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EXISTENCE THEOREMS FOR NONLINEAR INTEGRAL EQUATIONS OF HAMMERSTEIN TYPE[1]

BY HAIM BREZIS AND FELIX E. BROWDER

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Let Ω be a σ-finite measure space with measure denoted by dx. A nonlinear integral equation of Hammerstein type on Ω is an equation of the form

$$u(x) + \int_\Omega k(x, y) f(y, u(y))\, dy = v(x) \qquad (x \in \Omega), \tag{1}$$

where we seek a real-valued function u on Ω which satisfies the relation (1) for a given kernel $k(x, y)$, nonlinear function $f(y, u)$, and a given inhomogeneous term v. If u and v are r-vector functions with real components for an integer $r>1$, one speaks of a system of Hammerstein equations where f is a function from $\Omega \times R^r$ into R^r, and for each x and y in Ω, $k(x, y)$ is a linear transformation on R^r.

In two recent papers [1], [2], we have presented new methods of obtaining solutions of Hammerstein equations using techniques from the theory of monotone mappings between Banach spaces. In the present note, we present some new general results on the existence of solutions of Hammerstein equations and systems for the case in which the linear transformation defined by the kernel $k(x, y)$ is compact and in which, therefore, the methods of the theory of compact mappings in Banach

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spaces can be applied. The hypotheses of these existence theorems are drastically weaker than those of previously published results (cf. [3], [4] for extended discussions of results on Hammerstein equations obtained by compact operator methods).

We begin by transforming the Hammerstein equation or system into an equivalent functional equation in an appropriate Banach space. For the Hammerstein equations, we assume throughout that k is a measurable function on $\Omega\times\Omega$ and that $f(y, u)$ satisfies the Carathéodory condition. For the system, we assume similar conditions for the corresponding $(r\times r)$-matrix and r-vector functions. We introduce the operators

$$(Ku)(x) = \int_\Omega k(x, y)u(y)\,dy \qquad (x \in \Omega), \tag{2}$$

$$(Fv)(x) = f(x, v(x)) \qquad (x \in \Omega). \tag{3}$$

We impose one of the following sets of hypotheses:

(H_p): For a given p with $1<p<\infty$,
  (a) K is a compact linear mapping from $L^{p'}(\Omega)$ to $L^p(\Omega)$.
  (b) For each v in $L^p(\Omega)$, Fv lies in $L^{p'}(\Omega)$.

(H^∞): (a) K is a compact linear mapping of $L^1(\Omega)$ into $L^\infty(\Omega)$.
  (b) For each $R>0$, there exists a function g_R in $L^1(\Omega)$ such that $|f(y, u)|\leqq g_R(y)$ on Ω for $|u|\leqq R$.

THEOREM 1. *Suppose that the hypothesis* (H^∞) *is satisfied for a single Hammerstein equation, and that there exist a constant* $R_0>0$ *and* ψ *in* $L^1(\Omega)$ *such that* $[f(y, u)-\psi(y)]u\geqq 0$ *for* $|u|\geqq R_0$, y *in* Ω. *Then the Hammerstein equation* (1) *has a solution* u *in* $L^\infty(\Omega)$ *for each* v *in* $L^\infty(\Omega)$.

THEOREM 2. *Suppose* $\text{meas}(\Omega)<+\infty$, *the hypothesis* ($H_p$) *is satisfied for a single Hammerstein equation for a given* p, $1<p<\infty$, *and that there exist a constant* $R_0>0$ *and* ψ *in* $L^{p'}(\Omega)$ *such that* $[f(y, u)-\psi(y)]u\geqq 0$ *for* $|u|\geqq R_0$, y *in* Ω. *Then the Hammerstein equation* (1) *has a solution* u *in* $L^p(\Omega)$ *for each* v *in* $L^p(\Omega)$.

THEOREM 3. *Suppose that the hypothesis* (H_∞) *is satisfied for a system of Hammerstein equations, and that there exist constants* $R_0>0$, $c>0$ *and* ψ *in* $L^1(\Omega)$ *such that* $(f(y, u)-\psi(y))\cdot u\geqq c|f(y, u)-\psi(y)|\cdot|u|$ *for* $|u|\geqq R_0$ *and* y *in* Ω. *Then the Hammerstein system* (1) *has a solution* u *in* $L^\infty(\Omega)$ *for each* v *in* $L^\infty(\Omega)$.

THEOREM 4. *Suppose that* $\text{meas}(\Omega)<+\infty$ *and the hypothesis* (H_p) *is satisfied for a system of Hammerstein equations, and that there exist constants* $R_0>0$, $c>0$ *and an element* ψ *of* $L^{p'}(\Omega)$ *such that*

$$(f(y, u) - \psi(y))\cdot u \geqq c\,|f(y, u) - \psi(y)|\cdot|u|$$

for $|u| \geqq R_0$ *and* y *in* Ω. *Then the Hammerstein equation* (1) *has a solution* u *in* $L^p(\Omega)$ *for each* v *in* $L^p(\Omega)$.

We remark that the dot product on the left side of the inequalities in the hypotheses of Theorems 3 and 4 refers to the inner product in the Euclidean space R^r.

To simplify and normalize the proofs of Theorems 1 to 4, let us note first that the Hammerstein equation (1) is equivalent to the modified equation:

$$u(x) + \int_\Omega k(x, y) f_1(y, u(y))\, dy = v_1(x) \qquad (x \in \Omega) \tag{4}$$

with

$$f_1(y, u) = f(y, u) - \psi(y), \tag{5}$$

$$v_1(y) = v(y) - (K\psi)(y). \tag{6}$$

After this formal change, the inequalities in the hypotheses of the respective theorems have $\psi(y) \equiv 0$. We shall assume this henceforward in the argument. Second, let us observe that in the case of the scalar equation considered in Theorems 1 and 2, the hypothesis $f(y, u)u \geqq 0$ for $|u| \geqq R_0$ implies that $f(y, u)u \geqq |f(y, u)| \cdot |u|$, i.e., that the type of hypotheses imposed in Theorems 3 and 4 for systems holds for scalar equations with $c=1$ if the apparently weaker hypotheses of Theorems 1 and 2 are imposed. Thus, Theorems 1 and 2 follow from Theorems 3 and 4 for systems respectively.

LEMMA 1. *Under the hypothesis of* (H_p), F *is a continuous mapping of* $L^p(\Omega)$ *into* $L^{p'}(\Omega)$ *which maps bounded sets of* $L^p(\Omega)$ *into bounded sets of* $L^{p'}(\Omega)$. *Moreover, there exist a constant* c_1 *and* h *in* $L^{p'}(\Omega)$ *such that for all* y *and* u, $|f(y, u)| \leqq c_1 |u|^{p-1} + h(y)$.

PROOF OF LEMMA 1. This is established in the opening chapter of Krasnoselsky's book [4].

The proofs of Theorems 3 and 4 depend upon the proof of *a priori* inequalities for solutions of the Hammerstein system (1) which in turn rest upon the inequalities established in the following lemmas.

LEMMA 2. *Suppose that* $f(y, u)$ *satisfies the condition* (b) *of the hypothesis* (H_∞), *and that* $(f(y, u) \cdot u) \geqq c|f(y, u)| \cdot |u|$ *for* $|u| \geqq R_0$. *Then for each* $k > 0$, *there exists a constant* $c(k)$ *depending on* k *such that*

$$(F(u), u) \geqq k \, \|F(u)\|_{L^1} - c(k) \qquad (u \in L^\infty(\Omega)) \tag{7}$$

where $F(u)$ *is the Niemytskii operator defined by the equation* (3).

PROOF OF LEMMA 2. It suffices to prove the inequality (7) for all sufficiently large k. Let $R \geqq R_0$. Then

$$(F(u), u) = \int_{|u| \leqq R} f(y, u(x)) \cdot u(x)\, dx + \int_{|u| \geqq R} f(x, u(x)) \cdot u(x)\, dx = I_1 + I_2.$$

For I_1, we have the estimate

$$|I_1| \leqq \int_{|u|<R} g_R(x)\, |u(x)|\, dx \leqq R\, \|g_R\|_{L^1(\Omega)}.$$

For the integrand in I_2, we have the inequality

$$f(x, u(x)) \cdot u(x) \geqq |f(x, u(x))| \cdot |u(x)| \geqq cR\, |f(x, u(x))|.$$

Hence

$$I_2 \geqq \int_{|u(x)| \geqq R} cR\, |f(x, u(x))|\, dx \geqq cR\, \|F(u)\|_{L^1} - Rc \int_{|u(x)| \leqq R} |f(x, u(x))|\, dx$$
$$\geqq cR\, \|F(u)\|_{L^1} - cR\, \|g_R\|_{L^1}.$$

Combining these various inequalities, we see that

$$(F(u), u) \geqq cR\, \|F(u)\|_{L^1(\Omega)} - \chi(R).$$

Setting $k = cR$, we obtain the desired result. Q.E.D.

LEMMA 3. *Suppose that* $\operatorname{meas}(\Omega) < \infty$ *and that* $f(y, u)$ *satisfies the condition* (b) *of the hypothesis* (H_p), *and that* $(f(y, u)) \cdot u \geqq c|f(y, u)| \cdot |u|$ *for* $|u| \geqq R_0$. *Then there exist constants* $c_1 > 0$, $c_2 \geqq 0$ *such that*

$$(F(u), u) \geqq c_1\, \|F(u)\|_{L^{p'}}^{p'} - c_2 \qquad (u \in L^p(\Omega)), \tag{8}$$

and as a consequence for each $k > 0$, *there exists a constant* $c(k)$ *such that*

$$(F(u), u) \geqq k\, \|F(u)\|_{L^{p'}} - c(k) \qquad (u \in L^p(\Omega)). \tag{9}$$

PROOF OF LEMMA 3.

$$(F(u), u) = \int_{|u(x)| < R_0} f(x, u(x)) \cdot u(x)\, dx$$
$$+ \int_{|u(x)| \geqq R_0} f(x, u(x)) \cdot u(x)\, dx = I_1 + I_2.$$

By Lemma 1, we have the inequality

$$|f(x, u(x))| \leqq c_0\, |u(x)|^{p-1} + h(x),$$

with h in $L^{p'}(\Omega)$. Hence, we may estimate I_1 and obtain

$$\begin{aligned} |I_1| &\leqq \int_{|u|\leqq R_0} c_0 |u(x)|^p \, dx + \int_{|u|\leqq R_0} |u(x)| \, h(x) \, dx \\ &\leqq c_0 R_0^p \operatorname{meas}(\Omega) + \left(\int_{|u|\leqq R_0} |u(x)|^p \, dx \right)^{1/p} \|h\|_{L^{p'}} \\ &\leqq c_0 R_0^p \operatorname{meas}(\Omega) + \operatorname{meas}(\Omega)^{1/p} R_0 \|h\|_{L^{p'}}. \end{aligned}$$

For the integrand of I_2, we have the inequality

$$f(x, u(x)) \cdot u(x) \geqq c \, |f(x, u(x)| \cdot |u(x)|$$

while it follows from the inequality of Lemma 1 that

$$|u(x)| \geqq c_2 \, |f(x, u(x))|^{1/p-1} - c_3 h(x)^{1/p-1}.$$

Thus

$$f(x, u(x)) \cdot u(x) \geqq c_4 \, |F(u)(x)|^{1+1/p-1} - c_5 \, |F(u)(x)| \, h(x)^{1/p-1}.$$

If we integrate this last inequality and note that $p' = 1 + 1/p - 1$, we find that

$$\begin{aligned} I_2 &\geqq c_4 \|F(u)\|_{L^{p'}}^{p'} - c_4 \int_{|u(x)|\leqq R_0} |F(u)(x)|^{p'} \, dx - c_5 \|F(u)\|_{L^{p'}} \|h\|_{L^{p'}} \\ &\geqq c_4 \|F(u)\|_{L^{p'}}^{p'} - c_6 \|F(u)\|_{L^{p'}} - c_7. \end{aligned}$$

Since $A \leqq \varepsilon A^{p'} + c(\varepsilon)$ for each $\varepsilon > 0$, the inequality (8) follows immediately, while the inequality (9) is an obvious consequence of the inequality (8).

The general form of the argument by which we apply the results of Lemmas 2 and 3 to the proofs of Theorems 3 and 4 is given by the following abstract result.

THEOREM 5. *Let X and Y be two Banach spaces with a bilinear pairing to the reals denoted by (y, x) such that $|(y, x)| \leqq \|y\|_Y \|x\|_X$. Let F be a continuous mapping of X into Y which maps bounded sets of X into bounded sets of Y. Let K be a continuous mapping of Y into X which maps each bounded set in Y into a relatively compact subset of X. Suppose that:*

(1) *$(v, Kv) \geqq 0$ for all v in Y.*

(2) *For each $k > 0$, there exists a constant $c(k) \geqq 0$ such that*

$$(F(u), u) \geqq k \, \|F(u)\|_Y - c(k) \qquad (u \in X).$$

Then for each v in X, the equation $(I + KF)(u) = v$ has at least one solution u in X.

PROOF OF THEOREM 5. Let $C = KF$. Then C is a compact mapping of each ball in X into X. By the Leray-Schauder principle, it suffices to show that there exists a constant $R>0$ such that if u is a solution of the equation $(I+\xi KF)(u)=v$ for a given v and any ξ in $[0, 1]$, then $\|u\|_X \leqq R$. If this equation holds, however, we have

$$(F(u), u) + \xi(F(u), KF(u)) = (F(u), v),$$

and since $(F(u), KF(u)) \geqq 0$ by hypothesis (1), it follows that

$$(F(u), u) \leqq (F(u), v) \leqq \|F(u)\|_Y \, \|v\|_X.$$

On the other hand, we choose $k \geqq \|v\|_X + 1$, and from hypothesis (2), we see that

$$(F(u), u) \geqq k \, \|F(u)\|_Y - c(k).$$

Hence,

$$k \, \|F(u)\|_Y \leqq (F(u), u) + c(k) \leqq \|v\|_X \, \|F(u)\|_Y + c(k),$$

i.e., $\|F(u)\|_Y \leqq c(k)$.

Since $u = -\xi KF(u) + v$, we see that $\|u\|_X \leqq \|KF(u)\|_X + \|v\|_X$. K obviously maps each bounded subset of Y into a bounded subset of X. Hence $\|K(w)\|_X \leqq \beta(\|w\|_Y)$ for a suitable function β and all w in Y. Thus,

$$\|u\|_X \leqq \beta(c(k)) + \|v\|_X. \quad \text{Q.E.D.}$$

Theorems 3 and 4 follow from Theorem 5 by specializing X and Y to be $L^\infty(\Omega)$ and $L^1(\Omega)$ in the first case, and $L^p(\Omega)$ and $L^{p'}(\Omega)$ in the second case. The validity of the hypothesis (2) in the two cases follows from Lemmas 2 and 3 respectively. We note that the argument given in the proof of Theorem 5 shows that the assumption that K is linear and monotone is unnecessary provided that one takes $\psi=0$ and assumes merely that K is compact and that $(v, Kv) \geqq 0$ for all v in the appropriate space.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637