,.$$

We set $u=Kf(=A^{-1}f)$. We list some properties of K which are well known or easily verified:

$$\|Kf\|_{L^\infty}\leqq C\,\|f\|_{L^1}\,, \tag{7}$$

$$\int_\Omega (Kf)g=\int_\Omega f(Kg) \quad \text{for all} \quad f,g, \tag{8}$$

$$\|Kf\|_{C^{0,\alpha}}\leqq C\,\|f\|_{L^q} \quad \text{with} \quad \alpha=1-1/q\,. \tag{9}$$

In particular K is a compact selfadjoint operator in $\{f\in L^2;\ \int f\phi=0$ for all $\phi\in N\cap L^\infty\}$; its eigenvalues are $1/(j^2-k^2)$, $j=1,2,3,\cdots$, and $k=0,1,2,\cdots$, $j\neq k$.

Consider the space (here $1/p+1/p'=1$)

$$E=\left\{v\in L^{p'}(\Omega);\int_\Omega v\phi=0\quad\text{for all}\quad \phi\in N\cap L^p\right\}$$

provided with the $L^{p'}$ norm. It follows from (9) that

$$K:E\to L^p \text{ is compact}. \tag{9'}$$

On E we define

$$F(v)=\frac{1}{2}\int_\Omega (Kv)v+\frac{1}{p'}\int_\Omega |v|^{p'}.$$

It is clear that F is C^1 on E; in fact,

$$\langle F'(v),\zeta\rangle_{E^*,E}=\int_\Omega (Kv)\zeta+\int_\Omega |v|^{p'-2}v\zeta\quad\text{for all}\quad v\text{ and }\zeta\in E.$$

Using the Hahn–Banach theorem we may write

$$Kv+|v|^{p'-2}v=w+\chi \tag{10}$$

with $w\in L^p$, $\|w\|_{L^p}=\|F'(v)\|_{E^*}$, $\chi\in N\cap L^p$.

Mountain Pass Theorem (cf. [1]). *Assume F satisfies the Palais–Smale condition:*

(PS) *Whenever a sequence $\{v_j\}$ in E satisfies*
$|F(v_j)|\leqq M$ *and* $F'(v_j)\to 0$ *in* E^*,

there exists a subsequence of v_j which converges in E.

Assume also:

(11) *there are constants $r>0$ and $\rho>0$ such that $F(v)\geqq\rho$ for every $v\in E$ with $\|v\|=r$;*

(12) *$F(0)<\rho$ and $F(v_0)<\rho$ for some $v_0\in E$ with $\|v_0\|>r$.*

Then $c=\inf_{\gamma\in\mathcal{P}}\max_{v\in\gamma}F(v)\geqq\rho$ is a critical value. Here $\mathcal{P}$ denotes the class of all paths γ joining 0 to v_0.

VERIFICATION OF (PS). We write

$$Kv_j + |v_j|^{p'-2} v_j = w_j + \chi_j$$

with

$$w_j \in L^p, \qquad \|w_j\|_{L^p} \to 0, \qquad \chi_j \in N \cap L^p .$$

We have

$$\left|\frac{1}{2}\int_\Omega (Kv_j)v_j + \frac{1}{2}\int_\Omega |v_j|^{p'}\right| = \frac{1}{2}\left|\int_\Omega w_j v_j\right| \leqq \frac{1}{2}\|w_j\|_{L^p}\|v_j\|_{L^{p'}},$$

$$\left|\frac{1}{2}\int_\Omega (Kv_j)v_j + \frac{1}{p'}\int_\Omega |v_j|^{p'}\right| \leqq M .$$

Therefore,

$$\left(\frac{1}{p'} - \frac{1}{2}\right)\int_\Omega |v_j|^{p'} \leqq M + \frac{1}{2}\|w_j\|_{L^p}\,\|v_j\|_{L^{p'}}$$

and so v_j remains bounded in $L^{p'}$. We extract a subsequence—still denoted by v_j—such that v_j converges weakly to v in E. By the convexity of the function $|t|^{p'}$ we have

$$\begin{aligned}\frac{1}{p'}|v|^{p'} - \frac{1}{p'}|v_j|^{p'} &\geqq |v_j|^{p'-2} v_j(v - v_j) \\ &= (w_j + \chi_j - Kv_j)(v - v_j) .\end{aligned}$$

Thus

$$\frac{1}{p'}\int_\Omega |v|^{p'} - \frac{1}{p'}\int_\Omega |v_j|^{p'} \geqq \int_\Omega (w_j - Kv_j)(v - v_j) . \tag{13}$$

It follows from (9′) that the right-hand side of (13) goes to zero, and thus

$$\overline{\lim}\, \|v_j\|_{L^{p'}} \leqq \|v\|_{L^{p'}} .$$

Hence $v_j \to v$ strongly in $L^{p'}$.

VERIFICATION OF (11). Note that, by (7),

$$F(v) \geqq -C\|v\|_{L^1}^2 + \frac{1}{p'}\|v\|_{L^{p'}}^{p'} \geqq -C'\|v\|_{L^{p'}}^2 + \frac{1}{p'}\|v\|_{L^{p'}}^{p'}$$

and the conclusion follows immediately since $p' < 2$.

VERIFICATION OF (12). Simply choose v_0 of the form $v_0 = av_1$, where v_1 is any element in E such that $\int_\Omega (Kv_1)v_1 < 0$ and a is large enough.

We now deduce from the mountain pass theorem that there exists $v \in E$, $v \neq 0$, such that $F'(v) = 0$. Thus $Kv + |v|^{p'-2}v = \chi$ for some $\chi \in N \cap L^P$. Letting $u = \chi - Kv$ we find

$$Au + v = 0\,, \tag{14}$$

$$v = |u|^{p-2}u \equiv g(u)\,. \tag{15}$$

Finally, we show that $u \in L^\infty(\Omega)$. Indeed we have

$$\chi(x, t) = p(t + x) - p(t - x)\,,$$

where

$$p(t) = \frac{1}{2\pi}\int_0^\pi [\chi(x, t - x) - \chi(x, t + x)]\,dx$$

and $p \in L^p(0, 2\pi)$ since $\chi \in L^p(\Omega)$. Since $v \in E$, we have

$$\int_0^\pi [v(x, t - x) - v(x, t + x)]\,dx = 0 \qquad \text{a.e. } t\,. \tag{16}$$

Set $M = \|Kv\|_{L^\infty}$; then

$$-M + p(t + x) - p(t - x) \leqq u(x, t) \leqq M + p(t + x) - p(t - x)$$

and so

$$g(-M + p(t + x) - p(t - x)) \leqq v(x, t) \leqq g(M + p(t + x) - p(t - x))\,.$$

It follows easily from (16) that

$$\int_0^{2\pi} g(-M+p(t)-p(s))\,ds \leqq 0 \qquad \text{a.e. } t$$

and

$$\int_0^{2\pi} g(M+p(t)-p(s))\,ds \geqq 0 \qquad \text{a.e. } t$$

and consequently $p \in L^\infty(0, 2\pi)$.

Remark. Instead of using the "mountain pass" theorem one can give a still more elementary argument. One simply minimizes $\int_\Omega (Kv)v$ on the set $\{v \in E; \|v\|_{L^{p'}} \leqq 1\}$. The minimum is achieved at some $v_0 \in E$ with $\|v_0\|_{L^{p'}} = 1$ and $\int_\Omega (Kv_0)v_0 < 0$. Clearly, we have $Kv_0 + \lambda |v_0|^{p'-2} v_0 = \chi$ for some constant $\lambda > 0$ and some $\chi \in N \cap L^p$. Then $v = \alpha v_0$ satisfies $Kv + |v|^{p'-2} v \in N$ provided α is an appropriate constant. Note that *this proof works as well for* $1 < p < 2$. Unfortunately it relies heavily on the fact that $g(u)$ is homogenous.

2. The Case of a General Function g

Assume first that g is strictly increasing. Set $h = g^{-1}$ and

$$H(t) = \int_0^t h(s)\,ds = G^*(t)\,.$$

(G^* is the conjugate convex function of G.) Given $k > 0$ (k will be fixed, large, later) we set

$$h_k(t) = \begin{cases} h(k) & \text{for} \quad t \geqq k, \\ h(t) & \text{for} \quad |t| \leqq k, \\ h(-k) & \text{for} \quad t \leqq -k. \end{cases}$$

Set $H_k(t) = \int_0^t h_k(s)\,ds$. By (4) and (5) we have

$$\lim_{|s| \to \infty} \frac{h(s)}{s} = 0, \qquad \lim_{|s| \to \infty} \frac{H(s)}{s^2} = 0\,, \tag{17}$$

$$H(s) - \tfrac{1}{2} s h(s) \geqq \alpha |s| - c \quad \text{for all} \quad s\,. \tag{18}$$

(Recall Young's equality $tg(t)=G(t)+H(g(t))$ for all t.) It follows easily that

$$H_k(s)-\tfrac{1}{2}sh_k(s)\geqq \alpha|s|-C \quad \text{for all } s, \tag{19}$$

provided $k\geqq k_0$, where k_0 is large enough so that $h(k_0)\geqq 2\alpha$, $|h(-k_0)|\geqq 2\alpha$. Let $T=2\pi/n$, where n is a large integer to be chosen later. We shall seek a solution of (1), (2), and

$$u(x,t+T)=u(x,t)\,. \tag{20}$$

The kernel N of the operator $Au=u_{tt}-u_{xx}$ acting on functions satisfying (2) and (20) consists of functions of the form

$$N=\left\{p(t+x)-p(t-x);\ p \text{ is } T\text{-periodic, } p\in L^1_{\mathrm{loc}}(\mathbb{R}) \text{ and } \int_0^T p=0\right\}.$$

Set $\Omega=(0,\pi)\times(0,T)$. Given a function $f\in L^1(\Omega)$ such that $\int_\Omega f\phi=0$ for all $\phi\in N\cap L^\infty$, there exists a unique function $u\in C(\bar\Omega)$ such that $Au=f$ and $\int u\phi=0$, $\phi\in N$. In fact, u is given by the same expression as in Section 1 (formula (6)). Set $u=Kf$. We shall work in the Banach space

$$E=\left\{v\in L^1(\Omega);\ \int_\Omega v\phi=0 \qquad \text{for all} \quad \phi\in N\cap L^\infty\right\},$$

provided with the L^1 norm.

We list some properties of K which are easy to check:

$$\|Kf\|_{L^\infty(\Omega)}\leqq \frac{C}{T}\|f\|_{L^1(\Omega)} \qquad \text{for all} \quad f\in E \qquad (C \text{ independent of } T)\,, \tag{21}$$

$$\int_\Omega (Kf)g=\int_\Omega f(Kg) \qquad \text{for all} \quad f,\ g\in E, \tag{22}$$

$$\|Kf\|_{C^{0,\alpha}(\bar\Omega)}\leqq C_T\|f\|_{L^q} \qquad \text{for all} \quad f\in E, \qquad \alpha=1-1/q\,. \tag{23}$$

In particular, K is a compact selfadjoint operator in $E\cap L^2$, its eigenvalues are $1/(j^2-n^2k^2)$, $j=1,2,3,\cdots$ and $k=0,1,2,\cdots,j\neq nk$.

In addition we shall use the following

LEMMA 1. *We have*

$$\int_\Omega (Kf)f \geqq -2\|f\|^2_{L^1(\Omega)} \quad \textit{for all} \quad f \in E. \tag{24}$$

Proof: Recall that $\int_\Omega (Kf)f = \int_\Omega \psi f$, where ψ is defined in (6). Set $\eta(\xi) = \int_0^T f(\xi, \tau)\, d\tau$. Noting that

$$\int_{t+x-\xi}^{t-x+\xi} f(\xi, \tau)\, d\tau = \left[\frac{2(\xi - x)}{T}\right] \eta(\xi) + \text{remainder}$$

($[2(\xi - x)/T]$ denotes the integer part of $2(\xi - x)/T$), we may write

$$\begin{aligned}\psi(x, t) &= -\frac{1}{2}\int_x^\pi \frac{2(\xi - x)}{T}\eta(\xi)\, d\xi + c'\frac{(\pi - x)}{\pi} + R(x, t)\\ &\equiv \zeta(x) + R(x, t),\end{aligned}$$

where

$$c' = \frac{1}{2}\int_0^\pi \frac{2\xi}{T}\eta(\xi)\, d\xi$$

and

$$\|R\|_{L^\infty(\Omega)} \leqq 2\|f\|_{L^1(\Omega)}.$$

Hence

$$\begin{aligned}\int_\Omega \psi f &= \int_0^\pi \zeta(x)\eta(x)\, dx + \int_\Omega Rf\, dx\, dt\\ &\geqq \int_0^\pi \zeta(x)\eta(x)\, dx - 2\|f\|^2_{L^1}.\end{aligned}$$

But

$$-\zeta_{xx}(x) = \frac{1}{T}\eta(x), \qquad \zeta(0) = \zeta(\pi) = 0$$

and so

$$\int_0^{\pi} \zeta\eta = T\int_0^{\pi} (\zeta_x)^2 \geqq 0\,.$$

On the space E we define the function

$$F_k(v) = \tfrac{1}{2}\int_\Omega (Kv)v + \int_\Omega H_k(v)$$

and we shall now apply to F_k the following theorem—which is a variant of the mountain pass theorem. We first introduce a modified form of (PS). Assume F is a Gateaux differentiable function on a Banach space E and let $c \in \mathbb{R}$. We say that F satisfies condition $(\mathrm{PS})_c$ provided:

$(\mathrm{PS})_c$ Whenever a sequence $\{v_j\}$ in E is such that $F(v_j) \to c$ and $F'(v_j) \to 0$ in E^*, then c is a critical value.

Note the difference from (PS) on page 670.

THEOREM 2. *Assume F is Gateaux differentiable and $F' : E \to E^*$ is continuous from the strong topology of E into the weak* topology of E^*. Assume*

(25) *there exist a neighborhood U of 0 and a constant $\rho > 0$ such that $F(v) \geqq \rho$ for every v in the boundary of U,*

(26) $$F(0) < \rho \text{ and } F(v_0) < \rho \quad \text{for some } v_0 \notin U\,.$$

Assume $(\mathrm{PS})_c$ where $c = \inf_{p \in \mathcal{P}} \max_{v \in P} F(v) \geqq \rho$ and $\mathcal{P}$ denotes the class of paths joining 0 to v_0.

Conclusion: c is a critical value.

The proof of Theorem 2—which is an easy modification of a well-known argument—is sketched in the Appendix.

Since $H_k \in C^1(\mathbb{R})$, it is clear that F_k is Gateaux differentiable and that

$$\langle F_k'(v), \zeta\rangle_{E^*,E} = \int_\Omega (Kv)\zeta + \int_\Omega h_k(v)\zeta \quad \text{for all} \quad v \in E \quad \text{and} \quad \zeta \in E\,.$$

For fixed $\zeta \in E$, the mapping $v \to \langle F_k'(v), \zeta\rangle$ is obviously continuous; note that in general F_k is not C^1 on E. Using the Hahn–Banach theorem we may write

$$Kv + h_k(v) = w + \chi \tag{27}$$

with $w \in L^\infty$, $\|w\|_{L^\infty} = \|F_k'(v)\|_{E^*}$ and $\chi \in N \cap L^\infty$.

We shall now verify the assumptions of Theorem 2 for F_k provided k is sufficiently large.

VERIFICATION OF (25). We may take $\rho = \frac{1}{8}$ and $U = \{v \in E; \|v\|_{L^1} < \frac{1}{2}\}$. Indeed, by Young's inequality we have $\pm v \leqq H_k(v) + G_k(\pm 1)$, where G_k denotes the conjugate convex function of H_k so that $G_k(t) = G(t)$ for $t \in [g(-k), g(k)]$. Thus

$$|v| \leqq H_k(v) + G(1) + G(-1),$$

provided k is chosen large enough ($|g(\pm k)| \geqq 1$). Hence, if v lies on the boundary of U we have $\|v\|_{L^1} = \frac{1}{2}$ and

$$\|v\|_{L^1(\Omega)} \leqq \int H_k(v) + (G(1) + G(-1))\,|\Omega|\,.$$

On the other hand, by (24),

$$F_k(v) \geqq -\|v\|_{L^1}^2 + \|v\|_{L^1} - (G(1) + G(-1))\,|\Omega| \geqq \tfrac{1}{8}$$

when $\|v\|_{L^1} = \frac{1}{2}$ and the period T is so small that $(G(1) + G(-1))\pi T \leqq \frac{1}{8}$.

VERIFICATION OF (26). In fact we shall find a v_0 independent of k such that $F_k(v_0) \leqq 0$ and $\int_\Omega H_k(v_0) \geqq \frac{1}{8}$ for all k large enough. Indeed fix $v_1 \in E \cap L^\infty$ such that $\int_\Omega (Kv_1)v_1 < 0$ (choose for example for v_1 an eigenfunction of K corresponding to a negative eigenvalue of K). Assume $\|v_1\|_{L^2} = 1$. Next, fix $\varepsilon > 0$ with $\varepsilon < \frac{1}{2}\left|\int_\Omega (Kv_1)v_1\right|$. By (17) we have, $H(s) \leq \varepsilon s^2 + C$, for all s, for some constant C. Set $v_0 = av_1$ so that

$$\begin{aligned}
F_k(v_0) &= \tfrac{1}{2}a^2 \int_\Omega (Kv_1)v_1 + \int_\Omega H_k(av_1) \\
&\leqq \tfrac{1}{2}a^2 \int_\Omega (Kv_1)v_1 + \int_\Omega H(av_1) \\
&\leqq a^2\left[\varepsilon + \tfrac{1}{2}\int_\Omega (Kv_1)v_1\right] + C|\Omega| \leqq 0\,,
\end{aligned}$$

provided a is large enough. By further increasing a we may assume that $\|av\|_{L^1} > \frac{1}{2}$. Finally we assume that $k \geqq k_0 = \|v_0\|_{L^\infty}$ and then $H_k(v_0) = H(v_0)$.

In order to check $(\mathrm{PS})_c$ we shall use the following

LEMMA 2. *Given $M > 0$, there exist constants k_M and C_M such that, for all $k \geqq k_M$, the set $S_k = \{v \in E; F_k(v) \leqq M$ and $\|F_k'(v)\|_{E^*} \leqq \alpha\}$ is a bounded set in L^∞ and its norm is less than C_M (α occurs in (5)).*

Proof: Let $v \in S_k$; we have

$$Kv + h_k(v) = w + \chi \tag{28}$$

with $\|w\|_{L^\infty} \leqq \alpha$ and $\chi \in N \cap L^\infty$. Thus

$$\left|\tfrac{1}{2}\int_\Omega (Kv)v + \tfrac{1}{2}\int_\Omega h_k(v)v\right| = \tfrac{1}{2}\left|\int_\Omega wv\right| \leqq \tfrac{1}{2}\alpha\, \|v\|_{L^1}$$

and

$$\tfrac{1}{2}\int_\Omega (Kv)v + \int_\Omega H_k(v) \leqq M .$$

Consequently,

$$\int_\Omega [H_k(v) - \tfrac{1}{2}h_k(v)v] \leqq M + \tfrac{1}{2}\alpha\|v\|_{L^1}$$

and by (19)

$$\tfrac{1}{2}\alpha\|v\|_{L^1} \leqq M + C|\Omega| .$$

Hence

$$\|v\|_{L^1} \leqq C_M , \tag{29}$$

where, in what follows, C_M denote various constants depending only on M. Set

$$u = \chi - Kv \tag{30}$$

so that by (28) we have

$$h_k(v) = w + u . \tag{31}$$

Note that

$$g(h_k(s)) = \tau_k(s) = \begin{cases} k & \text{if} \quad s \geqq k\,, \\ s & \text{if} \quad |s| \leqq k\,, \\ -k & \text{if} \quad s \leqq -k\,. \end{cases}$$

Therefore, by (31), we have

$$\tau_k(v) = g(w+u)\,. \tag{32}$$

On the other hand, since $v \in E$, we have

$$\int_0^{\pi} [v(x, t-x) - v(x, t+x)]\, dx = 0 \qquad \text{for a.e. } t \tag{33}$$

and since $\chi \in N \cap L^\infty$ we may write

$$\chi(x, t) = p(t+x) - p(t-x)$$

for some $p \in L^\infty$ which is T-periodic and such that $\int_0^T p = 0$. It follows from (28), (29), and (17) that

$$\|\chi\|_{L^1} \leqq C_M \tag{34}$$

and therefore

$$\|p\|_{L^1(0,T)} \leqq C_M\,. \tag{35}$$

Next we estimate $\|p\|_{L^\infty}$ using the same device as in [2] (see also [4]).
Set

$$\mu = \operatorname*{ess\,sup}_{(0,T)} p.$$

We have

$$w + u = w - Kv + \chi$$

and since $\|w - Kv\|_{L^\infty} \leqq C_M$ it follows from (32) that

$$g(-C_M + p(t+x) - p(t-x)) \leqq \tau_k(v(x, t)) \leqq g(C_M + p(t+x) - p(t-x))\,.$$

In particular,

$$g(-C_M+p(t)-p(t-2x)) \leqq \tau_k(v(x,t-x)) . \tag{36}$$

Choosing t_0 such that $p(t_0) \geqq \mu-1$ we see that

$$g(-C_M-1) \leqq \tau_k(v(x,t_0-x)) \quad \text{for a.e. } x .$$

If we take $k \geqq |g(-C_M-1)|$ we find that

$$v(x,t_0-x) \geqq -k \quad \text{for a.e. } x ,$$

and in particular

$$\tau_k(v(x,t_0-x)) \leqq v(x,t_0-x) .$$

Therefore, by (36),

$$g(-C_M+p(t_0)-p(t_0-2x)) \leqq v(x,t_0-x) \quad \text{for a.e. } x .$$

Similarly if we choose $k \geqq g(C_M+1)$ we obtain

$$g(C_M+p(t_0+2x)-p(t_0)) \geqq v(x,t_0+x) \quad \text{for a.e. } x .$$

We deduce now from (33) that

$$\int_0^\pi [g(-C_M+p(t_0)-p(t_0-2x))-g(C_M+p(t_0+2x)-p(t_0))]\,dx \leqq 0 ,$$

i.e.,

$$\int_0^{2\pi} \tilde{g}(-C_M+p(t_0)-p(s))\,ds \leqq 0 , \tag{37}$$

where $\tilde{g}(u)=g(u)-g(-u)$. Let

$$\Sigma=\{s\in(0,2\pi);\, p(s)\geqq \tfrac{1}{2}\mu\} \quad \text{and} \quad \Sigma^c \text{ its complement.}$$

It follows from (35) that

$$\operatorname{meas}\Sigma \leqq \frac{2C_M}{\mu} \quad \text{and} \quad \operatorname{meas}\Sigma^c \geqq \left(2\pi-\frac{2C_M}{\mu}\right).$$

Splitting the integral in (37) on Σ and Σ^c we obtain

$$(2\pi-(2C_M/\mu))\tilde{g}(-C_M-1+\tfrac{1}{2}\mu)\leqq 2\pi\tilde{g}(C_M+1)$$

which provides a bound for μ in terms of C_M (assuming $k\geqq \max\{|g(-C_M-1)|, g(C_M+1)\}$). We estimate in the same way $\operatorname{ess\,inf}_{(0,T)} p$. Thus we have proved that $\|\chi\|_{L^\infty}\leqq C_M$ and also (by (30)) $\|u\|_{L^\infty}\leqq C_M$. Finally, we derive from (32) the bound $\|v\|_{L^\infty}\leqq C_M$ provided $k\geqq k_M$ (k_M sufficiently large).

VERIFICATION OF $(\mathrm{PS})_c$. Let

$$c_k=\inf_{P\in\mathscr{P}}\max_{v\in P} F_k(v)$$

($\mathscr{P}$ denotes the class of paths joining 0 to v_0—which we recall, is independent of k). In particular,

$$c_k\leqq \max_{s\in[0,1]} F_k(sv_0)\leqq \int_\Omega H(v_0)\,.$$

Set $M=\int_\Omega H(v_0)+1$. Let v_j be a sequence in E such that $F_k(v_j)\to c_k$ and $F_k'(v_j)\to 0$ in E^*. We wish to prove that c_k is a critical value of F_k. We may always assume that

$$F_k(v_j)\leqq M \quad\text{and}\quad \|F_k'(v_j)\|\leqq\alpha$$

and so, by Lemma 2, the v_j are uniformly bounded in L^∞ provided $k\geqq k_0$. Extracting a subsequence we may assume that $v_j\to v$ weakly in w^*L^∞ and also $Kv_j\to Kv$ in $C(\bar{\Omega})$ (by (23)). We use now the same monotonicity device as in [3], [4]. Let $\zeta\in E$; we have

$$\int_\Omega (h_k(v_j)-h_k(\zeta))(v_j-\zeta)\geqq 0\,.$$

On the other hand, we know that

$$Kv_j+h_k(v_j)=w_j+\chi_j$$

with $\|w_j\|_{L^\infty}\to 0$ and $\chi_j\in N\cap L^\infty$. It follows that

$$\int_\Omega (w_j-Kv_j-h_k(\zeta))(v_j-\zeta)\geqq 0$$

and in the limit

$$\int_\Omega (-Kv - h_k(\zeta))(v-\zeta) \geqq 0 \qquad \text{for all} \quad \zeta \in E .$$

Choosing $\zeta = v + t\eta$, $\eta \in E$, $t > 0$, we conclude easily that

$$\int_\Omega (Kv + h_k(v))\eta = 0 \qquad \text{for all} \quad \eta \in E ,$$

i.e., $F_k'(v) = 0$. Finally we have, by the convexity of H_k,

$$\begin{aligned}\int_\Omega H_k(v) - H_k(v_j) &\geqq \int_\Omega h_k(v_j)(v - v_j) \\ &= \int_\Omega (w_j - Kv_j)(v - v_j) .\end{aligned}$$

Since the right-hand side tends to 0 we conclude that

$$\overline{\lim} \int_\Omega H_k(v_j) \leqq \int_\Omega H_k(v) ,$$

and by lower semicontinuity, we have

$$\underline{\lim} \int_\Omega H_k(v_j) \geqq \int_\Omega H_k(v) .$$

Thus $\int_\Omega H_k(v_j) \to \int_\Omega H_k(v)$ and so $F_k(v_j) \to F_k(v)$. It follows that c_k is a critical value.

Conclusion of the Proof of Theorem 1: By Theorem 2 we know that for each $k \geqq k_0$ there exists $v_k \in E \cap L^\infty$ such that

$$F_k'(v_k) = 0 \quad \text{and} \quad F_k(v_k) = c_k \geqq \tfrac{1}{16} .$$

On the other hand, $c_k \leqq \int H(v_0)$ and therefore, by Lemma 2, v_k remains bounded in L^∞. Choosing k large enough we have

$$Kv_k + h(v_k) = Kv_k + h_k(v_k) = \chi_k \in N \cap L^\infty .$$

Letting $u = \chi_k - Kv_k$ we obtain a nontrivial solution of $Au + g(u) = 0$. Finally, in the case where g is not strictly monotone we replace $g(u)$ by $g_\varepsilon(u) = g(u) + \varepsilon u$ and then pass to the limit as $\varepsilon \to 0$ using the same technique as above. We omit the details.

Appendix

Proof of Theorem 2

The proof follows a well-known argument, as in P. Rabinowitz [8]. (K. C. Chang [5] has recently proved a very general result of this kind for functions F which are merely locally Lipschitz in a reflexive Banach space.) Before describing the proof of the theorem we observe first that F is locally Lipschitz, since F' is locally bounded by the uniform boundedness principle.

Suppose c is not a critical value. By condition $(\mathrm{PS})_c$, there exist $\bar{\varepsilon}, b > 0$ such that $\|F'(x)\| \geqq b$ in $\tilde{E} = \{x \in E; c - \bar{\varepsilon} \leqq F(x) \leqq c + \bar{\varepsilon}\}$. We may take $\bar{\varepsilon} < \min(\rho - F(0), \rho - F(v_0))$; then 0 and v_0 are not in $\tilde{E}$.

By Lemma 1.6 of [8], on $\tilde{E}$ there is a pseudogradient vector field v for F, i.e., a locally Lipschitz continuous vector field satisfying

$$\|v(x)\| \leqq 2\|F'(x)\|,$$
$$\langle F'(x), v(x)\rangle \geqq \|F'(x)\|^2,$$

so that

$$\|v(x)\| \geqq \|F'(x)\| \geqq b.$$

Lemma 1.6 of [8] assumes F to be in C^1 but the proof works exactly the same under our condition on F. Set $\tilde{\tilde{E}} = \{x \in E; c - \frac{1}{2}\bar{\varepsilon} < F(x) < c + \frac{1}{2}\bar{\varepsilon}\}$ and let $0 \leqq g(x) \leqq 1$ be a locally Lipschitz continuous function on E satisfying

$$g \equiv \begin{cases} 1 & \text{on } \tilde{\tilde{E}}, \\ 0 & \text{outside } \tilde{E}, \end{cases}$$

and define the vector field in E:

$$V(x) = \begin{cases} -g(x)\dfrac{v(x)}{\|v(x)\|^2} & \text{in } \tilde{E}, \\ 0 & \text{outside } \tilde{E}. \end{cases}$$

Clearly, $V(x)$ is locally Lipschitz and $\|V(x)\| \leqq b^{-1}$.

Consider the flow $y(t) = y(t, x)$ defined by

$$\frac{dy}{dt} = V(y), \quad y|_{t=0} = x \quad \text{for} \quad x \in E.$$

There is a unique solution $y(t) = y(t, x)$ for $0 \leqq t < \infty$ satisfying

$$\text{(A1)} \qquad \text{for all } t, \quad y(t, x) \equiv x \quad \text{if } x \text{ is not in } \tilde{E}.$$

Furthermore we have

$$\text{(A2)} \qquad \frac{d}{dt} F(y(t)) = \langle F'(y(t)), V(y(t)) \rangle \leqq -\tfrac{1}{4} g(y(t)).$$

Indeed, $y(t + \delta t) = y(t) + \delta t V(y(t)) + o(\delta t)$ and since F is locally Lipschitz

$$\begin{aligned} F(y(t + \delta t)) &= F(y(t)) + \delta t V(y(t)) + o(\delta t) \\ &= F(y(t)) + \delta t < F'(y(t)), V(y(t)) > + o(\delta t) \end{aligned}$$

and the result follows.

Now, by the definition of c, there is a path P joining 0 to v_0 such that $F(x) \leqq c + \frac{1}{2}\bar{\varepsilon}$ on P. From (A1) and (A2) it follows easily that the path $P(4\bar{\varepsilon})$, i.e., the points $y(4\bar{\varepsilon}, x)$ for $x \in P$, is also a path joining 0 to v_0, and on it we have $F(y) \leqq c - \frac{1}{2}\bar{\varepsilon}$—a contradiction.

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