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MAXIMAL MONOTONE OPERATORS IN NONREFLEXIVE BANACH SPACES AND NONLINEAR INTEGRAL EQUATIONS OF HAMMERSTEIN TYPE[1]

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Let Y be a Banach space, Y^* its conjugate space, X a weak*-dense closed subspace of Y^* with the induced norm. We denote the pairing between x in X and y in Y by (y, x). If T is a mapping from X into 2^Y, T is said to be monotone if for each pair of elements $[x, y]$ and $[u, w]$ of $G(T)$, the graph of T, we have $(y-w, x-u) \geqq 0$. T is said to be maximal monotone from X to 2^Y if T is monotone and maximal among monotone mappings in the sense of inclusion of graphs.

The theory of maximal monotone mappings has been intensively developed in the case in which Y is reflexive and $X = Y^*$. In this note, we present an extension of this theory to the case in which X and Y are not reflexive, and show that this extended theory can be used to give a new and more conceptual proof of a general existence theorem for solutions of nonlinear integral equations of Hammerstein type established by the writers in [2] by more concrete arguments.

An essential tool in our discussion is supplied by the following definitions:

DEFINITION 1. *Let T be a mapping from X into 2^Y. Then T is said to be X-coercive if for each real number k, the set $\{x \mid x \in X$, there exists w in $T(x)$ such that $(w, x) \leqq k\|x\|\}$ is contained in a convex weak* compact subset A_k of X.*

THEOREM 1. *Let T be a monotone mapping from X to 2^Y. Suppose that 0 lies in $D(T)$, the effective domain of T, and that T is X-coercive. Then the range $R(T)$ of T is all of Y.*

We use the following extension of the concept of pseudo-monotonicity [1]:

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DEFINITION 2. *Let f be a mapping of X into Y. Then f is said to be pseudo-monotone from X to Y if the following three conditions are satisfied:*

(a) *f is continuous from each finite-dimensional subspace of X to the weak topology on Y induced by the functionals from X.*

(b) *f maps each weak* compact subset of X into a subset of Y which is compact with respect to the weak topology induced by X on Y.*

(c) *Let $\{u_\gamma\}$ be a filter base on X contained in a weak* compact subset of X and converging in the weak*-topology to an element u of X, and suppose that* $\limsup(f(u_\gamma), u_\gamma - u) \leqq 0$. *Then $f(u_\gamma)$ converges to $f(u)$ in the weak topology on Y induced by X and* $\lim(f(u_\gamma)u_\gamma) = (f(u), u)$.

THEOREM 2. *Let T be a maximal monotone mapping from X to 2^Y which is X-coercive. Let f be a pseudo-monotone mapping from X to Y in the sense of Definition 2, such that for some constant c, $(f(u), u) \geqq -c\|u\|$. Suppose that $0 \in D(T)$. Then the range of $(T+f)$ is all of Y.*

LEMMA 1. *Let f be a monotone mapping from X to Y which satisfies conditions* (a), (b) *of Definition 2. Then f is pseudo-monotone from X to Y.*

COROLLARY TO THEOREM 2. *Let T be a maximal monotone mapping from X to 2^Y such that T is X-coercive, $0 \in D(T)$. Let f be a monotone mapping from X to Y satisfying conditions* (a) *and* (b) *of Definition 2. Suppose that for some constant c and all x in X, $(f(x), x) \geqq -c\|x\|$. Then the range of the mapping $(T+f)$ is all of Y.*

In the case of reflexive Banach spaces, the proof of the corresponding existence theorems rests upon the following monotone extension theorem of Debrunner and Flor [4] (cf. [3] for the corresponding multivalued generalization and a conceptual development of the proof).

PROPOSITION 1. *Let F be a finite subset of $X \times Y$ with F monotone, i.e., $[u_j, w_j]$, $[u_k, w_k]$ in F implies that $(u_j - u_k, w_j - w_k) \geqq 0$. Let C be the convex closure of the first components $\{u_1, \cdots, u_n\}$ of F, and suppose that f is a continuous mapping from C to Y endowed with the weak topology induced by X. Then there exists an element x of C such that*

$$(w_j - f(x), u_j - x) \geqq 0, \qquad (j = 1, \cdots, n).$$

To prove our new existence theorems in the nonreflexive case, we use a modification of the Debrunner-Flor result given in Proposition 2 below, which is suggested by the results of Minty [5].

PROPOSITION 2. *Let F be a finite monotone subset of $X \times Y$, and let C_0 be the convex closure of $\{0, u_1, u_2, \cdots, u_n\}$. Suppose that f is a continuous mapping from C_0 to Y endowed with the weak topology induced by X. Then there exists a point x of C_0 of the form $x = \sum_{j=1}^n \xi_j u_j$, with $0 \leqq \xi_j$, $\sum_{j=1}^n \xi_j \leqq 1$,* *such that*

$$(w_j + f(x), u_j - x) \geqq 0, \qquad (1 \leqq j \leqq n),$$

and

$$\sum_{j=1}^{n} \xi_j(w_j, u_j) + (f(x), x) \leqq 0.$$

PROOF OF PROPOSITION 2. For the case in which $f \equiv 0$, the result was obtained by Minty in [5] by a direct argument. We obtain the more general result for nonzero f by showing that Proposition 2 can be derived directly from Proposition 1. We note first that since the result really depends only upon the finite-dimensional space spanned by $\{u_1, \cdots, u_n\}$ and linear functionals upon this space, we may assume that $X = Y = H_0$, a finite-dimensional Hilbert space. We may assume that H_0 is contained in a larger Hilbert space H, and that H_1 is an n-dimensional Hilbert space in H which is orthogonal to H and with orthonormal basis $\{h_1, \cdots, h_n\}$. Let $v_j = u_j + h_j$ for each j, and let C_1 be the convex closure of $\{0, v_1, \cdots, v_n\}$. We form a new monotone set F_1 in $H \times H$ whose elements consist of $[v_j, w_j]$, $1 \leqq j \leqq n$, together with the single additional element $[0, w_0]$, where $w_0 = \sum_{j=1}^{n} (w_j, u_j)h_j$. Let ζ be the mapping of C_1 into C_0 given by $\zeta(\sum_j \xi_j v_j) = \sum_j \xi_j u_j$, and let g be the mapping of C_1 into H given by $g(x) = -f(\zeta(x))$.

We apply Proposition 1 to the monotone set F_1 and the mapping g. Then there exists an element y in C_1 of the form $y = \sum_{j=1}^{n} \xi_j v_j$ with $0 \leqq \xi_j$, $\sum_{j=1}^{n} \xi_j \leqq 1$, such that for $1 \leqq j \leqq n$,

$$(w_j - g(y), v_j - y) \geqq 0, \qquad (w_0 - g(y), 0 - y) \geqq 0.$$

Since for each j, $w_j - g(y)$ lies in H_0, the first n inequalities imply that if $x = \zeta(y) = \sum_{j=1}^{n} \xi_j u_j$, then $(w_j + f(x), v_j - x) \geqq 0$. For the last inequality, we see that $(g(y), y) = -(f(x), x)$, while

$$(w_0, y) = \sum_{j=1}^{n} (w_j, u_j)\xi_j(h_j, h_j) = \sum_{j=1}^{n} \xi_j(w_j, u_j).$$

Hence, this inequality becomes

$$\sum_{j=1}^{n} \xi_j(w_j, u_j) + (f(x), x) \leqq 0. \quad \text{Q.E.D.}$$

Theorem 1 is a special case of Theorem 2.

LEMMA 2. *Let X be a closed subspace of Y^*, A a subset of X. Suppose that for each $\varepsilon > 0$, there exists a weak* compact subset A_ε of X such that A is contained in the ε-neighborhood of A_ε in X. Then A is relatively weak* compact in X.*

PROOF. Since each A_ε must be bounded, A is bounded in X and hence relatively weak* compact in Y^*. Hence it suffices to show that the weak* closure A_1 of A in Y^* is contained in X. Since A is contained in $A_\varepsilon+B_\varepsilon$, where B_ε is the closed unit ball in Y^*, and since both A_ε and B_ε are weak* compact, it follows that A_1 must be contained in the weak* compact set $A_\varepsilon+B_\varepsilon$. Hence A_1 is contained in the ε-neighborhood of X for each $\varepsilon>0$, i.e. $A_1\subset\bigcap_{\varepsilon>0} N_\varepsilon(X)=X$ since X is a closed subspace of Y^*. Q.E.D.

PROOF OF THEOREM 2. Since $0\in D(T)$, we may add a constant to the values of T and assume that $[0, 0]$ lies in the graph of T by absorbing the constant into the mapping f without perturbing the pseudo-monotonicity of f. Similarly, if we wish to prove that an element w_0 of Y lies in the range of $(T+f)$, it suffices to prove that 0 lies in $R(T+f_0)$, where $f_0(x)=f(x)-w_0$. Thus, it suffices to show that $0\in R(T+f)$.

Let F be a finite subset of $G(T)$, the graph of T. We apply Proposition 2 to $F=\{[u_1, w_1],\cdots,[u_n, w_n]\}$ to obtain a point $x_F=\sum_{j=1}^n \xi_j u_j$ with $\xi_j\geqq 0$ for each j, and $\sum_{j=1}^n \xi_j\leqq 1$ satisfying the two systems of inequalities:

$$(1)\qquad (w_j+f(x_F), u_j-x_F)\geqq 0 \qquad (j=1,\cdots,n),$$

$$(2)\qquad \sum_{j=1}^n \xi_j(w_j, u_j)+(f(x_F), x_F)\leqq 0.$$

Corresponding to any $k>0$, we may group the set of first components of the subset F into two classes, so that for $1\leqq j\leqq r$, u_j lies in the set $\{x\,|\,x\in X$, there exists w in $T(x)$ such that $(w, x)\leqq k\|x\|\}$, and for $j>r$, u_j lies outside this set. We set

$$u_{k,F}=\sum_{j=1}^r \xi_j u_j, \qquad v_{k,F}=\sum_{j=r+1}^n \xi_j u_j.$$

Since the elements of the first set of u_j all lie by hypothesis in the weak* compact convex subset A_k of X and $0\in A_k$, it follows that $u_{k,F}$ lies in A_k. Since T is monotone, and by construction $0\in T(0)$, it follows that for all j, $(w_j, u_j)\geqq 0$. Moreover, for all x, $(f(x), x)\geqq -c\|x\|$ by hypothesis. For $r+1\leqq j\leqq n$, $(w_j, u_j)\geqq k\|u_j\|$. Hence

$$k\,\|v_{k,F}\|\leqq k\sum_{j=r+1}^n \xi_j\,\|u_j\|\leqq \sum_{j=r+1}^n \xi_j(w_j, u_j)\leqq c\,\|x_F\|.$$

If we apply the last inequality for a given value k_0 of k, $k_0>c$, we find that

$$(k_0-c)\,\|v_{k_0,F}\|\leqq c\,\|u_{k_0,F}\|,$$

while $u_{k_0,F}$ lies in the bounded subset A_{k_0} of X. Hence, $\|v_{k_0,F}\|\leqq c_0$ independently of F, so that $\|x_F\|\leqq c_1$ independently of F. If we employ the same inequality for a general large k, we see that $\|v_{k,F}\| \leqq cc_1k^{-1}$ so that

$$\operatorname{dist}(x_F, A_k) \leqq \|v_{k,F}\| \leqq cc_1k^{-1} \to 0 \qquad (k \to \infty).$$

It follows from Lemma 2 that the set A consisting of all solutions x_F of the inequalities (1) and (2) above for finite subsets F of $G(T)$ is contained in a weak* compact subset of X. By property (b) of Definition 2 for pseudo-monotonicity of f, it follows that the set B consisting of all elements of the form $f(x_F)$ is contained in a subset of Y which is compact in the X-weak topology of Y. Hence if we consider the filter base consisting of $\{[x_F, f(x_F)];$ F containing $F_1\}$ for finite subsets F_1 of $G(T)$, the corresponding filter has an adherent point $[x_0, y_0]$ in $X\times Y$ in the product of the weak* topology on X and the X-weak topology on Y. For any element $[x, y]$ in F_1 and $[x_F, f(x_F)]$ in the corresponding element of the filter base, we have $(y+f(x_F), x-x_F)\geqq 0$, i.e.,

$$(f(x_F)x_F) \leqq (f(x_F), x) + (y, x - x_F).$$

Hence, taking limits on the filter,

$$\begin{aligned}\limsup(f(x_F), x_F - x_0) &\leqq \lim(f(x_F), x - x_0) + (y, x - x_F)\\ &\leqq (y_0, x - x_0) + (y, x - x_0) = (y_0 + y; x - x_0).\end{aligned}$$

If the infimum of the bound on the right side for all elements $[x, y]$ of $G(T)$ is nonpositive, it follows from condition (c) of Definition 2 for the pseudo-monotonicity of f that $f(x_0)=y_0$ and $\limsup(f(x_F), x_F-x_0)=0$. It then follows that $(y_0+y, x-x_0)\geqq 0$ for all $[x, y]$ in $G(T)$, and since T is maximal monotone, $-y_0 \in T(x_0)$, i.e. $0 \in (T+f)(x_0)$. On the other hand, if $(y+y_0, x-x_0)\geqq\beta>0$ for all $[x, y]$ in $G(T)$, it follows that $-y_0 \in T(x_0)$, and hence $(-y_0+y_0, x_0-x_0)=0\geqq\beta>0$, which is a contradiction. Thus we have shown that 0 lies in $(T+f)(X)$, from which it follows as we have noted earlier that $Y=(T+f)(X)$. Q.E.D.

Theorem 2 may be applied to abstract Hammerstein equations through the following:

THEOREM 3. *Let f_0 be a monotone mapping of Y into X which is continuous from finite-dimensional subspaces of Y to the weak* topology of X, f a pseudo-monotone mapping of X into Y, with $(f(x), x)\geqq -c\|x\|$ for a suitable constant c and all x in X. Suppose that f_0 maps bounded subsets of Y into bounded subsets of X, $f_0(y_0)=0$ for some y_0 in Y, and f_0 satisfies conditions:*

(1) *f_0 is tricyclically monotone, i.e. for any three points y, u, and v of Y, we have*

$$(f_0(y), y - u) + (f_0(u), u - v) + (f_0(v), v - y) \geqq 0.$$

(2) *For any constant k, the set $\{f_0(y); (f_0(y), y)\leqq k, \|f_0(y)\|\leqq k\}$ is contained in a weak* compact convex subset of X.*

Then the range of the mapping $(I+ff_0)$ of Y into Y is the whole of the Banach space Y. In addition, if f is monotone, then $(I+ff_0)$ is one-to-one and has a continuous inverse.

PROOF OF THEOREM 3. For reasons of brevity, we shall only give the reduction of the existence assertion of Theorem 3 to the results of Theorem 2. Let y_1 be a given element of Y. If we seek to solve the equation $y+f(f_0(y))=y_1$, we introduce $u=f_0(y)$ as a new variable, and $y_1 \in (f_0^{-1}+f)(u)$, and it follows easily that if this latter equation has a solution u, then so does the original equation. Since f is pseudo-monotone from X to Y and satisfies the inequality $(f(u), u) \geqq -c\|u\|$, while f_0^{-1} is the inverse of a maximal monotone mapping from Y to X, and hence is itself maximal monotone from X to 2^Y, and $0 \in D(f_0^{-1})$, the applicability of Theorem 2 is reduced to showing that the mapping f_0^{-1} satisfies the basic hypothesis of X-coercivity.

We consider three points y, u, and y_0 in Y with $f_0(y_0)=0$, and obtain the inequality

$$(f_0(y), y-u) + (f_0(u), u-y_0) \geqq 0.$$

Hence

$$(f_0(y), u) \leqq (f_0(y), y) + (f_0(u), u-y_0).$$

Taking the supremum of $(f_0(y), u)$ over all u in the ball of radius R about 0 in Y and noting that f_0 is bounded on bounded sets, we see that

$$R\,\|f_0(y)\| \leqq (f_0(y), y) + c(R) \qquad (R > 0).$$

Thus if $[u, y]$ lies in $G(f_0^{-1})$, i.e., $u=f_0(y)$,

$$R\,\|u\| \leqq (u, y) + c(R).$$

Hence if $(u, y) \leqq k\|u\|$, it follows that

$$R\,\|u\| - c(R) \leqq k\,\|u\| \qquad (R > k),$$

from which it follows that $\|u\| \leqq s(k)$. Moreover, u lies in the set $\{f_0(y); \|f_0(y)\| \leqq s(k), (f_0(y), y) \leqq ks(k)\}$, which by condition (2) is contained in a weak* compact subset of X. Hence f_0^{-1} is X-coercive and the existence result follows from Theorem 2. Q.E.D.

To use Theorem 3 to obtain an existence theorem for the Hammerstein integral equation

$$u(t) + \int_\Omega k(t, s)h(s, u(s))\,ds = v(t) \qquad (t \in \Omega)$$

as in [2], we choose X to be $L^1(\Omega)$, $Y=L^\infty(\Omega)$. The mapping f is the monotone linear operator $(ku)(t)=\int_\Omega k(t, s)u(s)\,ds$, while f_0 is the Niemitskyii operator $(f_0(v))(t)=h(t, v(t))$. The condition (1) that f_0 is tricyclically monotone follows from the fact that f_0 is a potential operator, while condition (2) follows immediately from the criterion of Dunford and Pettis for a subset of $L^1(\Omega)$ to be relatively weakly compact.

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