# Forced Vibrations for a Nonlinear Wave Equation

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## Introduction

This paper treats forced vibrations for a nonlinear wave equation of the form

$$(1.1) \qquad u_{tt} - u_{xx} \pm F(x, t, u) = 0$$

on $0 < x < \pi$, under the boundary conditions

$$(1.2) \qquad u(0, t) = u(\pi, t) = 0 .$$

We seek solutions which are periodic in time with a prescribed period

$$T = 2\pi/\lambda , \qquad \lambda \text{ rational} .$$

Set $\Omega = (0, \pi) \times (0, T)$. $F$ is assumed to be periodic in $t$ with period $T$, continuous in $\bar{\Omega} \times$ , and to satisfy

HYPOTHESES. (i) $F$ is nondecreasing in $u$ for all $(x, t) \in \Omega$;

(ii) $|F(x, t, u)| \to \infty$ as $|u| \to \infty$ for all $(x, t) \in \Omega$, and there is some $u_0 \in L^2(\Omega)$ such that $F(x, t, u_0(x, t)) \equiv 0$.

In Sections 1–3 we treat the case $\lambda = 1$ and in Section 4 we indicate the necessary changes for any rational $\lambda = a/b$. With $\lambda = 1$, assume that $F$ satisfies, for some constants $\gamma$, $C$,

(iii) $|F| \leqq \gamma |u| + C$
with $\gamma < 3$ or $\gamma < 1$ according to whether we have + or − in (1.1).

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The kernel $N$ of the operator

$$A = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$$

acting on functions satisfying (1.2) and periodic in $t$ of period $2\pi$ consists of functions of the form

$$p(t+x) - p(t-x),$$

$p$ periodic of period $2\pi$; we may suppose that

$$\int_0^{2\pi} p(t)\, dt = 0 .$$

All the functions we consider are assumed to be time-periodic with period $T$. Any $u \in L^2(\Omega)$ has the orthogonal decomposition

$$u = u_1 + u_2 , \qquad u_2 \in N,\ u_1 \in N^\perp .$$

The range of $A$, $R(A)$, in $L^2$ is $N^\perp$.

The reason one can treat $\lambda$ rational but not irrational is that for $\lambda$ rational the range of $A$, acting on $D(A)$ *in* $L^2$, is *closed* while this is usually not the case for $\lambda$ irrational. For $\lambda$ rational the kernel of $A$ is infinite-dimensional, and this indeed is the main difficulty. But $A^{-1}$ acting on the *orthogonal complement* of the kernel is *compact.*

Our first main result is

THEOREM 1. *Assume* $\lambda = 1$ *and that* $F$ *satisfies conditions* (i), (ii), (iii). *Then there exists a generalized solution of* (1.1) *and* (1.2),

$$u = u_1 + u_2 ,$$

*with* $u_1 \in C^{0,1}$, $u_2 \in L^\infty$.

Here $C^{0,1}$ is the space of Lipschitz continuous functions in $\bar{\Omega}$.

*Remark.* In general, if $F$ is not strictly increasing, solutions need not be smooth. For instance if $F \equiv 0$ for $|u| \leqq k$, then there is a solution which is in $L^\infty$ but not smooth:

$$u = p(t+x) - p(t-x) , \qquad p \text{ periodic, period } 2\pi ,$$

with $p \in L^\infty$, $\sup|p| \leqq \frac{1}{2}k$ and $p$ not smooth. In Section 3, under further conditions on $F$, we prove that every such solution is regular.

A number of authors have treated the problem (1.1), (1.2) with $F$ of the form $\varepsilon f(x, t, u)$ and $\varepsilon$ small, i.e., a perturbation problem. See Vejvoda [32]–[34] ([33] in particular contains an extensive bibliography), Rabinowitz [22], [23], DeSimon and Torelli [4], and Lovicarova [16] as well as [2], [8], [5], [7], [14], [15], [31], [19]–[21], [29], [27], [17], [28]. In addition some papers take up higher-order operators in the space variables such as [9] and [12], [13]. A number of authors have described interesting formal expansions for the solutions, see for instance [11], [18] and [6].

Few papers study the global nonlinear problem, i.e., without $\varepsilon$. Rabinowitz [24] has treated them; see also [12], [13], [10].

In [1], Section 1.2, we proved that there is an $L^2$ solution of (1.1), (1.2) and that the solution is in $C^\infty(\bar{\Omega})$ in case $F_u > 0$ everywhere, and we shall use that result as a step in our proof. In a recent, striking paper, Rabinowitz [26] has treated (1.1), (1.2) requiring $F$ to be superlinear, i.e., to grow at infinity faster than a power ($>1$) of $u$. He obtains solutions as stationary points of a suitable functional. Furthermore, and this is most striking, he proves the existence of nontrivial solutions $u \not\equiv 0$ even if $F(x, t, 0) \equiv 0$. In general our existence theorem does not gurantee the existence of a nontrivial solution. In Section 5 we introduce some cases in which nontrivial solutions are assured. Our method of proof does not extend to superlinear $F$. In Section 6 we consider $F$ of the form $g(u) - f(x, t)$, with $g(0) = 0$ and $f$ small.

We wish to extend our thanks to P. Rabinowitz for several useful conversations.

Before tackling the theorem we recall some well known facts concerning the operator $A$ in $\Omega$ under (1.2) and its inverse, see Rabinowitz [22], Lovicovara [16], DeSimon and Torelli [4]. If $f \in L^2$ is in the range of $A$, then $f \in N^\perp$ and

$$u_1 = A^{-1} f \in H^1 \cap C^{1/2} \cap N^\perp,$$

i.e., $u_1$ has square integrable first derivatives in $\Omega$ and is Hölder continuous with exponent $\frac{1}{2}$. Furthermore, here $\| \ \|$ denotes $L^2$ norm in $\Omega$,

$$\|u_1\|_{H^1} + |u_1|_{C^{1/2}} \leqq C\|f\|. \tag{1.3}$$

These assertions follow easily from the Fourier series representation of the solution $u_1$, as a sine series in $x$: With

$$f = \sum_{j>0} \sum_{\substack{k \\ |k| \neq j}} f_{jk} \sin jx e^{ikt}, \qquad f_{j,-k} = \overline{f_{j,k}},$$

so that

$$\|f\|_{L^2}^2 = C \sum |f_{jk}|^2,$$

we have

$$A^{-1}f = u_1 = \sum_{j>0} \sum_{|k|\neq j} \frac{f_{jk}}{j^2-k^2} \sin jx e^{ikt}. \tag{1.4}$$

Then one derives, easily, the inequalities:

$$\text{for all } x, \quad \int_0^{2\pi} u_{1t}^2 \, dt \leqq C \|f\|^2,$$

$$\text{for all } t, \quad \int_0^{\pi} u_{1x}^2 \, dx \leqq C \|f\|^2.$$

These, in turn, yield (1.3). To continue, if $f \in C^k(\bar{\Omega})$, then $u_1 \in C^{k+1}(\bar{\Omega})$, $k = 0, 1, 2, \cdots$,

$$|u_1|_{C^{k+1}} \leqq C(k) \, |f|_{C^k}. \tag{1.3'}$$

Similarly, $f \in L^\infty \Rightarrow u_1 \in C^{0,1}$, i.e., $u_1$ is Lipschitz continuous;

$$f \in H^k \Rightarrow u_1 \in H^{k+1}. \tag{1.3''}$$

We shall prove Theorem 1 by first considering the equation, for $\varepsilon > 0$,

$$\pm \varepsilon u_{2\varepsilon} + (\partial_t^2 - \partial_x^2) u_\varepsilon \pm F(x, t, u_\varepsilon) = 0, \qquad u_\varepsilon(0, t) = u_\varepsilon(\pi, t) = 0, \tag{1.5}$$

and then letting $\varepsilon \to 0$. According to Theorem I.8 in [1], under the conditions of Theorem 1 there is a $C^\infty(\bar{\Omega})$ solution $u_\varepsilon$ of (1.5). If $P_2$ is the orthogonal projection of $L^2$ onto $N$, we find from (1.5) (dropping the $\varepsilon$ in $u_\varepsilon$)

$$\varepsilon u_2 + P_2 F(x, t, u) = 0.$$

We shall use the following simple form of this relation (see [4] and [16]): for $u = u_1 + u_2$, $u_2 = p(t+x) - p(t-x)$,

$$\begin{aligned} 0 = \varepsilon p(s) + \frac{1}{2\pi} \int_0^\pi [&F(x, s-x, p(s) - p(s-2x) + u_1(x, s-x)) \\ &- F(x, s+x, p(s+2x) - p(s) + u_1(x, s+x))] \, dx. \end{aligned} \tag{1.6}$$

For convenience we write the integral as

$$\frac{1}{2\pi} \int_0^\pi [F(I) - F(II)] \, dx,$$

using $(I)$ and $(II)$ to denote the respective arguments.

Our aim is to derive estimates for $u_\varepsilon$ independent of $\varepsilon$ and let $\varepsilon \to 0$, to obtain a limit $u$ which solves (1.1) and (1.2). In doing this we follow the setup of DeSimon–Torelli [4] which makes use of (1.6), rather than the method of Rabinowitz in [22] which is set in the variational formulation of the problem and makes use of special variations.

## 2. Proof of Theorem 1

We have to consider the cases + and − in (1.1). Since the arguments differ only slightly we shall suppose that we have the + sign in (1.5):

$$\varepsilon u_2 + Au_1 + F(x, t, u) = 0 . \tag{2.1}$$

(a) We shall first establish a bound for the $L^2$ norm of $F(x, t, u(x, t))$. From now on the letter $C$ is used to denote various constants independent of $\varepsilon$. Taking $L^2$ scalar with $u$ we find

$$(Au_1, u_1) + (F(x, t, u), u) \leqq 0 .$$

From the Fourier series representation formulas in the preceding section we see easily that

$$3(Au_1, u_1) + \|Au_1\|^2 \geqq 0 .$$

On the other hand,

$$F(x, t, u) \cdot (u - u_0) = |F(x, t, u)|\,|u - u_0| \geqq |F(x, t, u)|\,|u| - |F(x, t, u)|\,|u_0|$$

and so

$$F(x, t, u) \cdot u \geqq |F(x, t, u)|\,|u| - 2\,|F(x, t, u)|\,|u_0| .$$

Hence

$$\begin{aligned}\int_\Omega |F|\,|u| &\leqq (F(x, t, u), u) + 2\int |F(x, t, u)|\,|u_0| \\ &\leqq \tfrac{1}{3}\|Au_1\|^2 + 2\int |F(x, t, u)|\,|u_0| \\ &\leqq \tfrac{1}{3}\|F\|^2 + 2\int |F(x, t, u)|\,|u_0| .\end{aligned}$$

Therefore,

$$\frac{1}{\gamma}\int_\Omega |F|\,(|F|-C) \leqq \tfrac{1}{3}\|F\|^2 + 2\int |F(x,t,u)|\,|u_0|$$

and since $\gamma < 3$ we infer that

$$\|F\| \leqq C. \tag{2.2}$$

It now follows from (1.3) that $\|u_1\|_{H^1} + |u_1|_{C^{1/2}} \leqq C$ independent of $\varepsilon$. In particular,

$$\rho = \max|u_1| \leqq C. \tag{2.3}$$

(b) Next we wish to establish the estimate

$$\max|u_2| \leqq C. \tag{2.4}$$

Recall that

$$u_2(x,t) = p(t+x) - p(t-x) \quad \text{and} \quad \int_0^{2\pi} p(t)\,dt = 0.$$

Set $M = \max|p|$; we shall derive an upper bound for $M$ independent of $\varepsilon$.

In (1.6), fix $s$ at the point where $|p(t)|$ takes its maximum. We may suppose $p(s) > 0$. Let

$$\Sigma = \{x \in [0,\pi] \mid p(s) - p(s-2x) \geqq \tfrac{1}{2}M\}.$$

Since

$$0 = \int_0^\pi p(s-2x)\,dx = \int_\Sigma + \int_{\Sigma^c} \geqq -M \operatorname{meas}\Sigma + \tfrac{1}{2}M(\pi - \operatorname{meas}\Sigma),$$

we find that $\operatorname{meas}\Sigma \geqq \frac{1}{3}\pi$. Note that

$$p(s+2x) - p(s) + u_1(x, s+x) \leqq \rho$$

and consequently

$$F(II) \leqq F(x, s+x, \rho) \leqq \gamma\rho + C,$$

so that

$$-\frac{1}{2\pi}\int_0^{\pi} F(II) \geqq -C_\rho .$$

Also

$$p(s)-p(s-2x)+u_1(x, s-x) \geqq -\rho$$

and so

$$F(I) \geqq F(x, s-x, -\rho) \geqq -\gamma\rho - C .$$

Hence

$$\begin{aligned}\frac{1}{2\pi}\int_0^{\pi} F(I) &= \frac{1}{2\pi}\int_{\Sigma} + \frac{1}{2\pi}\int_{\Sigma^c} \\ &\geqq \frac{1}{2\pi}\int_{\Sigma} F(x, s-x, \tfrac{1}{2}M-\rho)\, dx - \tfrac{1}{2}(\gamma\rho + C) .\end{aligned}$$

Using (1.6) we conclude that

$$\int_{\Sigma} F(x, s-x, \tfrac{1}{2}M-\rho)\, dx \leqq C_\rho .$$

In order to emphasize the dependence on $\varepsilon$ we write

$$\int_{\Sigma_\varepsilon} F(x, s_\varepsilon - x, \tfrac{1}{2}M_\varepsilon - \rho)\, dx \leqq C_\rho .$$

Arguing by contradiction suppose that $M_{\varepsilon_n} \to +\infty$; extracting a subsequence we may also assume $s_{\varepsilon_n} \to \bar{s}$. It follows from Egorov's lemma and the assumption $F(x, t, u) \to +\infty$ as $u \to +\infty$, that

for all $L > 0$, there are $\theta$ and $E \subset [0, \pi]$ with meas $E < \frac{1}{6}\pi$

such that

$$F(x, \bar{s}-x, \theta) \geqq L \quad \text{for} \quad x \in [0, \pi]\backslash E .$$

Therefore,

$$F(x, s_{\varepsilon_n} - x, \theta) \geqq \tfrac{1}{2}L \quad \text{for} \quad x \in [0, \pi]\backslash E \quad \text{and} \quad n \geqq N .$$

Write

$$\begin{aligned}\int_{\Sigma_{\varepsilon_n}} F(x, s_{\varepsilon_n} - x, \tfrac{1}{2}M_{\varepsilon_n} - \rho)\, dx &= \int_{\substack{x\in\Sigma_n \\ x\in E}} + \int_{\substack{x\in\Sigma_n \\ x\in E}} \\ &\geqq -(\gamma\rho + C)\tfrac{1}{6}\pi + \tfrac{1}{2}L\tfrac{1}{6}\pi \qquad \text{for } n \text{ large enough} .\end{aligned}$$

This is impossible if we fix $L$ so large that

$$\tfrac{1}{12}L\pi > C_\rho + \tfrac{1}{6}\pi(\gamma\rho + C) .$$

(c) Since we have a bound for $\max|u|$, it follows that

$$\max|F(x, t, u)| \leqq C$$

and hence from the properties of $A^{-1}$

$$|u_1|_{C^1} \leqq C .$$

Thus we have established the following bounds independent of $\varepsilon$:

$$|u_{1\varepsilon}|_{C^1} + \max|u_{2\varepsilon}| \leqq C . \tag{2.5}$$

(d) PROOF OF EXISTENCE IN THEOREM 1. We wish to pass to the limit as $\varepsilon \to 0$. For a suitable sequence of values of $\varepsilon$ tending to zero we have $u_{1\varepsilon}$ converging uniformly to $u_1$ with $u_1 \in C^{0,1}$ and $u_{2\varepsilon}$ converging weakly to $u_2$ in $L^2$ with $u_2 \in L^\infty$. We repeat an argument from [1], Section 1.3, to show that $u$ is a generalized solution of (1.1), (1.2).

For any $\xi \in L^2$ we have, by monotonicity of $F$,

$$(F(x, t, u_\varepsilon) - F(x, t, \xi), u_\varepsilon - \xi) \geqq 0 .$$

Thus

$$(-\varepsilon u_{2\varepsilon} - Au_{1\varepsilon} - F(x, t, \xi), u_\varepsilon - \xi) \geqq 0 .$$

Since $A^{-1}$ is a compact map: $N^\perp \to N^\perp$, it follows that $Au_{1\varepsilon} \rightharpoonup Au_1$. Going to the limit in the preceding inequality we obtain

$$(-Au_1 - F(x, t, \xi), u - \xi) \geqq 0 .$$

We now use Minty's trick: for $v \in L^2$ and $\tau > 0$, set $\xi = u - \tau v$. After dividing by $\tau$ we find

$$(Au_1 + F(x, t, u - \tau v), v) \leqq 0 .$$

Letting $\tau \to 0$ and using the fact that $v$ is arbitrary in $L^2$ we conclude that $Au + F(x, t, u) = 0$.

## 3. Regularity

THEOREM 2. *Let $\lambda = 1$. Assume $F \in C^\infty(\bar{\Omega} \times \mathbb{R})$ is periodic in $t$ with period $2\pi$ and*

(iv) *$F$ is strictly increasing in $u$ for all $(x, t) \in \bar{\Omega}$. Then every $L^\infty$ generalized* *solution $u$ of* (1.1), (1.2) *is continuous on $\bar{\Omega}$ (more precisely, there is a continuous function on $\bar{\Omega}$ which coincides* a.e. *with $u$).*

*Assume in addition:*

(v) *Each connected component of the set $\{(x, t, u) \mid F_u(x, t, u) = 0\}$ admits a $C^\infty$ representation $u = \sigma(x, t)$.*

*Then every continuous solution $u$ of* (1.1), (1.2) *belongs to $C^\infty(\bar{\Omega})$.*

*Remark.* (v) holds in particular when $F(x, t, u) = F(u) - f(x, t)$, and $F(u)$ is strictly increasing in $u$. It also holds of course if $F_u(x, t, u) > 0$.

Proof: Recall that since $F(x, t, u(x, t)) \in L^\infty$ we have $u = u_1 + u_2$ with $u_1 \in C^{0,1}$ and $u_2 \in L^\infty$,

$$u_2(x, t) = p(t+x) - p(t-x), \qquad \int_0^{2\pi} p(t)\, dt = 0.$$

Since

$$p(s) = \frac{1}{2\pi} \int_0^\pi [u_2(x, s-x) - u_2(x, s+x)]\, dx,$$

it follows that $p \in L^\infty$.

(a) We first prove that $p$ is continuous. For fixed $h$ with $|h| < \frac{1}{2}$, set $\hat{p}(t) = p(t+h) - p(t)$ and let $M = M_h = \sup \operatorname{ess}_t |\hat{p}(t)|$. Fix $s$ such that $|\hat{p}(s)| > M(1-|h|)$; we may suppose that $\hat{p}(s) > 0$. We can also assume that (1.6) holds at $s$ and $s+h$. Taking the difference we find, for $\tilde{F}(x, t, r) = F(x, t, r + u_1(x, t))$,

$$\begin{aligned}
0 = \int_0^\pi & \tilde{F}(x, s+h-x, p(s+h) - p(s+h-2x)) \\
& \qquad - \tilde{F}(x, s-x, p(s+h) - p(s+h-2x))\, dx \\
& + \int_0^\pi \tilde{F}(x, s-x, p(s+h) - p(s+h-2x)) - \tilde{F}(x, s-x, p(s) - p(s-2x))\, dx \\
& - \int_0^\pi \{\tilde{F}(x, s+h+x, p(s+h+2x) - p(s+h)) \\
& \qquad - \tilde{F}(x, s+x, p(s+h+2x) - p(s+h))\}\, dx \\
& - \int_0^\pi \{\tilde{F}(x, s+x, p(s+h+2x) - p(s+h)) - \tilde{F}(x, s+x, p(s+2x) - p(s))\}\, dx \\
= K_1 & + K_2 - K_3 - K_4 .
\end{aligned}$$

Clearly, $|K_1| \leqq C|h|$, $|K_3| \leqq C|h|$ since $\tilde{F}$ is $C^{0,1}$. On the other hand,

$$p(s+h+2x)-p(s+2x) \leqq p(s+h)-p(s)+M|h|$$

and since $\tilde{F}$ is increasing in $u$ we see that

$$\begin{aligned}\tilde{F}(x, s+x, p(s+h+2x)-p(s+h)) &\leqq \tilde{F}(x, s+x, p(s+2x)-p(s)+M|h|)\\ &\leqq \tilde{F}(x, s+x, p(s+2x)-p(s))+C|h|.\end{aligned}$$

Thus $K_4 \leqq C|h|$ and consequently $K_2 \leqq C|h|$. For real $z$ define

$$\phi(z) = \min_{\substack{(x,t)\in\bar{\Omega}\\ |\zeta|\leqq 2N}} [\tilde{F}(x, t, z+\zeta)-\tilde{F}(x, t, \zeta)],$$

where $N = \text{sup ess}\,|p|$.

Clearly, $\phi$ is strictly increasing in $z$, Lipschitz continuous on bounded intervals, and $\phi(0)=0$. Since

$$\begin{aligned}\phi(\hat{p}(s)-\hat{p}(s-2x)) \leqq \tilde{F}(x, s-x, p(s+h)-p(s+h-2x))&\\ -\tilde{F}(x, s-x, p(s)-p(s-2x))&,\end{aligned}$$

we obtain by integration

$$\int_0^{\pi} \phi(\hat{p}(s)-\hat{p}(s-2x))\,dx \leqq C|h|.$$

As in the preceding section, let

$$\Sigma = \{x\in[0,\pi] \mid \hat{p}(s)-\hat{p}(s-2x) \geqq \tfrac{1}{2}M\}.$$

Since

$$\begin{aligned}0 = \int_0^{\pi} \hat{p}(s-2x)\,dx = \int_{\Sigma} + \int_{\Sigma^c}&\\ \geqq -M \operatorname{meas}\Sigma + (\tfrac{1}{2}M - M|h|)(\pi-\operatorname{meas}\Sigma)&,\end{aligned}$$

we see that $\operatorname{meas}\Sigma \geqq \pi(1-2|h|)/(3-2|h|)$.

Since $\hat{p}(s)-\hat{p}(s-2x) \geqq -M|h|$, we also have

$$\int_{\Sigma} \phi(\hat{p}(s)-\hat{p}(s-2x))\,dx \leqq C|h|$$

and so

$$\phi(\tfrac{1}{2}M) \leqq C\,|h|\,.$$

It follows that $\sup \mathrm{ess}_t\,|p(t+h)-p(t)| \to 0$ as $h \to 0$—which implies that $p$ coincides a.e. with a continuous function (use for example mollifiers). As a consequence $F(x, t, u)$ is continuous and by the properties of $A^{-1}$ it follows that $u_1 \in C^1$.

(b) Assuming (v) we prove now that $u \in C^\infty(\bar{\Omega})$. As in (a), we set $\tilde{F}(x, t, r) = F(x, t, r + u_1(x, t))$ so that now $\tilde{F}$ is $C^1$ in $(x, t, r)$. Let $M_h = \sup_t |p(t+h)-p(t)|$ and set

$$\Phi(x, s) = \tilde{F}(x, s-x, p(s)-p(s-2x)) - \tilde{F}(x, s+x, p(s+2x)-p(s))\,,$$

$$\Psi(x, s) = \tilde{F}(x, s-x, p(s)-p(s-2x)) + \tilde{F}(x, s+x, p(s+2x)-p(s))\,.$$

Since $u$ is a solution of (1.1), we have

$$\int_0^\pi \Phi(x, s)\,dx = 0 \quad \text{for all} \quad s\,. \tag{3.1}$$

Let $h > 0$ and $\tfrac{1}{2}h < x < \pi$; consider

$$\begin{aligned}
&[\Phi(x, s+h) - \Phi(x, s)] - [\Psi(x-\tfrac{1}{2}h, s) - \Psi(x, s)] \\
&\quad = [h\tilde{F}_t(x, s-x, p(s)-p(s-2x)) - h\tilde{F}_t(x, s+x, p(s+2x)-p(s)) + o(h) \\
&\qquad\qquad + \tilde{F}_r(x, s-x, p(s)-p(s-2x))(\hat{p}(s)-\hat{p}(s-2x)) \\
&\qquad\qquad - \tilde{F}_r(x, s+x, p(s+2x)-p(s))(\hat{p}(s+2x)-\hat{p}(s)) + M_h\varepsilon_h] \\
&\qquad + [\tfrac{1}{2}h\tilde{F}_x(x, s-x, p(s)-p(s-2x)) + \tfrac{1}{2}h\tilde{F}_x(x, s+x, p(s+2x)-p(s)) \\
&\qquad\qquad - \tfrac{1}{2}h\tilde{F}_t(x, s-x, p(s)-p(s-2x)) + \tfrac{1}{2}h\tilde{F}_t(x, s+x, p(s+2x)-p(s)) + o(h) \\
&\qquad\qquad + \tilde{F}_r(x, s-x, p(s)-p(s-2x))\hat{p}(s-2x) \\
&\qquad\qquad + \tilde{F}_r(x, s+x, p(s+2x)-p(s))\hat{p}(s+2x) + M_h\varepsilon_h]\,,
\end{aligned}$$

where $\varepsilon_h \to 0$ as $h \to 0$ uniformly in $x$ and $s$ ($o(h)$ is also uniform). After some rearrangement we find

$$\begin{aligned}
&[\Phi(x, s+h) - \Phi(x, s)] - [\Psi(x-\tfrac{1}{2}h, s) - \Psi(x, s)] \\
&\quad = \tfrac{1}{2}hG(x, s) + \hat{p}(s)[\tilde{F}_r(x, s-x, p(s)-p(s-2x)) + \tilde{F}_r(x, s+x, p(s+2x)-p(s))] \\
&\qquad + o(h) + M_h\varepsilon_h\,,
\end{aligned}$$

where

$$G(x,s)=\tilde F_t(x,s-x,p(s)-p(s-2x))-\tilde F_t(x,s+x,p(s+2x)-p(s))$$
$$+\tilde F_x(x,s-x,p(s)-p(s-2x))+\tilde F_x(x,s+x,p(s+2x)-p(s)).$$

Integrating in $x$ on $[\frac12 h,\pi]$, and recalling (3.1), we obtain for all $s$

$$-\int_0^{h/2}\Psi(x,s)\,dx+\int_{\pi-h/2}^{\pi}\Psi(x,s)\,ds=\tfrac12 h\int_0^{\pi}G(x,s)\,dx+\hat p(s)H(s)+o(h)+M_h\varepsilon_h,$$

where

$$H(s)=\int_0^{\pi}[\tilde F_r(x,s-x,p(s)-p(s-2x))+\tilde F_r(x,s-x,p(s+2x)-p(s))]\,dx$$

(so that $H\geqq 0$).

Finally, we write

$$\int_0^{h/2}\Psi(x,s)\,dx=\tfrac12 h\Psi(0,s)+o(h),$$

$$\int_{\pi-h/2}^{\pi}\Psi(x,s)\,dx=\tfrac12 h\Psi(\pi,s)+o(h),$$

with

$$\Psi(0,s)=2F(0,s,0),\qquad \Psi(\pi,s)=2F(\pi,s-\pi,0).$$

Thus we have

$$\hat p(s)H(s)=-hF(0,s,0)+hF(\pi,s-\pi,0)$$
$$-\tfrac12 h\int_0^{\pi}G(x,s)\,dx+o(h)+M_h\varepsilon_h. \tag{3.2}$$

We now distinguish two cases:

*Case* (i). $H(s)$ vanishes for some $\bar s$ and in particular $F_u(x,\bar s-x,p(\bar s)-p(\bar s-2x)+u_1(x,\bar s-x))$ is identically zero for $x\in[0,\pi]$. From hypothesis (v) it follows that, for some $C^\infty$ function $\sigma(x,t)$,

$$p(\bar s)-p(\bar s-2x)+u_1(x,\bar s-x)=\sigma(x,\bar s-x).$$

In this case the function $u_2$ is as smooth as $u_1$. In particular $u_2\in C^1$. But then

$F(x, t, u) \in C^1$ and by the properties of $A^{-1}$ it follows that $u_1 \in C^2$. Then $u_2 \in C^2$, and so on. Thus $u \in C^\infty$.

*Case* (ii). $H(s) \geqq c_0 > 0$ for all $s$.

In this case we derive from (3.2) that $M_h \leqq Ch$ and so $p \in C^{0,1}$. Therefore $p$ is differentiable a.e. and we have for a.e. $s$

$$\dot{p}(s)H(s) = -F(0, s, 0) + F(\pi, s-\pi, 0) - \frac{1}{2}\int_0^\pi G(x, s)\, dx, \tag{3.3}$$

so that in fact $p$ is $C^1$. Therefore $u_2 \in C^1$, and $u_1 \in C^2$. It follows that $\bar{F}$ is $C^2$ and $G$, $H$ are $C^1$. Hence $\dot{p}$ is $C^1$ and $p$ is $C^2$, and so on. Thus $u \in C^\infty$.

## 4. Solutions with Other Periods

We now extend Theorems 1 and 2 in order to obtain solutions with period

$$T = \frac{2\pi}{\lambda}, \qquad \lambda = \frac{a}{b},$$

$a$, $b$ being coprime. We shall assume as before that $F$ satisfies hypotheses (i), (ii), (iii) but we shall require a different bound on the constant $\gamma$ in (iii).

First, some remarks about the operator $A = \partial_t^2 - \partial_x^2$ acting on functions satisfying (1.2) and with period $2\pi b/a$ in time. We need extensions of the results cited on pages 3 and 4. Only brief sketches of their proofs will be given.

1. $N = \ker A$ consists of functions of the form

$$u_2 = p(t+x) - p(t-x)$$

with $p(t)$ having period $2\pi$ and period $2\pi b/a$. Thus $p(t)$ has period $2\pi/a$ and we may suppose that

$$\int_0^{2\pi/a} p(t)\, dt = 0.$$

Then the range of $A$ in $L^2$ is $R(A)$,

$$R(A) = N^\perp.$$

This is most easily seen with the aid of Fourier series. If $Au = f$ and

$$u = \sum_{j>0} \sum_k u_{jk} \sin jx e^{i\lambda kt}, \qquad u_{j,-k} = \overline{u_{j,k}},$$

$$f = \sum_{j>0} \sum_k f_{jk} \sin jx e^{i\lambda kt},$$

then

$$f_{jk} = (j^2 - \lambda^2 k^2) u_{jk}.$$

Thus $N$ is spanned by functions of the form $\sin \lambda kx \cos \lambda kt$, $\sin \lambda kx \sin \lambda kt$, with $k$ and $\lambda k$ positive integers; also $R(A) = N^\perp$. In particular we see that if $f \in R(A)$, then the solution $u_1 \in N^\perp$ of $Au_1 = f$ is

$$A^{-1} f = u_1 \sum_{\substack{j>0 \\ j \neq \lambda |k|}} \frac{f_{jk}}{j^2 - \lambda^2 k^2} \sin jx e^{i\lambda kt}. \tag{4.1}$$

We see furthermore that, for a fixed constant $C$,

$$(Au, u) = C \sum (j^2 - \lambda^2 k^2) |u_{jk}|^2,$$

$$(Au, Au) = C \sum |j^2 - \lambda^2 k^2|^2 |u_{jk}|^2.$$

Let $\alpha$ be the largest number such that

$$\alpha(Au, u) + (Au, Au) \geqq 0 \quad \text{for all} \quad u \in D(A),$$

that is

$$\alpha = \min_{\substack{j>0 \\ k>j/\lambda}} (\lambda^2 k^2 - j^2). \tag{4.2}$$

Also let $\alpha'$ be the largest number such that

$$-\alpha'(Au, u) + (Au, Au) \geqq 0 \quad \text{for all} \quad u \in D(A),$$

i.e.,

$$\alpha' = \min_{\substack{j>0 \\ k<j/\lambda}} (j^2 - \lambda^2 k^2). \tag{4.2'}$$

In case $a = 1$, i.e., $\lambda = 1/b$, it is easy to verify that

$$\alpha = \frac{2b+1}{b^2}, \qquad \alpha' = \frac{2b-1}{b^2}.$$

With $\alpha$, $\alpha'$ so defined, our extensions of Theorems 1 and 2 are:

THEOREM 1′. *Assume* $\lambda = a/b$ *and that* $F$ *satisfies conditions* (i), (ii), (iii) *with* $\gamma < \alpha$ *or* $\gamma < \alpha'$ *according to whether we have* $+$ *or* $-$ *in* (1.1). *Then there exists a generalized solution of* (1.1), (1.2),

$$u = u_1 + u_2 , \qquad u_1 \in N^{\perp} , \; u_2 \in N ,$$

*with* $u_1 \in C^{0,1}$, $u_2 \in L^{\infty}$.

THEOREM 2′. *Theorem* 2 *holds also if* $\lambda = a/b$ *and* $F$ *is periodic of period* $2\pi/\lambda$ *in* $t$.

2. We shall make use of a simple formula for the orthogonal projection $P_2$ of $L^2$ onto $N$. If $v(x, t) \in L^2(\Omega)$ (period $T$ in $t$ of course), then $P_2 v = p(t+x) - p(t-x)$ with, for $0 \leqq s \leqq 2\pi/a$,

$$p(s) = \frac{1}{2\pi a} \sum_{r=0}^{a-1} \int_0^{\pi} \left[ v\left(x, s + 2\pi \frac{r}{a} - x\right) - v\left(x, s + 2\pi \frac{r}{a} + x\right) \right] dx . \tag{4.3}$$

In such terms we also give a different representation of $u_1 = A^{-1} f$ for $f \in N^{\perp}$ which is the analogue of formula (2.5) in Lovicavorá [16]. One may verify directly that the following is a particular solution of $Au = f$:

$$u(x, t) = -\frac{1}{2\pi} \int_x^{\pi} \int_{t+x-\xi}^{t-x+\xi} f(\xi, \tau)\, d\tau\, d\xi + r(t+x) - r(t + 2\pi - x) + c \frac{\pi - x}{\pi} , \tag{4.4}$$

where

$$r(t) = -\frac{1}{2b} \sum_{m=0}^{b-1} m \int_0^{\pi} \int_{t+2\pi m - \xi}^{t+2\pi m + \xi} f(\xi, \tau)\, d\tau\, d\xi ,$$

and

$$c = \frac{1}{T} \int_0^{\pi} \int_0^{T} \xi f(\xi, \tau)\, d\tau\, d\xi .$$

In particular the solution $u_1 \in N^{\perp}$ of $Au_1 = f$ is then given by

$$u_1 = u - P_2 u$$

with $P_2$ described in (4.3).

Using these explicit formulas one verifies easily that the properties (1.3), (1.3′), (1.3″) of $A^{-1}$ cited earlier continue to hold.

3. Theorem I.8 of [1] applies once again (with our respective conditions $\gamma < \alpha$ or $\gamma < \alpha'$) to ensure the existence of a $C^{\infty}$ solution of (1.5):

$$\pm \varepsilon u_{2\varepsilon} + (\partial_t^2 - \partial_x^2) u_\varepsilon \pm F(x, t, u_\varepsilon) = 0 , \qquad u_\varepsilon(0, t) = u_\varepsilon(\pi, t) = 0 .$$

Again, we wish to obtain estimates for $u_\varepsilon$ independent of $\varepsilon$ and then let $\varepsilon \to 0$. The relation (writing $u_\varepsilon$ as $u$)

$$\varepsilon u_2 + P_2 F(x, t, u) = 0$$

now takes the form—the analogue of (1.6)—for $u_2 = p(t+x) - p(t-x)$:

$$\begin{aligned} 0 = \varepsilon p(s) + \frac{1}{2\pi a} \sum_{r=0}^{a-1} \int_0^\pi \Big[ & F\Big(x, s-x+2\pi\frac{r}{a}, p(s)-p(s-2x)+u_1\Big(x, s-x+2\pi\frac{r}{a}\Big)\Big) \\ & - F\Big(x, s+x+2\pi\frac{r}{a}, p(s+2x)-p(s)+u_1\Big(x, s+x+2\pi\frac{r}{a}\Big)\Big)\Big]\, dx \qquad (4.5) \\ \equiv G[p, u_1]\,. \end{aligned}$$

The following is a direct extension of a result in DeSimon-Torelli [4] and has the same proof:

LEMMA 4.1. *For $\varepsilon > 0$, and for $u_1 \in C(\bar{\Omega})$* (and $T$-periodic in time) *the equation*

$$G[p, u_1] = 0$$

*has a unique continuous solution $p(s)$. If $u_1 \in C^k$, then $p \in C^k$, $k = 0, 1, \cdots$.*

This is proved in the following manner: For $u_1$ fixed, one shows that the map $p \to G[p, u_1]$ is one–one, surjective, on the space of continuous functions $q(t)$ with period $2\pi/a$ and zero average. This is done by proving that the image of $G$ is open (using the implicit function theorem) and closed (using estimates for the solution of $G[p, u_1] = q$ of the type we obtained earlier; these are easy to derive in case $\varepsilon > 0$). Regularity is then readily established.

With the aid of the properties of $A^{-1}$ discribed above, and (4.5), the proofs of Theorems 1′, 2′ just follow those of Theorems 1, 2. The expression (4.5) is a bit more complicated than (1.6) but it is treated in the same way.

We shall consider Theorems 1′, 2′ as proved.

## 5. Existence of Nontrivial Solutions

Suppose

$$F(x, t, u) \equiv 0\,. \qquad (5.1)$$

Then $u \equiv 0$ is a solution of (1.1), (1.2) and Theorem 1′ does not ensure the existence of any other solution. It is sometimes possible, under additional conditions, to prove that there are more. In this section we shall illustrate this by treating some simple model problems. Here we follow some ideas from Cronin [3] and Tavantzis [30]; these use degree theory and invoke an analysis of the solutions near $u=0$. We shall assume for convenience that $F$ satisfies all the conditions of Theorems 1′ and 2′ including (iv) and (v). In addition we suppose that

(vi) $$F_u(x, t, 0) = \beta, \quad \text{a positive constant}.$$

Assume first

(5.2) $$\beta \neq j^2 - \lambda^2 k^2 \quad \text{for all integers} \quad j > 0, k.$$

For $\beta$ lying in certain intervals we obtain nontrivial solutions for (1.1) with the minus sign,

(5.3) $$u_{tt} - u_{xx} - F(x, t, u) = 0.$$

We shall explain later why the argument yields nothing in the case of the + sign and we shall present a different result with the plus sign.

We obtained a solution of (5.3) as a limit, through a sequence of $\varepsilon \to 0$, of solutions of

(5.4) $$-\varepsilon u_2 + Au - F(x, t, u_1 + u_2) = 0.$$

Our aim now is to show that (5.4) has a solution $u_\varepsilon$ which is bounded away from zero as $\varepsilon \to 0$. Going to the limit as before we shall then obtain a nontrivial solution. Rewrite (5.4) in the form

(5.5) $$u_1 - A^{-1} P_1 F(x, t, u_1 + u_2) = 0,$$

(5.6) $$\varepsilon u_2 + P_2 F(x, t, u_1 + u_2) = 0.$$

We shall first study this in the space of continuous functions $u_1 \in N^\perp \cap C$, $u_2 \in N \cap C$ (always satisfying the boundary and periodicity conditions). The nonlinear operators in (5.5), (5.6) are then smooth operators, and we may use the implicit function theorem to analyze the solutions near the origin.

LEMMA 5.1. *Assuming* (5.2), *there are positive numbers* $r$, $\varepsilon_0$ *such that for* $\varepsilon < \varepsilon_0$ *the only continuous solution* $u = u_1 + u_2$ *of* (5.5), (5.6) *with*

$$\max |u_1| + \max |u_2| \leqq r,$$

*is* $u \equiv 0$.

Proof: By the implicit function theorem we have only to verify that the linearized operator at $u_1+u_2=0$, $\varepsilon=0$:

$$(u_1, u_2) \to \begin{Bmatrix} u_1 - \beta A^{-1} u_1 \\ \beta u_2 \end{Bmatrix}$$

is bijective. Since $A^{-1}$ is compact, this is the case provided $\beta^{-1}$ is not an eigenvalue of $A^{-1}$—which is assured by condition (5.2).

Next we wish to obtain nontrivial continuous solutions of (5.5), (5.6). By Theorem 2′ the solutions are then automatically in $C^\infty(\bar{\Omega})$. We shall rewrite these equations. Using Lemma 4.1 for given $u_1 \in N^\perp \cap C$ there is a unique solution $u_2 \in N \cap C$, $u_2(u_1)$ of (5.6). Inserting this into (5.5) we obtain the equation

$$0 = u_1 - A^{-1} P_1 F(x, t, u_1 + u_2(u_1)) \equiv u_1 - K[u_1] \tag{5.7}$$

with $K$ continuous and compact.

Since we have an *a priori* bound

$$\max |u_1| \leqq C \quad \text{independent of} \quad \varepsilon \tag{5.8}$$

(obtained in the proof of Theorems 1′ and 2′), we see that the Leray–Schauder degree

$$\nu = \deg (I - K, \|u_1\| \leqq C + 1, 0)$$

is defined. A look at the derivation of (5.8) shows that the same estimate also holds if $F$ is replaced by $\tau F$, $0 \leqq \tau \leqq 1$. Thus the degree $\nu$ is also the degree for $I - \tau K$ and it follows (taking $\tau = 0$) that $\nu = 1$.

We know that $u = 0$ is an isolated solution of (5.7). If the Leray–Schauder index of $I - K$ at $u = 0$ is different from one, we may infer that (5.7) and hence (5.4), and consequently (5.3), have nontrivial solutions. We may summarize this in

LEMMA 5.2. *Assume* $F(x, t, 0) = 0$ *and that* $F$ *satisfies hypotheses* (i)–(vi) *and* (5.2). *In* $N^\perp \cap C$ *assume that the Leray–Schauder index at the origin of the linear operator*

$$u_1 \to u_1 - \beta A^{-1} u_1$$

*is different from one. Then, equation* (5.3) *possesses a nontrivial solution* $u$ *which belongs to* $C^\infty(\bar{\Omega})$.

The Leray–Schauder index of $I-\beta A^{-1}$ is given by

$$\text{ind}=(-1)^{\rho}, \tag{5.9}$$

where

$$\rho=\sum_{\mu>1} n_{\mu}. \tag{5.10}$$

Here we sum over all eigenvalues $\mu>1$ of $\beta A^{-1}$, $n_{\mu}$ being the multiplicity of $\mu$.

To see what Lemma 5.2 says let us compute the index for various values of $\beta$. The eigenvalues of $A^{-1}$ are of the form

$$(j^2-\lambda^2 k^2)^{-1}, \qquad j>0, \quad k \text{ integers}.$$

Thus if $\mu>1$ is an eigenvalue of $\beta A^{-1}$, we have

$$\mu=\beta(j^2-\lambda^2 k^2)^{-1}.$$

Furthermore if $k\neq 0$, the contribution to the multiplicity of $\mu$ corresponding to $j$, $k$ is *even* since we obtain a contribution from $-k$ as well as from $k$. Hence

$$(-1)^{n_\mu}=-1 \quad \text{if} \quad \mu=\beta/j^2 \quad \text{for some integer } j,$$

and $(-1)^{n_\mu}=1$ otherwise. Thus we find

$$\text{ind}=\begin{cases}-1 & \text{if the number of positive squares } <\beta \text{ is odd},\\ 1 & \text{otherwise}.\end{cases}$$

Consequently we have

THEOREM 3. *Assume $F(x, t, 0)\equiv 0$ and that $F$ satisfies hypotheses* (i)–(vi) *and* (5.2). *If the number of squares less than $\beta$ is odd, then* (5.3) *possesses a nontrivial $C^\infty$ solution satisfying the boundary conditions and having period $2\pi b/a$ in time.*

*Remark.* For the equation with the $+$ sign, $u_{tt}-u_{xx}+F(x, t, u)=0$, the preceding analysis gives no results since the index of $I+\beta A^{-1}$ at the origin is always 1.

We should also remark that in case $F$ is independent of $t$ our nontrivial solution may also be independent of $t$. We have no way of excluding this, somewhat trivial, solution.

What happens if condition (5.2) is not satisfied? It may still be possible to prove the existence of nontrivial solutions. Let us suppose that the set $\Sigma$ of pairs of integers $(j>0, k)$ such that

$$\beta = j^2 - \lambda^2 k^2$$

is nonempty. Naturally it is finite. We wish to show under further conditions on $F$ that $u_1 = 0$ is an isolated solution of (5.7) and that a nontrivial solution of (5.3) exists. Suppose that the Taylor series of $F$ with respect to $u$ at $u=0$ takes the form

$$F(x, t, u) = \beta u + a(x, t)u^r + \cdots . \tag{5.11}$$

Denote the (finite-dimensional) null space of $A - \beta I$ by $N_1$. We may then decompose any $u_1 \in N^\perp \cap C$ into two components:

$$u_1 = u_1' + u_1'' \equiv P_1' u_1 + P_1'' u_1 , \qquad u_1' \perp N_1 , \qquad u_1'' \in N_1 ,$$

with $P_1'$, $P_1''$ the corresponding orthogonal projections in $L^2$. In the following $| \ |_0$ denotes maximum norm.

Suppose $u_1$ is a solution of (5.7) having small norm $|u_1|_0$. By the implicit function theorem the solution $u_2(u_1)$ near 0 depends smoothly on $u_1$ and if we compute the Frechet derivative at the origin we find

$$(\varepsilon + \beta) \frac{\partial}{\partial u_1} u_2(0) = 0 .$$

Thus $|u_2(u_1)|_0 = O(|u_1|_0^2)$ uniformly in $\varepsilon$ for $\varepsilon$ small.

We rewrite (5.7) in the form

$$\left.\begin{array}{ll} (5.7') & 0 = u_1' - (A - \beta I)^{-1} P_1'[F(x, t, u_1 + u_2(u_1)) - \beta u] \\ (5.7'') & 0 = P_1''[u - \beta^{-1} F(x, t, u_1 + u_2(u_1))] \end{array}\right\} \equiv u_1 - K_1[u_1] ,$$

$K_1$ compact; we see furthermore from (5.7′) that

$$|u_1'|_0 = O(|u_1''|_0^r) .$$

Hence from (5.7″) we find that

$$P_1'' a(x, t) u_1''^r = O(|u_1''|_0^{r+1}) . \tag{5.12}$$

Let us now assume

(vii) For every $v \in N_1$, $v \neq 0$,

$$P_1''[a(x, t)v'] \neq 0 .$$

Under this assumption we see from (5.12) that the analogue of Lemma 5.1 holds, i.e., there are positive numbers $\rho$, $\varepsilon_0$ such that for $\varepsilon < \varepsilon_0$ the only solution of (5.7) with $\max |u_1| \leqq \rho$ is $u_1 = 0$. Furthermore, if we deform [ ] in (5.7′) *via* $\tau$[ ], $0 \leqq \tau \leqq 1$, and deform $u_2$ *via* $\tau u_2$ we find that the Leray–Schauder index at the origin of $I - K_1$ is equal to the degree at the origin of the finite-dimensional mapping

$$u_1'' \to -P_1''[a(x, t)u_1''] \quad \text{for} \quad |u_1''|_0 = \text{some small } \delta > 0 . \tag{5.13}$$

Call this degree $d_0$.

We have to compare $d_0$ with the degree $d$ at the origin of the map $I - K_1$ in a large ball. Since (5.7′) and (5.7″) are equivalent to (5.4), for which we have *a priori* bounds for the solution, we infer that this degree of $I - K_1$ is defined in some large ball. Furthermore the degree is the same for each map $I - K_1(\tau)$, $0 \leqq \tau \leqq 1$, given by

$$(I - K_1(\tau))[u_1] = \begin{cases} u_1' - (A - \beta I)^{-1} P_1'[\tau F(x, t, u_1 + u_2(u_1)) - \beta u] , \\ \qquad P_1''[u - \tau \beta^{-1} F(x, t, u_1 + u_2(u_1))] . \end{cases}$$

For if $u_1 = u_1' + u_1''$ is a solution of $u_1 - K_1(\tau)[u_1] = 0$, then it is a solution of

$$(A - \beta)u_1' - P_1' \tau F + \beta u_1' = 0 ,$$

$$(A - \beta)u_1'' + P_1''(\beta u - \tau F) = 0 .$$

Thus $u = u_1 + u_2(u_1)$ is a solution of

$$Au_1 - \tau P_1 F(x, t, u_1 + u_2) = 0 ,$$

$$\varepsilon u_2 + \tau P_2 F(x, t, u_1 + u_2) = 0 ,$$

i.e., of (5.7), in which $F$ has been replaced by $\tau F$. Our *a priori* bounds hold for this, independent of $\tau$, and our assertion then follows.

Since

$$(I - K_1(0))u_1 = \begin{cases} u_1' + (A - \beta I)^{-1} \beta u_1' , \\ u_1'' , \end{cases}$$

we conclude that

$$\begin{aligned} d &= \text{degree of the linear map in } H_1' : I+\beta(A-\beta I)^{-1} \\ &= (-1)^\rho , \\ \rho &= \sum_{\mu>1} n_\mu , \end{aligned}$$

where we sum over all eigenvalues $\mu > 1$ of

$$-\beta(A-\beta I)^{-1} \quad \text{on} \quad H_1' ,$$

and $n_\mu$ is the multiplicity of $\mu$. Thus $\mu$ has the form

$$1 < \mu = \frac{\beta}{\beta-(j^2-\lambda^2 k^2)}$$

with

$$0 < j^2 - \lambda^2 k^2 < \beta .$$

In case $k>0$, $k$ and $-k$ contribute an even number to $\rho$. So we need only consider $k=0$. Thus $(-1)^{n_\mu}=1$ except:

$$(-1)^{n_\mu} = -1 \quad \text{if} \quad \mu = \beta/(\beta-j^2)$$

for some positive integer $j<\beta$. Hence we find

$$d = \begin{cases} -1 & \text{if the number of positive squares} < \beta \text{ is odd} , \\ 1 & \text{otherwise} . \end{cases} \tag{5.14}$$

We now have

THEOREM 4. *Assume $F(x,0,0)\equiv 0$ and that $F$ satisfies hypotheses* (i)–(vi), *with $\Sigma$ nonempty, and also* (5.11) *and* (vii). *If the degree $d_0$ of the finite-dimensional map* (5.13) *is different from $d$ given by* (5.14), *then* (5.3) *possesses a nontrivial $C^\infty$ solution satisfying the boundary and periodicity conditions.*

EXAMPLE. Suppose $\beta=1$. Then $\Sigma$ consists of one element $(1,0)$, and the space $N_1$ consists of functions of the form $c\sin x$. In this case, condition (vii) simply takes the form

$$a_0 \equiv \int_0^T \int_0^\pi a(x,t) \sin^{r+1} x \, dx \, dt \neq 0 , \tag{vii'}$$

and we see that the degree $d_0$ of the map (5.13) is the same as that of the mapping

$$c \longrightarrow -a_0 c^r .$$

Thus

$$d_0 = \begin{cases} 0 & \text{if } r \text{ is even}, \\ 1 & \text{if } r \text{ is odd and } a_0 < 0, \\ -1 & \text{if } r \text{ is odd and } a_0 > 0. \end{cases}$$

Hence in this case we see that *we have a nontrivial solution in each of the following cases*, assuming $a_0 \neq 0$:

(i) $r$ is even,
(ii) $r$ is odd and $a_0 > 0$.

We turn now to the equation with the plus sign:

$$u_{tt} - u_{xx} + F(x, t, u) = 0 \tag{5.15}$$

and assume that $F$ satisfies (i)–(vi) with $\Sigma'$ nonempty. Here $\Sigma'$ is the set of pairs of integers $j > 0$, $k$ such that $\beta + j^2 - \lambda^2 k^2 = 0$. Assume also that $F$ satisfies (5.11) and (vii). In analogy with the preceding, we consider

$$u_1 - K_2[u_1] = \begin{cases} u_1' + (A + \beta I)^{-1}[P_1' F(x, t, u_1 + u_2(u_1)) - \beta u], \\ P_1''[u - \beta^{-1} F(x, t, u_1 + u_2(u_1))]. \end{cases}$$

As before, the local index of $I - K_2$ equals $d_0$, the degree of (5.13). A similar calculation to that for $d$ shows that the degree at the origin of $I - K_2$ in a large ball is one. Thus we obtain

THEOREM 5. *Assume $F(x, t, 0) \equiv 0$ and that $F$ satisfies* (i)–(vi) *with $\Sigma'$ nonempty, and* (5.11) *and* (vii). *If the degree $d_0$ of* (5.13) *is different from one, then* (5.15) *has a nontrivial $C^\infty$ solution satisfying the periodicity and boundary conditions.*

EXAMPLE. Suppose $\lambda = 1$, i.e., $T = 2\pi$, and $\beta = 3$. Then the null space of $A + \beta I$ consists of functions of the form $w = c_1 \sin x \sin 2t + c_2 \sin x \cos 2t$.

Condition (vii) takes the form: if $c_1^2 + c_2^2 > 0$, then $c_1'^2 + c_2'^2 > 0$, where

$$c_1' = -\int_0^T \int_0^{2\pi} a(x, t) w^r \sin x \sin 2t \, dx \, dt,$$

$$c_2' = -\int_0^T \int_0^{2\pi} a(x, t) w^r \sin x \cos 2t \, dx \, dt.$$

Then $d_0$ = degree at the origin of the map

$$(c_1, c_2), \quad \text{with} \quad c_1^2 + c_2^2 = 1, \mapsto (c_1', c_2') .$$

In particular if $r$ is even and $a = q(x, t) \sin 2t$ or $a = q(x, t) \cos 2t$, where $q$ never vanishes, we find $d_0 = 0$ and thus (5.15) has a nontrivial solution.

Paul Rabinowitz has pointed out to us that one may also obtain the existence of small nontrivial solutions with the aid of known bifurcation theorems for equations coming from variational problems. Consider for example the equation (5.15),

$$u_{tt} - u_{xx} + F(x, t, u) = 0 ,$$

with $F$ satisfying (i)–(vi) and $\beta \neq \lambda^2 k^2 - j^2$ for all integers $j > 0$, $k$, but $\beta$ close to some $\beta_0 = \lambda^2 k_0^2 - j_0^2$ for some integers $j_0 > 0$, $k_0$. In this case, for $u = u_1 + u_2$ near zero we may rewrite (5.15) in the form

$$u_{1tt} - u_{1xx} + P_1 F(x, t, u_1 + u_2(u_1)) = 0 ,$$

where $u_2$ is the unique solution near zero (obtained for instance with the aid of the implicit function theorem) of $P_2 F(x, t, u_1 + u_2(u_1)) = 0$. This in turn may be rewritten as

$$u_{1tt} - u_{1xx} + P_1[F(x, t, u_1 + u_2(u_1)) - \beta u_1] = -\lambda u_1 \tag{5.16}$$

with $\lambda = \beta$. We wish to find nontrivial solutions of this in case $\lambda = \beta$. Let us first permit $\lambda$ to vary in an interval near $\beta_0$.

Problem (5.16) may be expressed as a variational problem: $u_1$ is a stationary point in $H_1 \cap N^\perp$ of the functional

$$\int_0^T \int_0^\pi [\tfrac{1}{2}(u_{1x}^2 - u_{1t}^2) + P_1 G(x, t, u_1) + \tfrac{1}{2}\lambda u_1^2] \, dx \, dt .$$

where

$$G(x, t, w) = \int_0^w F(x, t, s - u_2(s)) \, ds - \tfrac{1}{2}\beta w^2 .$$

According to the theorem in Rabinowitz [25], if for $\lambda = \beta_0$, $u_1 = 0$ is an isolated solution of (5.16—this will be the case, say, if $F$ satisfies (5.11) and (vii)—then, for some small $\delta > 0$, for every $\lambda$, on the interval $I_- : \beta_0 - \delta < \lambda < \beta_0$ *or* on the interval $I_+ : \beta_0 < \lambda < \beta_0 + \delta$, (5.16) has a nontrivial solution near zero. So if $\beta$ is in the appropriate interval we obtain a nontrivial solution of (5.15).

In general, for nonlinearities $F$ of the kind considered in this paper with $F(x, t, 0) = 0$, and having at most small linear growth as $|u| \to \infty$, one should not always expect to have nontrivial solutions. For instance in the situation of Theorem 1 consider the problem

$$u_{tt} - u_{xx} + F(x, t, u) = 0 \quad \text{in} \quad \Omega,$$

$$u(0, t) = u(\pi, t) = 0, \quad u \text{ is } 2\pi \text{ periodic in } t,$$

with $F(x, t, 0) \equiv 0$, $F$ strictly increasing in $u$ and $F_u \leqq \gamma < 3$. In this case we have uniqueness, by Theorem I.9 in [1], and so $u = 0$ is the only solution.

## 6. Nonlinear Vibrations with Small Forcing

Consider a nonlinear wave equation of the form

$$\begin{aligned} &u_{tt} - u_{xx} \pm g(u) = f(x, t) \quad \text{on} \quad \Omega = (0, \pi) \times (0, 2\pi), \\ &u(0, t) = u(\pi, t) = 0, \quad \text{and } u \text{ is } 2\pi \text{ periodic in } t. \end{aligned} \tag{6.1}$$

We assume that $g(0) = 0$ and that the forcing term $f$ is small, so that small vibrations are "expected." It is therefore logical to impose conditions on $g$ only near $u = 0$. This is the purpose of our next result.

Assume $g : [-L + L] \to R$ is a continuous function satisfying:

$$g \text{ is nondecreasing in } u, \tag{6.2}$$

$$|g(u)| \leqq \gamma |u| \quad \text{for} \quad u \in [-L, +L], \tag{6.3}$$

where $\gamma < 3$ or $\gamma < 1$ according to whether we have $+$ or $-$ in (6.1),

$$g(-\tfrac{1}{2}L) < 0 < g(\tfrac{1}{2}L). \tag{6.4}$$

THEOREM 6. *Assume* (6.2)–(6.4). *Then, there is a* $\delta > 0$ (*depending only on* $g$) *such that, for each* $f \in L^\infty(\Omega)$ *with* $\|f\|_{L^\infty} < \delta$, *there exists a generalized solution of* (6.1) *with* $\|u\|_{L^\infty} \leqq L$. *If $f$ is smooth and if $g$ is strictly increasing and smooth, then every such solution $u$ is smooth. If $g$ is strictly increasing and* $g' \leqq \gamma$ *in* $[-L, +L]$, *with* $\gamma < 3$ *or* $\gamma < 1$ (*respectively*), *then the solution of* (6.1) *is unique.*

*Remarks.* 1. Most of the papers dealing with small perturbations are concerned with equations of the form $u_{tt} - u_{xx} + \varepsilon(g(u) - f) = 0$, but in Theorem 6 the nonlinear term $g(u)$ is not required to be small—except for the natural restriction (6.3). This is only required of the forcing term $f$.

2. Theorem 6 implies the existence, uniqueness, and smoothness, of small solutions for the equations

$$u_{tt}-u_{xx}\pm u^3=f$$

or

$$u_{tt}-u_{xx}+\sin u=f$$

provided the term $f$ is small.

In view of the uniqueness result (in the class of solutions satisfying $\|u\|_{L^\infty}\leqq L$) we remark that nontrivial solutions of $u_{tt}-u_{xx}+u^3=0$—which exist by [26]—must satisfy $\|u\|_{L^\infty}\geqq 1$.

Proof: (a) *Existence.*

Extend $g$ outside $[-L,+L]$ by

$$\bar{g}(u)=\begin{cases}\gamma(u-L)+g(L) & \text{for} \quad u\geqq L,\\ \gamma(u+L)+g(-L) & \text{for} \quad u\leqq -L,\end{cases}$$

so that $|\bar{g}(u)|\leqq \gamma|u|$ for all $u\in R$.

Theorem 1 implies the existence of a solution $u$ of

$$u_{tt}-u_{xx}\pm\bar{g}(u)=f \tag{6.5}$$

with boundary and periodicity conditions as in (6.1).

We shall verify now that if $\|f\|_{L^\infty}<\delta$, for $\delta$ small enough, then $\|u\|_{L^\infty}\leqq L$. We consider only the + case in (6.5) (the − case is similar). Let $u_0(x,t)$ be such that $\bar{g}(u_o)=f$. Multiplying (6.5) by $u$ leads as in the proof of Theorem 1 to (integration is over $\Omega$):

$$\int(\bar{g}(u)-f)u\leqq\frac{1}{3}\int|Au|^2=\frac{1}{3}\int|\bar{g}(u)-f|^2.$$

But, as is easily verified,

$$(\bar{g}(u)-f)u\geqq|\bar{g}(u)-f|\,|u|-2|f|\,|u_0| \quad \text{for all} \quad u\in R,$$

and so

$$\int|\bar{g}(u)-f|\,|u|\leqq\frac{1}{3}\int|\bar{g}(u)-f|^2+2\int|f|\,|u_0|.$$

Thus

$$\frac{1}{\gamma}\int|\bar{g}(u)-f|^2 \leqq \frac{1}{3}\int|\bar{g}(u)-f|^2+\frac{1}{\gamma}\int|\bar{g}(u)-f|\,|f|+2\int|f|\,|u_0|\,.$$

Hence

$$\|\bar{g}(u)-f\|_{L^2}^2 \leqq C(\|f\|_{L^2}^2+\|f\|_{L^2})\,.$$

As in the proof of Theorem 1 we conclude that

$$\rho=\|u_1\|_{L^\infty} \leqq C(\|f\|_{L^2}+\|f\|_{L^2}^{1/2})\,.$$

Set

$$u_2(x,t)=p(t+x)-p(t-x)\,, \qquad \int_0^{2\pi} p(s)\,ds=0\,,$$

$$\mathrm{I}=p(s)-p(s-2x)+u_1(x,s-x)\,,$$

$$\mathrm{II}=p(s+2x)-p(s)+u_1(x,s+x)\,,$$

so that (6.5) leads to

$$0=\frac{1}{2\pi}\int_0^{\pi}[g(\mathrm{I})-f(x,s-x)]\,dx-\frac{1}{2\pi}\int_0^{\pi}[g(\mathrm{II})-f(x,s+x)]\,dx\,. \tag{6.6}$$

Set $M=\|p\|_{L^\infty}$ and fix a point $s$ where $|p|$ takes its maximum (for a rigorous proof, since $p$ need not be continuous, one should introduce $\varepsilon u_\varepsilon$ in (6.5) and then pass to the limit as $\varepsilon\to 0$). Assume $p(s)>0$.

We have $\mathrm{II}\leqq\rho$ and by (6.6) we find

$$\frac{1}{2\pi}\int_0^{\pi} g(\mathrm{I}) \leqq \tfrac{1}{2}\gamma\rho+\|f\|_{L^\infty}\,. \tag{6.7}$$

Denote by $[\underline{\theta},\bar{\theta}]$ the largest interval on which $\bar{g}$ vanishes so that by (6.4) we have $-\frac{1}{2}L<\theta\leqq\bar{\theta}<\frac{1}{2}L$. Fix $k$ with $2\bar{\theta}/L<k<1$ and set

$$\Sigma=\{x\in[0,\pi]\mid p(s)-p(s-2x)\geqq kM\}\,.$$

Since

$$0=\int_0^{\pi} p(s-2x)\,dx=\int_{\Sigma}+\int_{\Sigma^c} \geqq -M\operatorname{meas}\Sigma+(1-k)M(\pi-\operatorname{meas}\Sigma)\,,$$

we see that meas $\Sigma \geqq ((1-k)/(2-k))\pi$. Using the fact that $\mathrm{I} \geqq -\rho$ we find that

$$\frac{1}{2\pi}\int_0^{\pi} g(\mathrm{I}) = \frac{1}{2\pi}\int_{\Sigma} g(\mathrm{I}) + \frac{1}{2\pi}\int_{\Sigma^c} g(\mathrm{I})$$

$$\geqq \frac{1}{2\pi} g(kM-\rho)\,\mathrm{meas}\,\Sigma - \tfrac{1}{2}\gamma\rho\,.$$

We conclude from (6.7) that

$$\frac{(1-k)}{2(2-k)} g(kM-\rho) \leqq \gamma\rho + \|f\|_{L^\infty}$$

and so $kM-\rho \leqq \bar{\theta} + \omega(\|f\|_{L^\infty})$, where $\omega(r)\to 0$ as $r\to 0$. Hence

$$\|u\|_{L^\infty} \leqq \|u_1\|_{L^\infty} + 2M \leqq \rho + \frac{2\bar{\theta}}{k} + \frac{2\rho}{k} + \frac{2}{k}\,\omega(\|f\|_{L^\infty})\,.$$

Consequently $\|u\|_{L^\infty} \leqq L$ provided $\|f\|_{L^\infty}$ is small enough.

(b) *Smoothness and uniqueness.* Smoothness follows from Theorem 2. For the uniqueness we proceed as in [1] (Corollary 1.6). Assume $u$ and $\tilde{u}$ are two solutions of (6.1). We have

$$\int A(u-\tilde{u})\cdot(u-\tilde{u}) + \int (g(u)-g(\tilde{u}))(u-\tilde{u}) = 0$$

and so

$$\frac{1}{\gamma}\int |g(u)-g(\tilde{u})|^2 \leqq \int (g(u)-g(\tilde{u}))(u-\tilde{u})$$

$$\leqq \frac{1}{3}\int |A(u-\tilde{u})|^2 = \frac{1}{3}\int |g(u)-g(\tilde{u})|^2$$

Hence $g(u)=g(\tilde{u})$ and therefore $u=\tilde{u}$.

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