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# A General Principle on Ordered Sets in Nonlinear Functional Analysis

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## INTRODUCTION

We put forward in the present paper a general and very simple principle concerning order relations which unifies a number of diverse results in nonlinear functional analysis. These results include the normal solvability theorems, generalizations of the Bishop–Phelps theorem, invariance theorems for closed sets under flows in metric spaces, and a number of other new related results. It had been realized earlier that there were mutual interrelations between such theorems and vague similarities in their method of proof. Our discussion exhibits their mutual relation in an explicit way and derives them from a relatively transparent general argument.

## 1. A THEOREM ON ORDERED SETS

Let $X$ be an ordered set; for $x \in X$ we denote $S(x) = \{y \in X; y \geqslant x\}$. A sequence $\{x_n\}$ in $X$ is said to be increasing provided $x_n \leqslant x_{n+1}$ for all $n$.

We begin with our basic result.

THEOREM 1. *Let $\phi: X \to \mathbb{R}$ be a function satisfying*

(1) *$x \leqslant y$ implies $\phi(x) \leqslant \phi(y)$;*

(2) *for any increasing sequence $\{x_n\}$ in $X$ such that $\phi(x_n) \leqslant C < \infty$ for all $n$, there exists some $y \in X$ such that $x_n \leqslant y$ for all $n$;*

(3) *for every $x \in X$ there exists $u \in X$ such that $x \leqslant u$ and $\phi(x) < \phi(u)$.*

*Then for each $x \in X$, $\phi(S(x))$ is unbounded.*



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*Proof.* For $a \in X$ let $\rho(a) = \mathrm{Sup}_{b \in S(a)} \phi(b)$. We want to show that $\rho(x) = +\infty$ for each $x \in X$. Suppose $\rho(x) < \infty$ for some $x \in X$. We define by induction a sequence $(x_n)$ such that $x_1 = x$, $x_{n+1} \in S(x_n)$ and $\rho(x_n) \leqslant \phi(x_{n+1}) + (1/n)$ for all $n \geqslant 1$. Since $\phi(x_{n+1}) \leqslant \rho(x) < \infty$, it follows from (2) that there exists some $y \in X$ such that $x_n \leqslant y$ for all $n$. By (3) we can find $u \in X$ such that $y \leqslant u$ and $\phi(y) < \phi(u)$. Since $x_n \leqslant u$ we have $\phi(u) \leqslant \rho(x_n)$ for all $n$. We also have $x_{n+1} \leqslant y$ so that $\phi(x_{n+1}) \leqslant \phi(y)$. Thus $\phi(u) \leqslant \rho(x_n) \leqslant \phi(x_{n+1}) + (1/n) \leqslant \phi(y) + (1/n)$ for all $n$; therefore $\phi(u) \leqslant \phi(y)$, a contradiction.

As a direct consequence of Theorem 1, we have the following.

COROLLARY 1. *Let $\phi: x \to \mathbb{R}$ be a function, bounded above, and satisfying* (1). *Assume*

(4) *for any increasing sequence $\{x_n\}$ in $X$, there exists some $y \in X$ such that $x_n \leqslant y$ for all $n$.*

*Then, for each $a \in X$, there exists some $\bar{a} \in X$ such that $a \leqslant \bar{a}$ and $\phi(S(\bar{a})) = \phi(\bar{a})$.*

*In particular, if we strengthen assumption* (1) *to* (1′) *$x \leqslant y$ and $x \neq y$ imply $\phi(x) < \phi(y)$, then for each $a \in X$ there exists $\bar{a} \in X$ such that $a \leqslant \bar{a}$ and $\bar{a}$ is maximal* (i.e., $S(\bar{a}) = \{\bar{a}\}$).

*Proof.* We apply Theorem 1 to $X = S(a)$; since the assumptions (1) and (2) of Theorem 1 are satisfied and the conclusion of Theorem 1 does not hold, we deduce that (3) is violated at some $\bar{a} \in S(a)$. Therefore, we have $\phi(S(\bar{a})) = \phi(\bar{a})$.

COROLLARY 2. *Let $\phi: X \to \mathbb{R}$ be a function satisfying* (1) *and*

(2′) *for any increasing sequence $\{x_n\}$ in $X$ such that $\phi(x_n) \leqslant C < \infty$ for all $n$, there exists some $y \in X$ such that $x_n \leqslant y$ for all $n$ and $\phi(x_n) \to \phi(y)$ as $n \to \infty$;*

(3′) *for every $x \in X$ and for every $\epsilon > 0$, there exists $x' \in X$ such that $x \leqslant x'$ and $\phi(x) < \phi(x') < \phi(x) + \epsilon$.*

*Then for each $x \in X$, $\phi(S(x)) = [\phi(x), +\infty)$.*

*Proof.* Let $x \in X$ and let $T > \phi(x)$ be fixed. Consider

$$X_0 = \{z \in S(x); \phi(z) \leqslant T\}.$$

Assumption (4) holds on $X_0$ ; indeed let $\{x_n\}$ be an increasing sequence in $X_0$ . By (2′) there exists some $y \in X$ such that $x_n \leqslant y$ and $\phi(x_n) \to \phi(y)$

as $n \to \infty$; hence $\phi(y) \leqslant T$ and thus $y \in X_0$. It follows from Corollary 1 (applied in $X_0$) that there is some $\bar{a} \in X_0$ such that $\phi(S_{X_0}(\bar{a})) = \phi(\bar{a})$. We must have $\phi(\bar{a}) = T$; otherwise $\phi(\bar{a}) < T$ would contradict (3').

## 2. Applications to Nonlinear Semigroups

Let $M$ be a complete metric space. Let $S(t)$ be a semigroup on $M$, i.e., for each $t \geqslant 0$, $S(t)$ is a mapping from $M$ into $M$ such that $S(0) = I$ and $S(t_1 + t_2) = S(t_1) \circ S(t_2)$ for all $t_1, t_2 \geqslant 0$. We assume that

(5) $d(S(t)u, S(t)v) \leqslant e^{\omega t} d(u, v)$ *for all* $t \geqslant 0$, $u, v \in M$ (*and some fixed* $\omega \in \mathbb{R}$);

(6) *for each* $u \in M$, $t \to S(t)u$ *is continuous on* $[0, +\infty)$.

We shall prove the following two theorems.

Theorem 2. *Let* $F \subset M$ *be a closed set and let* $C \geqslant 0$. *Assume*

(7) *for every* $u \in F$,

$$\liminf_{\substack{t \to 0 \\ t>0}} \frac{d(S(t)u, F)}{t} \leqslant C.$$

*Then, for every* $u \in M$ *and every* $t \geqslant 0$ *we have*

$$d(S(t)u, F) \leqslant e^{\omega t} d(u, F) + (C/\omega)(e^{\omega t} - 1) \qquad (\omega \neq 0),$$

(*resp.* $d(S(t)u, F) \leqslant d(u, F) + Ct$, *when* $\omega = 0$).

*Remark.* Theorem 2 for $C = 0$ is due to Martin [15]. It shows in particular that if (7) holds with $C = 0$, then $F$ is stable under $S(t)$.

Theorem 3. *Let* $F \subset M$ *be a closed set. Assume* $\omega < 0$, *so that there exists a unique common fixed point* $p$ *for* $S(t)$, *i.e.*, $S(t)p = p$ *for all* $t \geqslant 0$. *Then we have*

$$(8) \quad d(p, F) = \sup_{u \in F} \inf_{t>0} (d(S(t)u, F)/(1 - e^{\omega t})).$$

The following lemma will play a crucial role in applying the results of Section 1. Let $F \subset M$ be a closed set and let $X = F \times [0, +\infty)$; let $L \geqslant 0$. We define on $X$ the following relation. *Let* $x = (u, p)$ *and* $y = (v, q)$.

(9) $x \leqslant y$ iff $p \leqslant q$ and $d(S(q-p)u, v) \leqslant (L/\omega)(e^{\omega(q-p)} - 1)$, $(\omega \neq 0)$, *(resp.* $p \leqslant q$ *iff* $d(S(q-p)u, v) \leqslant L(q-p)$ *when* $\omega = 0$).

LEMMA 1. *The relation* $x \leqslant y$ *provides an ordering on* $X$. *In addition if for* $x = (u, p)$ *we set* $\phi(x) = p$, *then* $\phi$ *satisfies properties* (1) *and* (2′).

*Proof of Lemma* 1. Clearly $x \leqslant x$; also the relations $x \leqslant y$ and $y \leqslant x$ imply $x = y$. Next let $z = (w, r)$ and assume $x \leqslant y$, $y \leqslant z$. Hence we have $p \leqslant q \leqslant r$ and

(10) $d(S(q-p)u, v) \leqslant (L/\omega)(e^{\omega(q-p)} - 1)$,

(11) $d(S(r-q)v, w) \leqslant (L/\omega)(e^{\omega(r-q)} - 1)$.

It follows from (10) and (5) that

(12) $d(S(r-q)S(q-p)u, S(r-q)v) \leqslant e^{\omega(r-q)}(L/\omega)(e^{\omega(q-p)} - 1)$.

Combining (11) and (12) we have

$$d(S(r-p)u, w) \leqslant (L/\omega)(e^{\omega(r-p)} - 1),$$

i.e., $x \leqslant z$.

Clearly $\phi$ satisfies (1); finally verify (2′). Let $x_n = (u_n, p_n)$ be an increasing sequence in $X$ such that $\phi(x_n) = p_n \leqslant C < +\infty$. Therefore $p_n$ is nondecreasing and converges to some $p$. Since $x_n \leqslant x_{n+k}$, we have for all $n \geqslant 0$ and all $k \geqslant 0$,

(13) $d(S(p_{n+k} - p_n)u_n, u_{n+k}) \leqslant (L/\omega)(e^{\omega(p_{n+k}-p_n)} - 1)$.

We now prove that $\{u_n\}$ is a Cauchy sequence. For every $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $n \geqslant N(\epsilon)$ and all $k \geqslant 0$,

$$(L/\omega)(e^{\omega(p_{n+k}-p_n)} - 1) < \epsilon.$$

It follows from (13) that for all $n \geqslant N(\epsilon)$ and all $k \geqslant 0$, $l \geqslant 0$ we have

(14) $d(u_{n+k}, u_{n+l}) \leqslant d(S(p_{n+k} - p_n)u_n, S(p_{n+l} - p_n)u_n) + 2\epsilon$.

For a *fixed* $n \geqslant N(\epsilon)$, the right-hand side in (14) converges to $2\epsilon$ as $k \to \infty$ and $l \to \infty$. Hence there exists $N'(\epsilon)$ such that for $k \geqslant N'(\epsilon)$ and $l \geqslant N'(\epsilon)$ we have $d(u_{n+k}, u_{n+l}) \leqslant 3\epsilon$. Therefore $\{u_n\}$ is a Cauchy sequence. Let $u_n \to u$ in $F$. Passing to the limit as $k \to +\infty$ in (13) we obtain

$$d(S(p - p_n)u_n, u) \leqslant (L/\omega)(e^{\omega(p-p_n)} - 1) \qquad (n \geqslant 0),$$

i.e., $(u_n, p_n) \leqslant (u, p)$ for all $n \geqslant 0$. Consequently $y = (u, p)$ satisfies $x_n \leqslant y$ for all $n$ and $\phi(x_n) \to \phi(y)$ as $n \to \infty$.

*Proof of Theorem* 2. We apply Corollary 2 in the ordered space $X = F \times [0, +\infty)$ defined by (9) where we fix $L > C$. We have only to check that (3′) holds. Let $x = (u, p)$ and let $\epsilon > 0$ be fixed. It follows from (7) that

$$\liminf_{\substack{t \to 0 \\ t>0}} \left[\frac{d(S(t)\,u, F)}{((e^{\omega t} - 1)/\omega)}\right] \leqslant C.$$

Hence there exists $0 < t < \epsilon$ such that

$$d(S(t)\,u, F) < (L/\omega)(e^{\omega t} - 1).$$

Consequently we can find $u' \in F$ such that $d(S(t)\,u, u') \leqslant (L/\omega)(e^{\omega t} - 1)$ and so we get

$$x = (u, p) \leqslant (u', p') = x',$$

where $p' = p + t$, i.e., (3′) holds.

It follows from the conclusion of Corollary 2 that for each $u \in F$ and each $T > 0$, there exists $v \in F$ such that $(u, 0) \leqslant (v, T)$ or $d(S(T)\,u, v) \leqslant (L/\omega)(e^{\omega T} - 1)$, and since this is true for any $L > C$ we get the conclusion of the theorem in case $u \in F$.

In the general case, let $u \in M$ and let $f \in F$; we have $d(S(t)\,u, S(t)f) \leqslant e^{\omega t} d(u, f)$ and thus $d(S(t)u, F) \leqslant d(S(t)f, F) + e^{\omega t}\, d(u, f)$.

By the previous result $d(S(t)f, F) \leqslant (C/\omega)(e^{\omega t} - 1)$ and so

$$d(S(t)\,u, F) \leqslant (C/\omega)(e^{\omega t} - 1) + e^{\omega t}\, d(u, f)$$

for all $f \in F$. By taking the infimum of the right-hand side over all $f \in F$ we obtain the desired conclusion.

*Proof of Theorem* 3. Let $\Lambda = \mathrm{Sup}_{u \in F} \mathrm{Inf}_{t>0} (d(S(t)\,u, F)/(1 - e^{\omega t}))$. Since

$$\inf_{t>0} \frac{d(S(t)\,u, F)}{1 - e^{\omega t}} \leqslant \lim_{t \to \infty} \frac{d(S(t)\,u, F)}{1 - e^{\omega t}} = d(p, F)$$

we always have $\Lambda \leqslant d(p, F)$.

Suppose now $\Lambda < d(p, F)$ and let $\Lambda < \Lambda' < d(p, F)$. We apply Theorem 1 in the ordered space $X = F \times [0, \infty)$ defined by (9), where $L = -\Lambda'\omega$. We have only to verify that (3) holds. Let $x = (u, p)$ be fixed; since $\Lambda < \Lambda'$ we have

$$\inf_{t>0} (d(S(t)\,u, F)/(1 - e^{\omega t})) < \Lambda'$$

and hence there exists $t > 0$ such that

$$d(S(t)\,u, F) < \Lambda'(1 - e^{\omega t}) = (L/\omega)(e^{\omega t} - 1).$$

Thus there exists some $u' \in F$ such that

$$d(S(t)\,u, u') \leqslant (L/\omega)(e^{\omega t} - 1),$$

i.e., $x = (u, p) \leqslant (u', p') = x'$, where $p' = p + t$. Therefore (3) holds.

The conclusion of Theorem 1 implies that given $u \in F$ there exists a sequence $\{u_n\} \subset F$ and a sequence $t_n \to \infty$ such that $(u, 0) \leqslant (u_n, t_n)$. Thus $d(S(t_n)\,u, u_n) \leqslant \Lambda'(1 - e^{\omega t_n})$ and in particular $d(S(t_n)\,u, F) \leqslant \Lambda'(1 - e^{\omega t_n})$. Passing to the limit as $n \to \infty$ we have $d(p, F) \leqslant \Lambda'$, which is a contradiction.

*Remark.* Using the same argument as in Theorem 3, we can prove the following.

*Assume* $d(S(t)\,u, S(t)\,v) \leqslant d(u, v)$ *for all* $u$ *and* $v$ *and all* $t \geqslant 0$. *Given* $F \subset M$ *closed, we define*

$$\lambda = \lim_{t \to +\infty} \inf\,(d(S(t)\,u, F)/t)$$

(*note that* $\lambda$ *is independent of* $u \in M$). *Then*

$$\operatorname*{Sup}_{u \in F}\ \operatorname*{Inf}_{t > 0}\,(d(S(t)\,u, F)/t) = \lambda.$$

## 3. Other Applications

In this section we point out the relationship between the results of Sections 1 and 2 and previously known results.

First we derive a simple Corollary from Corollary 1.

Corollary 3. *Let* $X$ *be a Hausdorff topological space with an ordering structure. Let* $\psi\colon X \to \mathbb{R}$ *be a function bounded below. Assume*

(15) $S(x)$ *is closed for each* $x \in S$;

(16) $x \leqslant y$ *and* $x \neq y$ *imply* $\psi(y) < \psi(x)$;

(17) *any nondecreasing sequence is relatively compact.*

*Then for each* $a \in X$ *there exists* $\bar{a} \in X$ *such that* $a \leqslant \bar{a}$ *and* $\bar{a}$ *is maximal.*

*Proof of Corollary* 3. We apply Corollary 1 to $\phi = -\psi$. Note that if $\{x_n\}$ is an increasing sequence, then we can choose a subsequence $x_{n_k} \to y$. We have to verify that $x_n \leqslant y$ for all $n$. Indeed, given $n$, we have $n_k \geqslant n$ for $k$ large enough to that $x_n \leqslant x_{n_k}$ for $k$ large enough. Thus by (15), $y \in S(x_n)$ for all $n$.

An immediate consequence is Ekeland's theorem [10]:

COROLLARY 4. *Let $X$ be a complete metric space. Let $\psi: X \to \mathbb{R}$ be a l.s.c. function bounded below. Then there exists some $\bar{a} \in X$ such that*

$$(18) \quad \psi(u) - \psi(\bar{a}) > -d(\bar{a}, u) \text{ for all } u \in X,\ u \neq \bar{a}.$$

To prove Corollary 4 define on $X$ the ordering $x \leqslant y$ iff $\psi(y) - \psi(x) \leqslant -d(x, y)$. For any increasing sequence $\{x_n\}$, $\psi(x_n)$ converges and therefore $\{x_n\}$ is a Cauchy sequence. Thus we can apply Corollary 3.

Ekeland's theorem has been put by Brondsted [2] in a slightly more general form, which includes also a well-known lemma of Bishop and Phelps (see [1, 16]):

COROLLARY 5. *Let $\mathcal{U}$ be a Hausdorff uniformity on an ordered $X$. Let $\psi: X \to \mathbb{R}$ be a function bounded below. Assume*

(i) *$S(x)$ is complete for each $x \in X$;*

(ii) *$x \leqslant y$ implies $\psi(y) \leqslant \psi(x)$;*

(iii) *for each $U \in \mathcal{U}$ there exists $\delta > 0$ such that $x_1 \leqslant x_2$ and $\psi(x_1) - \psi(x_2) < \delta$ implies $(x_1, x_2) \in U$.*

*Then for each $a \in X$, there exists $\bar{a} \in X$ such that $a \leqslant \bar{a}$ and $\bar{a}$ is maximal.*

We mention also the Caristi–Kirk fixed point theorem (see [7, 13]) which is a reformulation of Ekeland's theorem:

COROLLARY 6. *Let $X$ be a complete metric space. Let $\psi: X \to \mathbb{R}$ be a l.s.c. function, bounded below. Let $T: X \to X$ be a mapping satisfying*

$$(19) \quad d(u, Tu) \leqslant \psi(u) - \psi(Tu) \text{ for all } u \in X.$$

*Then $T$ has a fixed point.*

Indeed, it follows from Corollary 4 that there exists some $\bar{a} \in X$ satisfying (18). We get $T\bar{a} = \bar{a}$; otherwise $T\bar{a} \neq \bar{a}$ would imply $\psi(T\bar{a}) - \psi(\bar{a}) > -d(\bar{a}, T\bar{a})$ and would contradict (19). Note that Corollary 4 could also be deduced from Corollary 6. Indeed suppose that the con-

clusion of Corollary 4 does not hold. Then for each $x \in X$ there exists some $y \in X$, $y \neq x$ such that $\psi(y) - \psi(x) \leqslant -d(x, y)$. Hence we could build a mapping $T: X \to X$ satisfying (19) and having no fixed point.

We conclude with some geometrical applications connected to the theory of normal solvability (see [3, 5, 6]). The Drop theorem [10] is a direct consequence of Theorem 3.

COROLLARY 7. *Let $S$ be a closed subset of a Banach space $E$ and let $z \in E \backslash S$. Let $0 < r < d(z, S) < R$. Then there exists a point $u \in S$ such that $\| u - z \| \leqslant R$ and $S \cap \operatorname{conv}(B_r(z) \cup \{u\}) = \{u\}$, where we denote $B_r(z) = \{x \in E; \| x - z \| \leqslant r\}$.*

*Proof.* We can always assume that $z = 0$. We apply Theorem 3 to $M = B_R(0)$, $F = S \cap M$, $S(t)\, v = e^{-t}v$. Let $r < r' < d(0, S)$; it follows from (8) that there exists some $u \in F$ such that

$$(d(e^{-t}u, F)/(1 - e^{-t})) \geqslant r' \qquad \text{for all} \quad t > 0,$$

i.e., $d(\sigma u, F) \geqslant r'(1 - \sigma)$ for all $0 \leqslant \sigma \leqslant 1$. We prove now that $S \cap \operatorname{conv}(B_r(0) \cup \{u\}) = \{u\}$.

Let $x \in S \cap \operatorname{conv}(B_r(0) \cup \{u\})$, so that $x = \alpha u + (1 - \alpha)v$ for some $\alpha \in [0, 1]$ and some $v \in B_r(0)$. Thus in particular $\| x \| \leqslant R$ and thus $x \in F$. Therefore $\| \alpha u - x \| \geqslant d(\alpha u, F) \geqslant r'(1 - \alpha)$, i.e., $(1 - \alpha)\| v \| \geqslant r'(1 - \alpha)$. Hence $\alpha = 1$ and $x = u$.

*Remark.* Corollary 7 can be proved directly from Corollary 5 (see [2]) or from Corollary 6 (see [14, Theorem 2]).

A stronger variant of this result was obtained earlier by one of the authors [6, 13]:

THEOREM 4. *Let $S$ be a closed subset of a Banach space $E$ and let $z \in E \backslash S$. Let $0 < \rho < d(z, S)$ and let $s \in S$. Let*

$$K = \bigcup_{\lambda \geqslant 0} \lambda B_\rho(z - s), \qquad K_1 = \bigcup_{0 \leqslant \lambda \leqslant 1} \lambda B_\rho(z - s).$$

*Then there exists some $u \in S \cap (K_1 + s)$ such that $S \cap (K + u) \cap B_\epsilon(u) = \{u\}$ for*

$$\epsilon < d(z, S) - \rho.$$

*Proof of Theorem* 4. Let $F = S \cap (K_1 + s)$. We apply Corollary 1 to the bounded set $F$ with the ordering relation: $x \leqslant y$ provided $y - x \in K$. The function $\phi$ is chosen as an element of $E^*$ such that

$\phi(x) \geqslant \alpha \| x \|$ ($x \in K$ and $\alpha > 0$). With these choices Corollary 1 implies the existence of a maximal element $u \in F$ with respect to the ordering. For this $u$ and $\epsilon < d(z, S) - \rho$, we have $S \cap (K + u) \cap B_\epsilon(u) = \{u\}$. Indeed, let $s_1 \in S \cap (K + u) \cap B_\epsilon(u)$. Since $\epsilon < d(z, S) - \rho$ one easily verifies that $B_\epsilon(u) \cap (K + s) \subset K_1 + s$ so that $s_1 \in K_1 + s$. By the maximality property of $u$, it follows that $s_1 = u$.

*Remark.* Theorem 4 implies Corollary 7 by a simple direct argument.

Indeed, let $d_0 = d(z, S), 0 < r < d_0 < R$. Choose $\rho$ with $r < \rho < d_0$, and apply Theorem 4 to a point $s$ in $S$ with $d(z, s) \leqslant (1 + \eta) d_0$ for $\eta > 0$ to be chosen later. By Theorem 4, there exists $u$ in $S \cap (K_1 - s)$ such that

$$S \cap (K + u) \cap B_\epsilon(u) = \{u\}$$

for $\epsilon < d_0 - \rho$. Since $u = (1 - \lambda) s + \lambda\xi$, $0 \leqslant \lambda \leqslant 1$ with $\xi \in B_\rho(z)$, we have $d_0 \leqslant \| u - z \| \leqslant (1 - \lambda) \| s - z \| + \lambda \| \xi - z \| \leqslant (1 - \lambda)(1 + \eta) d_0 + \lambda\rho$, so that

$$\lambda((1 + \eta) d_0 - \rho) \leqslant \eta d_0 .$$

For any $\delta > 0$, we can make $\lambda((1 + \eta) d_0 + \rho) < \delta$ by making $\eta$ sufficiently small. Hence

$$\| u - s \| = \lambda \| \xi - s \| < \delta$$

for such a choice of $\eta$.

The convex $K_1 + u$ contains the drop $\operatorname{conv}(B_r(z) \cup \{u\})$ for $r < \rho$ for $\| u - s \|$ sufficiently small.

*Remark.* Either Corollary 7 or Theorem 4 implies a nonconvex generalization of Bishop–Phelps theorem [1]; namely, the points $u$ in bdry $S$ (the boundary of a closed subset $S$ in a Banach space) having the local supporting cone property are dense in bdry $S$. We recall that $u$ has the local supporting cone property if there exists a cone $K$ with nonempty interior and $\epsilon > 0$ such that $S \cap (K + u) \cap B_\epsilon(u) = \{u\}$. This result was first proved in [3].

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